Paper is in preparation

Grassmann tensor renormalization group approach to (1+1)-dimensional two-color QCD with staggered fermion

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Content

TN approach to lattice field theories

- Tensor renormalization group (TRG), and tensor network (TN) methods in general, are free from the sign problem

- TN studies on (1+1)-D QCD have been active recently, as the first step towards (3+1)-D QCD:

  ✓ Hamiltonian approach:

    [S. Kuhn+, JHEP 07 (2015) 130] [P. Silvi+, Quantum 1 (2017) 9] [M. C. Banuls+, PRX 7 (2017) 041046]
    [P. Sala+, PRD 98 (2018) 034505] [P. Silvi+, PRD 100 (2019) 074512] [M. Rigobello+, 2308.04488]
    [H. Liu+, 2312.17734] [T. Hayata+, JHEP 07 (2024) 106]

  ✓ Lagrangian approach:

    [J. Bloch & R. Lohmayer, Nucl. Phys. B 986 (2023) 116032] [M. Asaduzzaman+, JHEP 05 (2024) 195] [Thomas' talk on 31/7/2024]

    SU(3), infinite coupling, finite density

    SU(2) with reduced staggered fermion, finite coupling, zero density

    SU(3) with staggered fermion, finite coupling, finite density

- Some issues to consider
  1) How to discretize the gauge group integration? (e.g. character expansion, Gauss-Legendre, Taylor expansion…)
  2) How to handle the numerous degrees of freedom in practical computation?

The initial bond dimension is inevitably large!

<u>Today:</u> *1+1-D* SU(2) lattice gauge theory at finite coupling with standard staggered fermions at finite density

Lattice theory

- 1+1-dimensional SU(2) Yang-Mills theory coupled with staggered fermion on a square lattice

What we calculate with TRG

$$Z = \int \mathcal{D}U \mathcal{D}\chi \mathcal{D}\bar{\chi}\, \mathrm{e}^{-S}$$

$$S = S_f + S_g + S_\lambda$$

$$S_f = \sum_{n\in\Lambda,\, \nu=1,2} \frac{p_\nu(n)}{2}\left[\mathrm{e}^{\mu\delta_{\nu,2}}\bar{\chi}(n)U_\nu(n)\chi(n+\hat{\nu}) - \mathrm{e}^{-\mu\delta_{\nu,2}}\bar{\chi}(n+\hat{\nu})U_\nu^\dagger(n)\chi(n)\right] + m\sum_n \bar{\chi}(n)\chi(n),$$

$$p_1(n) = 1 \qquad p_2(n) = (-1)^{n_1}$$

$$S_g = -\frac{\beta}{N}\sum_n \mathrm{ReTr} U_1(n)U_2(n+\hat{1})U_1^\dagger(n+\hat{2})U_2^\dagger(n)$$

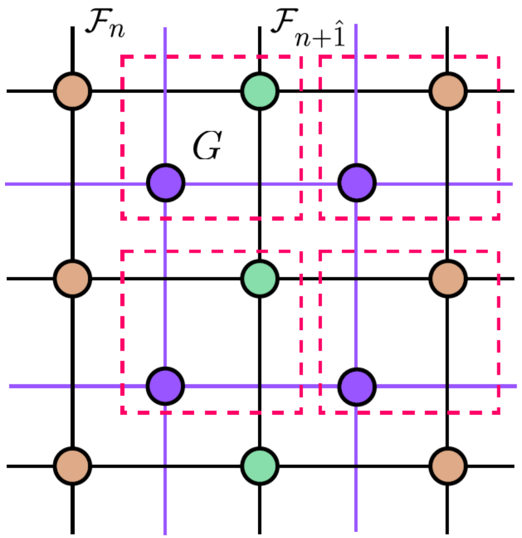
$$S_\lambda = \frac{\lambda}{2}\sum_n \left[\chi^T(n)\sigma_2\chi(n) + \bar{\chi}(n)\sigma_2\bar{\chi}^T(n)\right]$$

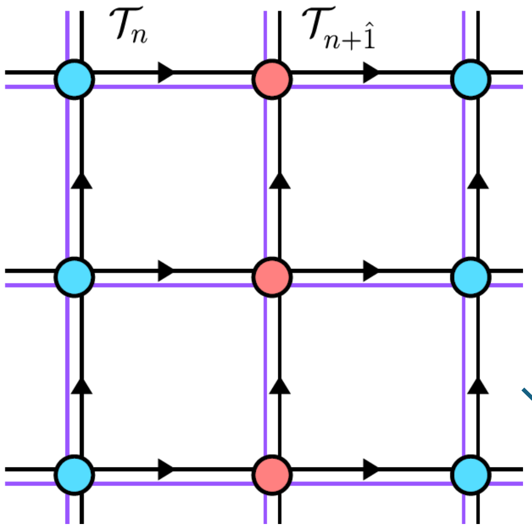
- Mermin-Wagner-Coleman theorem: no spontaneous breaking of continuous symmetry occurs in 2D
- In this study, we always employ finite m or λ which explicitly breaks $U_A(1)$ or $U_V(1)$ symmetry respectively

- Grassmann path integral is expressed as the trace of a Grassmann tensor network by introducing a two-component auxiliary Grassmann field on edges to decompose each of the hopping terms [Akiyama, S., & Kadoh, D., JHEP, 2021(10), 1-16]

- The gauge group integration is dicretized by a summation with group elements sampled uniformly from the group manifold (The sample size is denoted as K) [Fukuma, M.+, PTEP, 2021(12),123B03]

$$\int \mathrm{d}U\, f(U) \simeq \frac{1}{K}\sum_{i=1}^{K} f(U_i)$$





$$Z = \int \mathcal{D}U\mathcal{D}\chi\mathcal{D}\bar{\chi}\, \mathrm{e}^{-S}$$

Bond dimension: $2^{2N}K$

Fermion bond dimension

- Our construction: $\boldsymbol{2^{2N}}$
- [M. Asaduzzaman+, JHEP 05 (2024) 195]: $\boldsymbol{2^{2N^2}}$

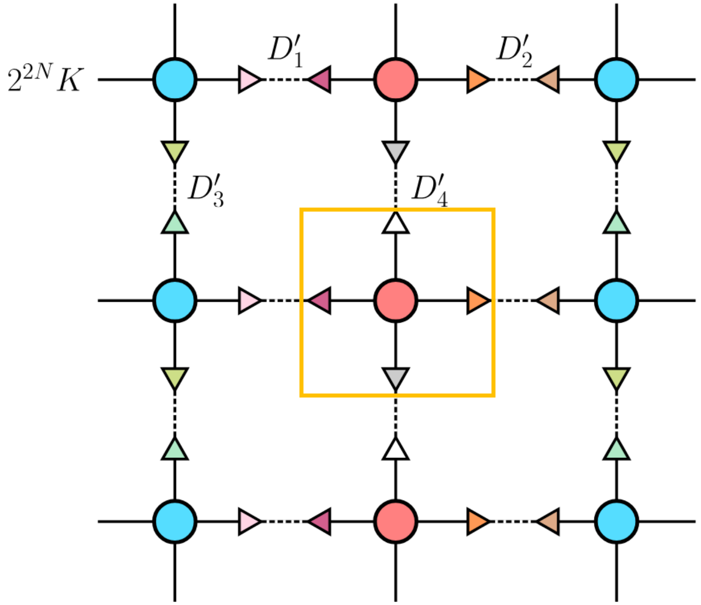
Infinite-coupling limit ($\beta = 0$)

The link variables can be integrated exactly, and the bond dimension of the tensor is 2^{2N} only

Compression of initial tensors

[Adachi, D.+, PRB, 105(6), L060402 (2022)]
[Akiyama, S., JHEP, 2022(11), 1-14]

- We use bond-weighted tensor renormalization group to coarse-grain the tensor network and reach the thermodynamic limit
- The bond dimension cutoff in TRG algorithms usually depends on the bond dimension of initial tensors.
  In our case (two-color i.e., *N=2*), the initial bond dimension is *16K* ⇨ tensor compression scheme is needed
- Main idea: insert a pair of squeezers, which acts as a good approximation of identity, on every bond of the tensor network.



$$m = 0.1, \quad \beta = 1.6, \quad \mu = 0.4, \quad K = 14$$

Accuracy of squeezers	Original bond dimension (16*K*)	D_2', D_4', D_1', D_3'	# of elements in compressed tensor / # of elements in original tensor
0.99	224	33, 32, 33, 32	0.044%
0.999		64, 63, 64, 63	0.646%
0.9995		75, 72, 75, 72	1.158%
0.9999		100, 97, 100, 97	3.737%
0.99995		112, 109, 112, 109	5.920%
0.99999		137, 133, 137 133	13.187%

The initial bond dimension can be reduced to less than half while keeping the Frobenius norm of the contraction well enough

Definition of observables

Free energy density:

$$f = \frac{\ln Z}{V}$$

What we calculate with TRG

Number density:

$$\langle n \rangle = \frac{\partial f}{\partial \mu}$$

Fermion condensate:

$$\langle \bar{\chi}\chi \rangle = \frac{\partial f}{\partial m}$$

Diquark condensate:

$$\langle \chi\chi \rangle = \frac{\partial f}{\partial \lambda}$$

The derivatives are computed by forward differences:

$$\langle n \rangle = [f(\mu + \Delta\mu) - f(\mu)]/\Delta\mu$$

$\lambda = 0$

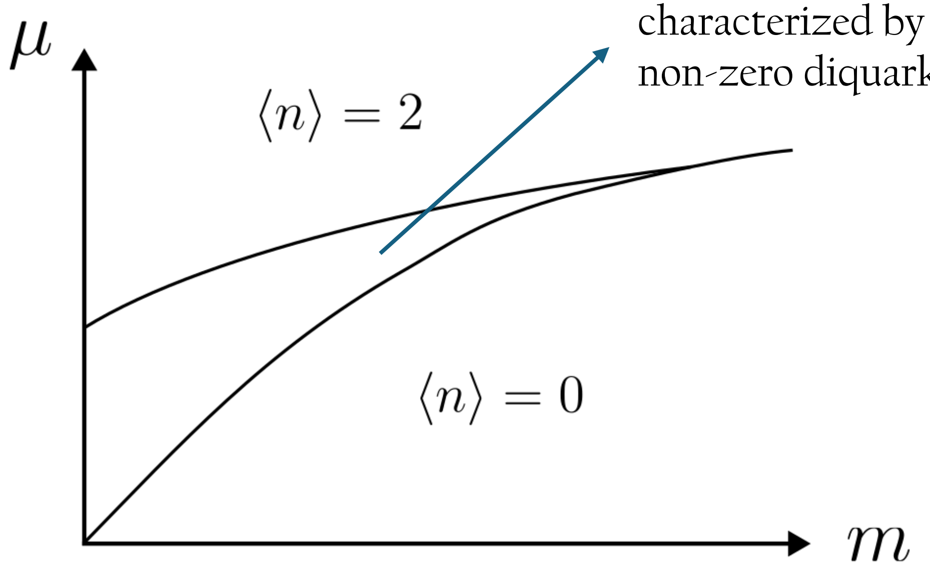
$$\langle \bar{\chi}\chi \rangle = [f(m + \Delta m) - f(m)]/\Delta m$$

$\Delta m = 10^{-4}$ $\lambda = 0$

$$\langle \chi\chi \rangle = [f(\lambda + \Delta\lambda) - f(\lambda)]/\Delta\lambda$$

$\lambda = \Delta\lambda = 10^{-4}$

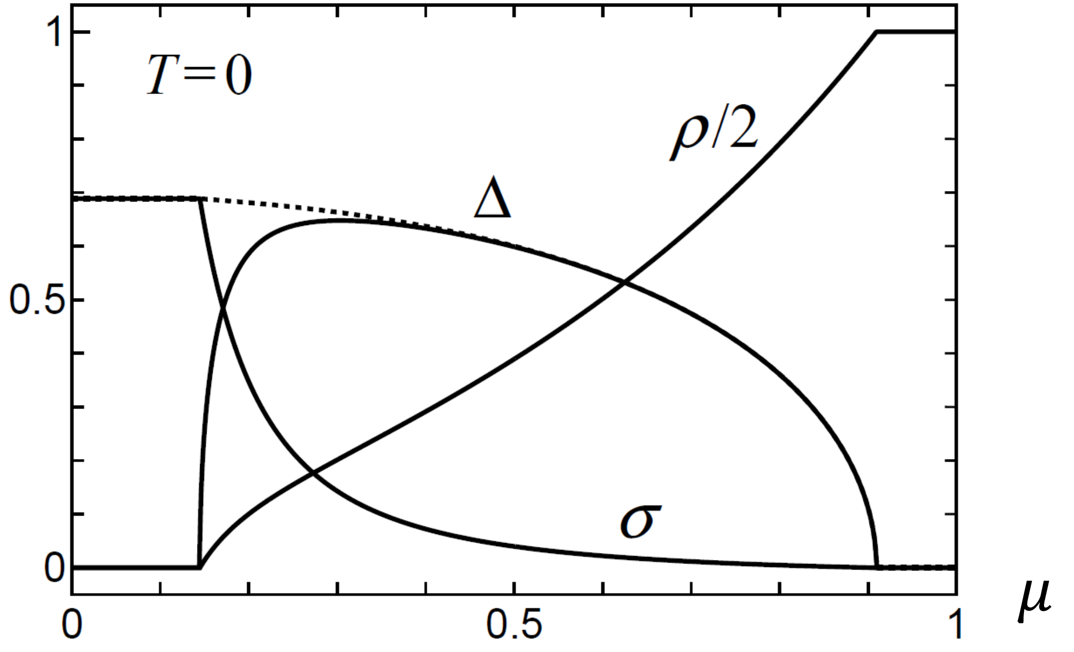
Phase structure of higher dimensional two-color QCD



An intermediate phase characterized by $0 < \langle n \rangle < 2$ and non-zero diquark condensate

- The phase diagram of infinite coupling two-color QCD in (3+1)-D obtained by *1/d* expansion and mean-field approximation [Y. Nishida+, Phys. Rept. 398 (2004) 281–300]

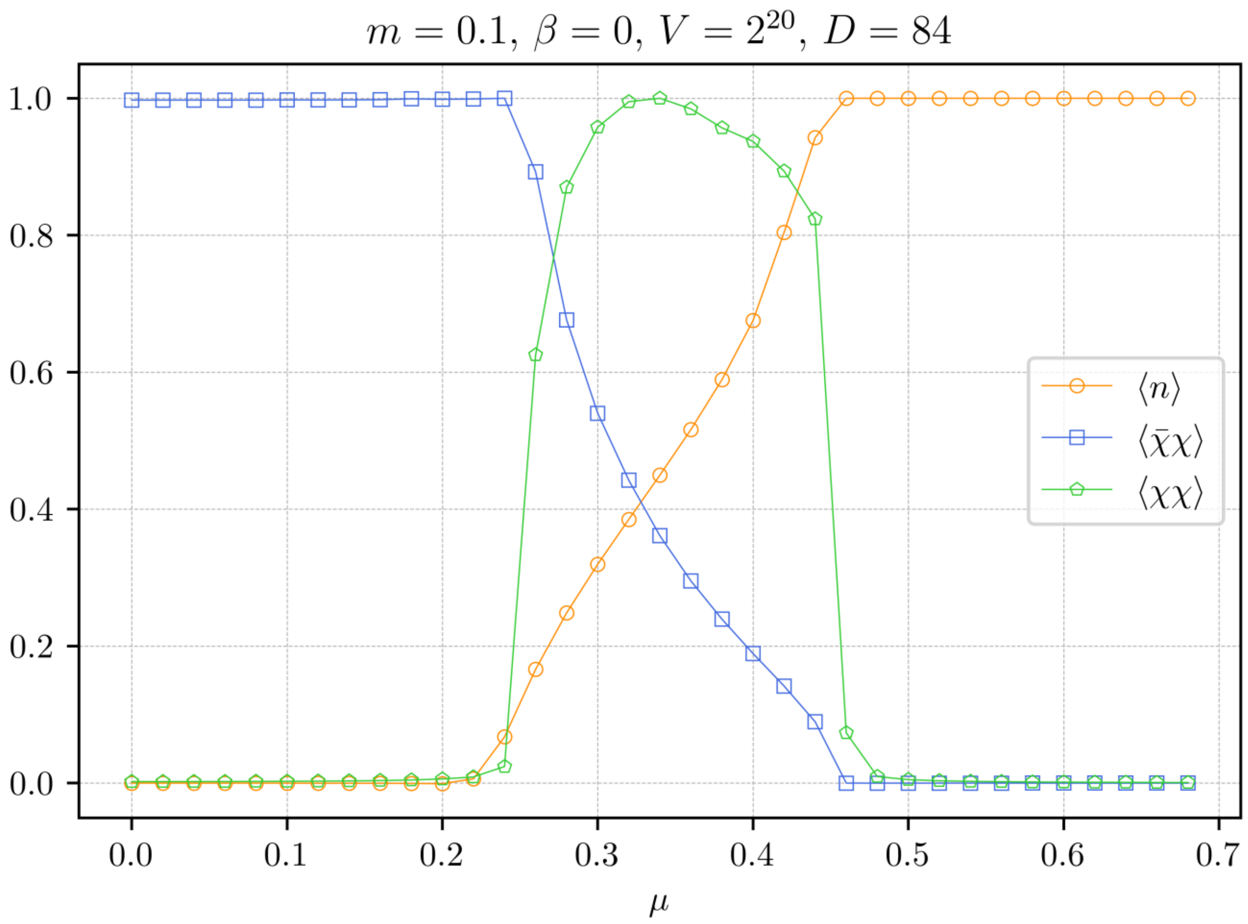
Captured from [Y. Nishida+, Phys. Rept. 398 (2004) 281–300]
Mass = 0.02, spatial dimension = 3

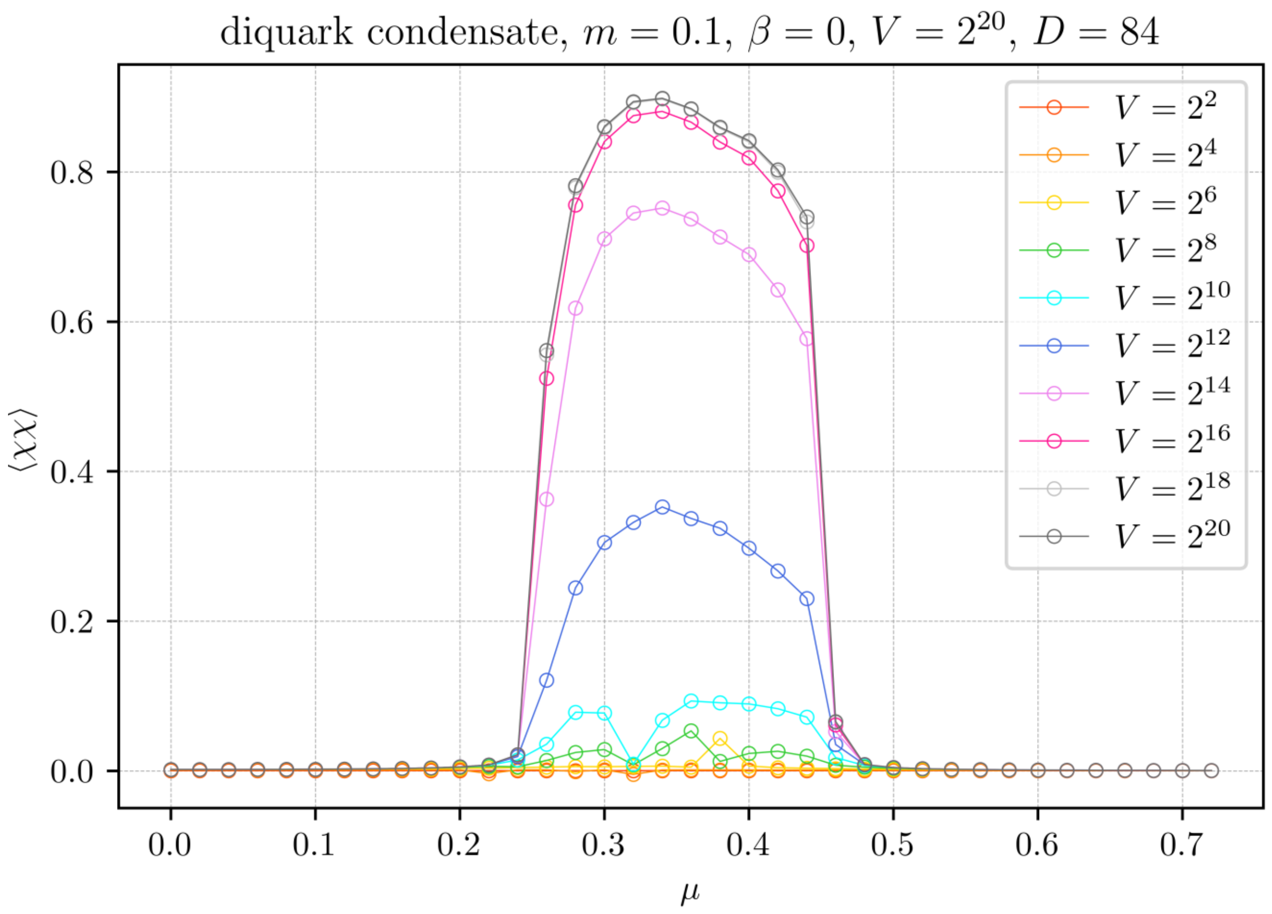


- Can we use TRG to calculate similar quantities in (1+1)-D, with finite m and/or λ?

Numerical results: infinite coupling limit

The bond dimension is 16 only, so no initial tensor compression here

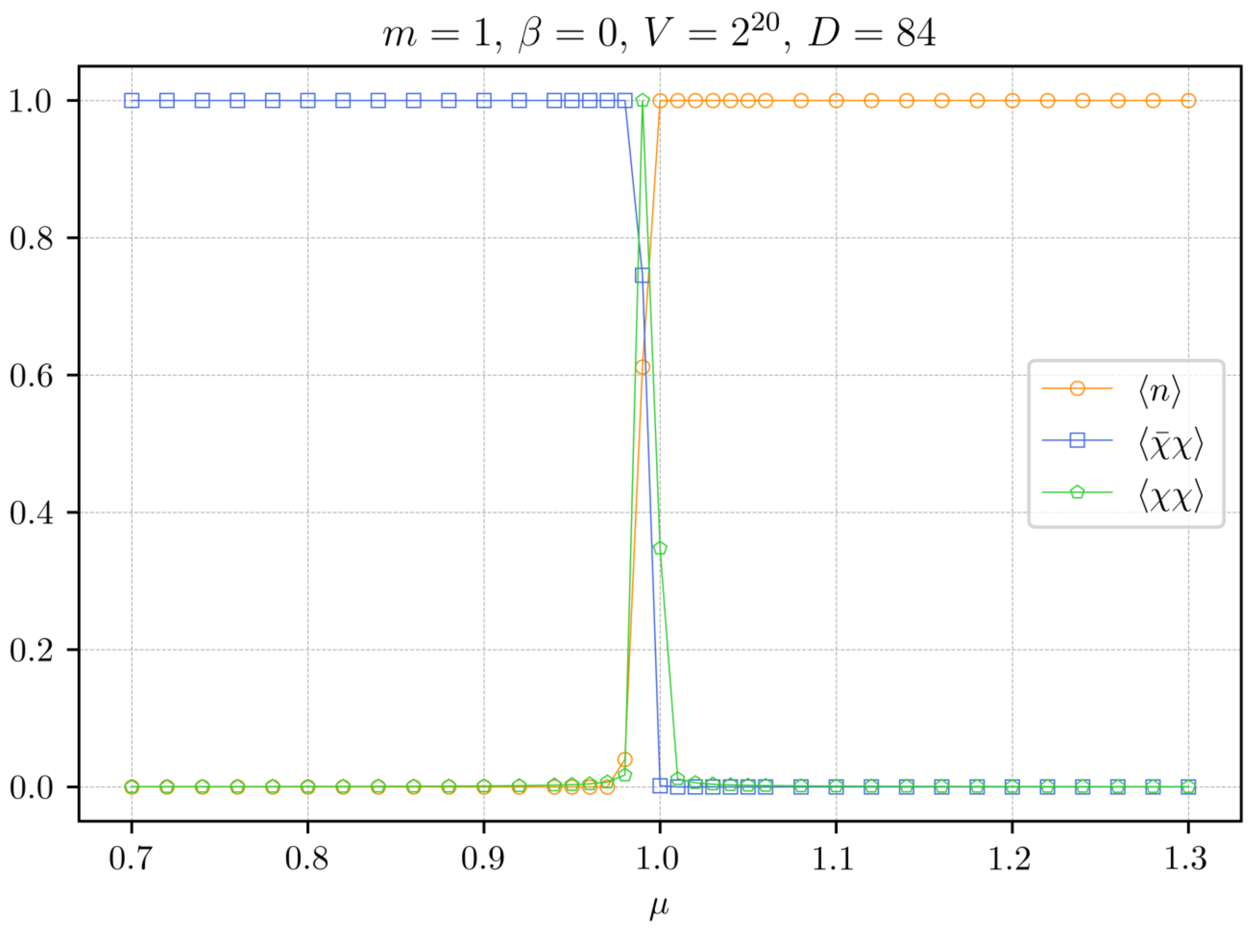


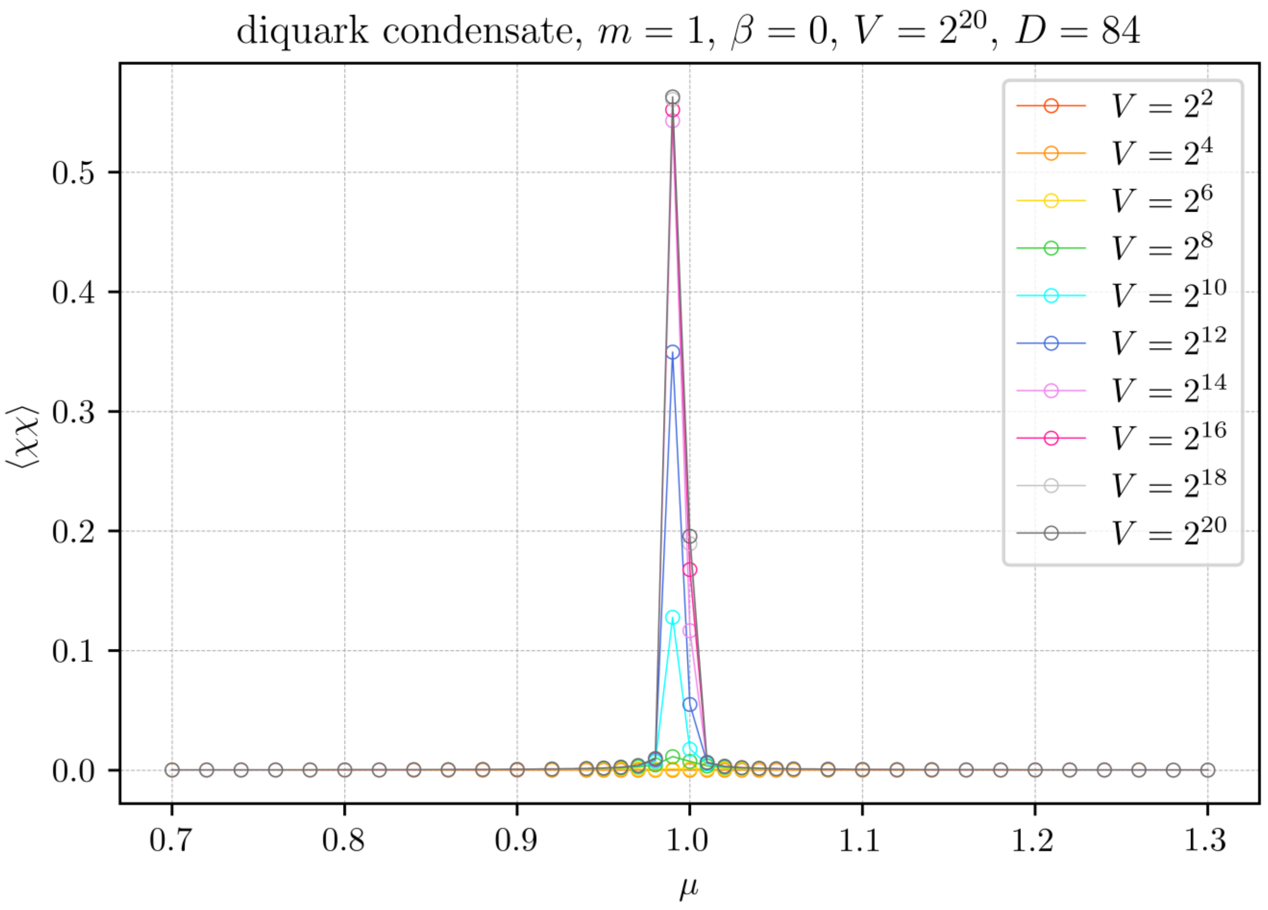


- An intermediate phase is observed in a finite region of μ at $m = 0.1$
- The qualitative behavior of the observables at finite m and/or λ is similar to that in the (3+1)-D case

Numerical results: infinite coupling limit

The bond dimension is 16 only, so no initial tensor compression here

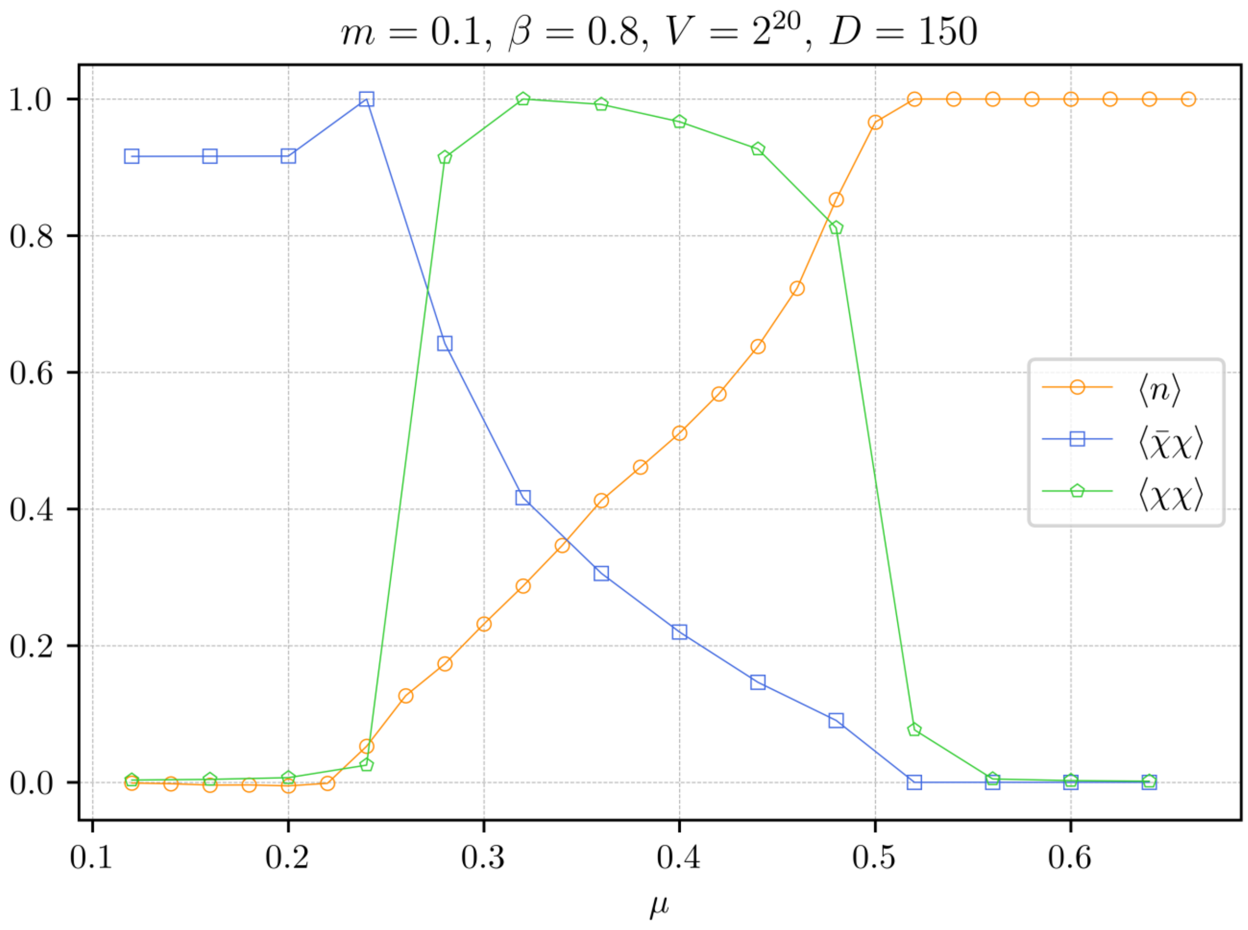


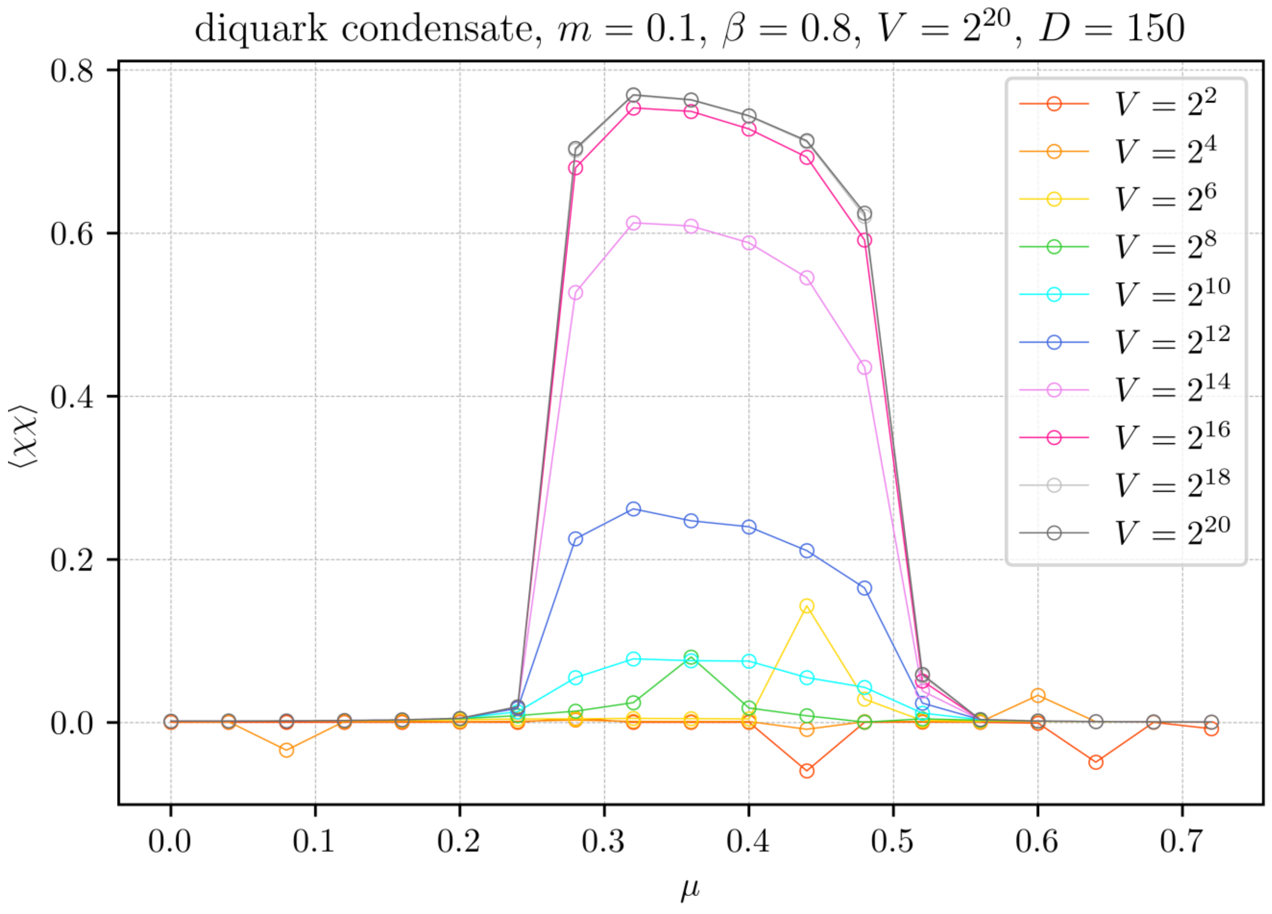


- A sharp transition is observed, and the intermediate phase becomes a very narrow region at $m = 1$

Numerical results: finite β regime

K=14

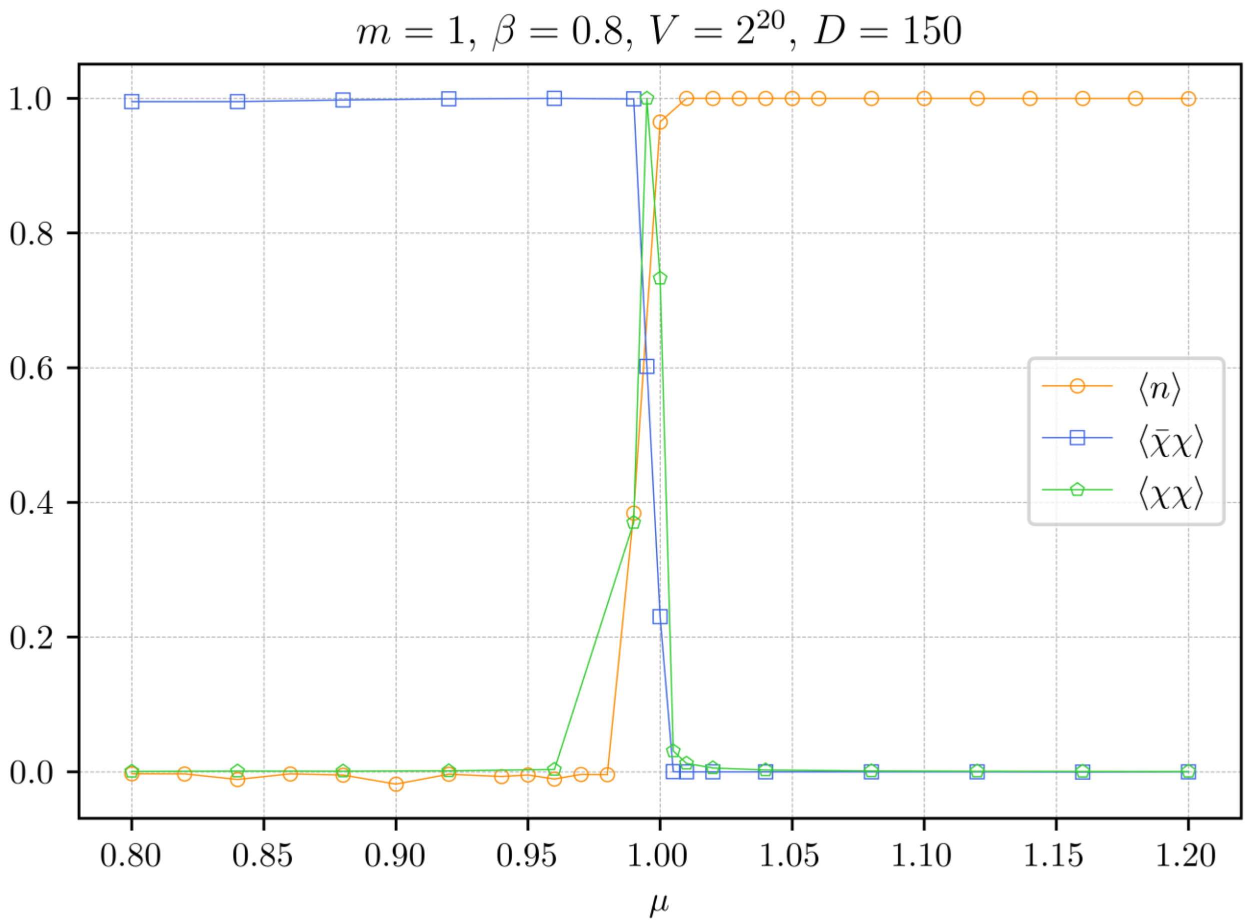


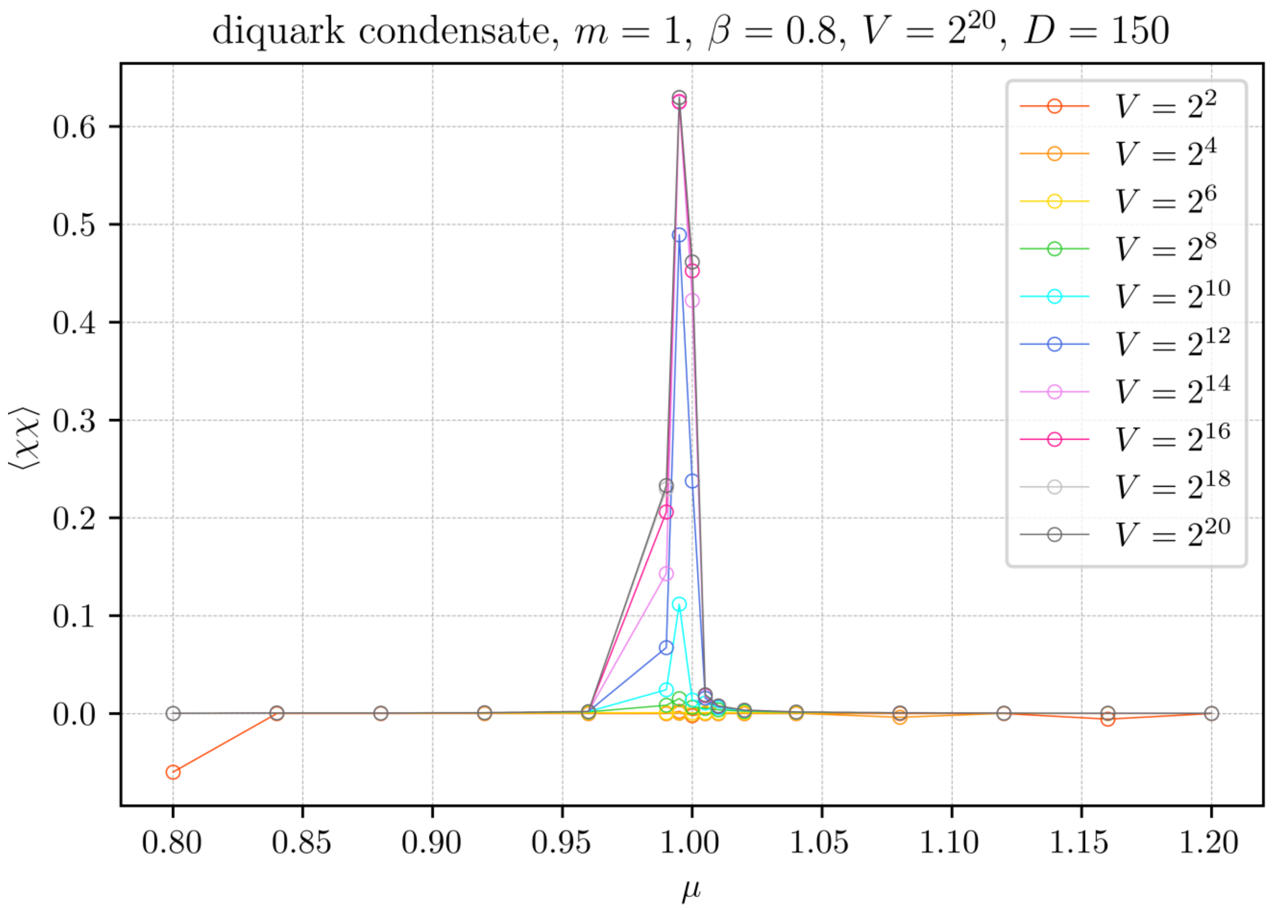


- As β becomes nonzero, the intermediate phase becomes broader at $m = 0.1$

Numerical results: finite β regime

K=14

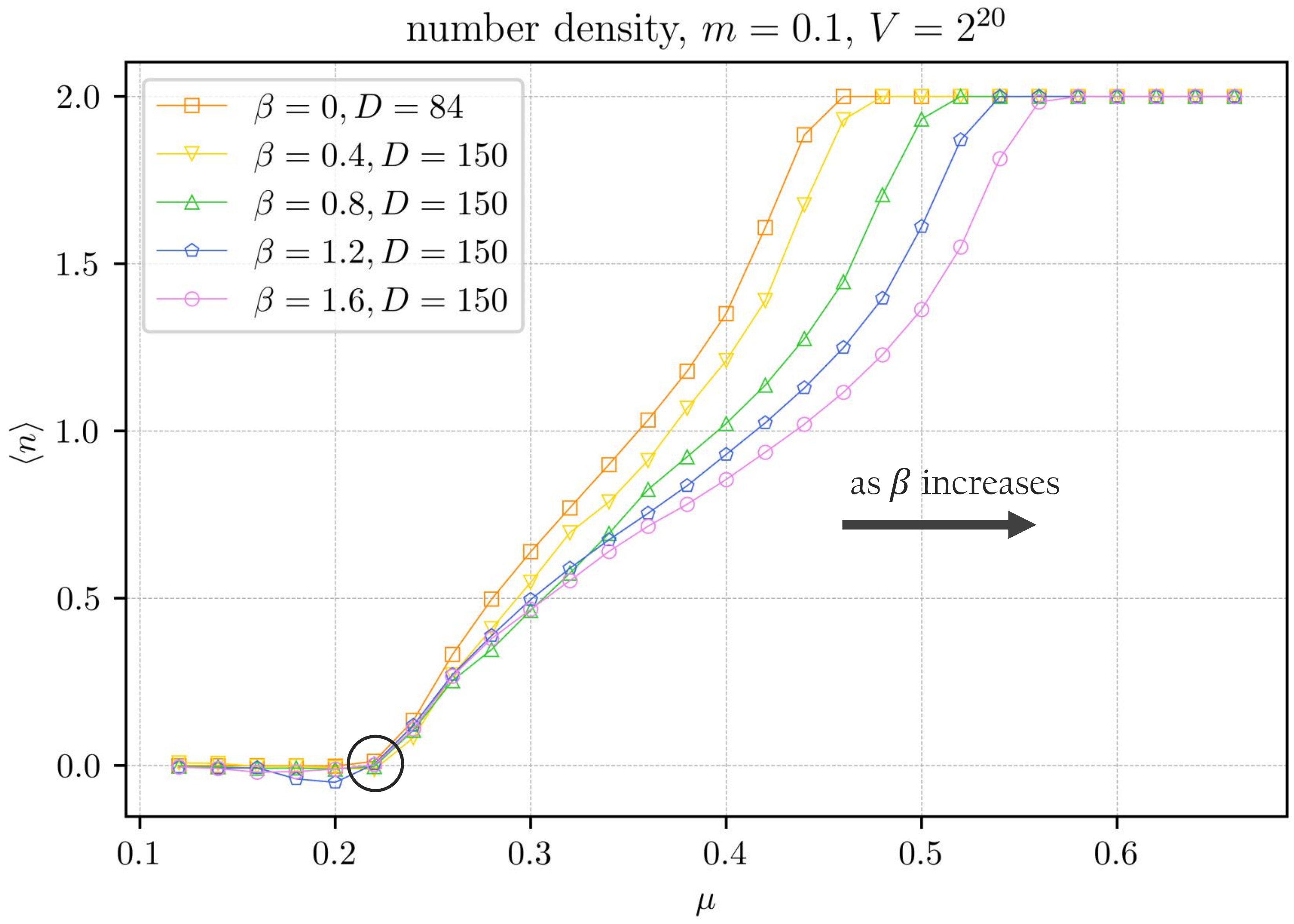




- For $m = 1$, the behavior at finite coupling is similar to that at infinite coupling

β dependence on transition position

K=14



- The first transition point (the one with smaller μ) seems to be robust against β
- The second transition point locates at larger chemical potential as β increases
- $\langle n \rangle$ does not saturate in regions of larger chemical potential as the gauge interaction is weakened, approaching the continuum limit

Summary

- This is a TRG study on non-Abelian gauge theory coupled with standard staggered fermions at finite density and finite coupling
- Tensor network calculation for this kind of theories is computationally challenging because of the very large initial bond dimension
- We introduce an efficient initial tensor compression scheme to deal with this issue
- TRG enables the calculation of several physical quantities at the infinite coupling limit and finite β regime
- Future directions:
  1) Construction of tensors which allows a larger sample size K for the discretization of gauge group
  2) Chiral limit and vanishing λ limit in higher dimensions
  3) Extension to the SU(3) gauge group
  4) Investigation of inhomogeneous phase [T. Kojo, Nucl. Phys. A 877(2012) 70-94] [T. Hayata+, JHEP 07 (2024) 106]

Thank you for listening!

Backup slides

Backup: tensor network representation

Grassmann TN representation for fermionic sector

[Akiyama, S., & Kadoh, D., JHEP, 2021(10), 1-16]

- We introduce two-component auxiliary Grassmann fields on each link of the lattice to decompose each of the forward and backward hopping term
- Integrate out the original staggered fermion fields on each site
- The Grassmann path integral becomes the trace of a Grassmann tensor network with the auxiliary Grassmann fields
- *2N* binary indices are assigned on every link to specify the occupation number of the auxiliary Grassmann fields

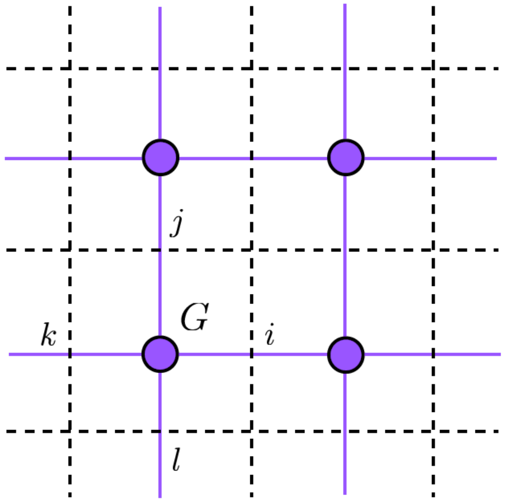
$$Z_f[U] = \int \mathcal{D}\chi \mathcal{D}\bar{\chi}\ \mathrm{e}^{-S_f}$$

$$= \prod_{n,\nu} \int_{\bar{\eta}_\nu(n),\eta_\nu(n)} \int_{\bar{\zeta}_\nu(n),\zeta_\nu(n)} \left[\prod_n \mathcal{F}_n[U] \right]$$

Discretization of gauge group integration

[Fukuma, M., Kadoh, D., & Matsumoto, N., PTEP, 2021(12),123B03]

- The gauge group integration is dicretized by a summation with group elements sampled uniformly from the group manifold

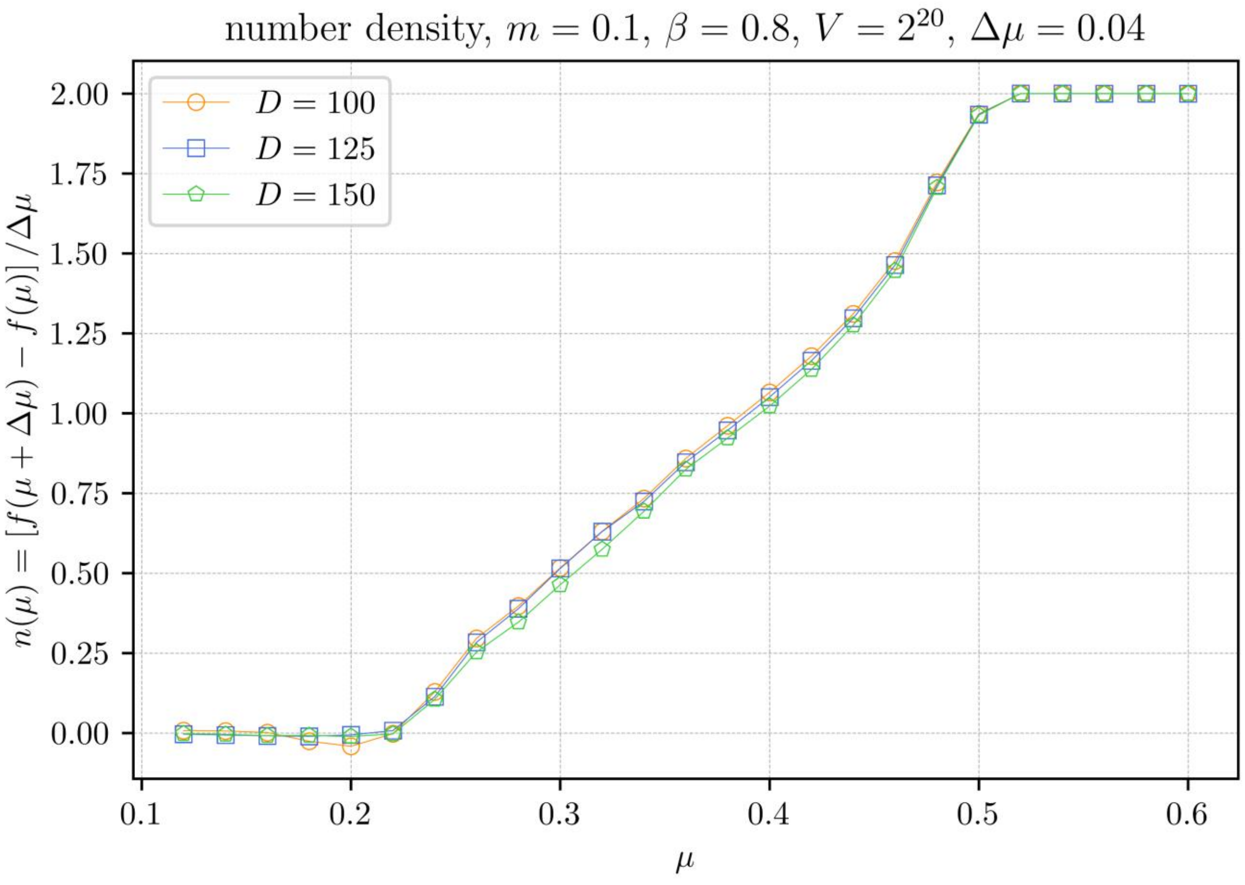


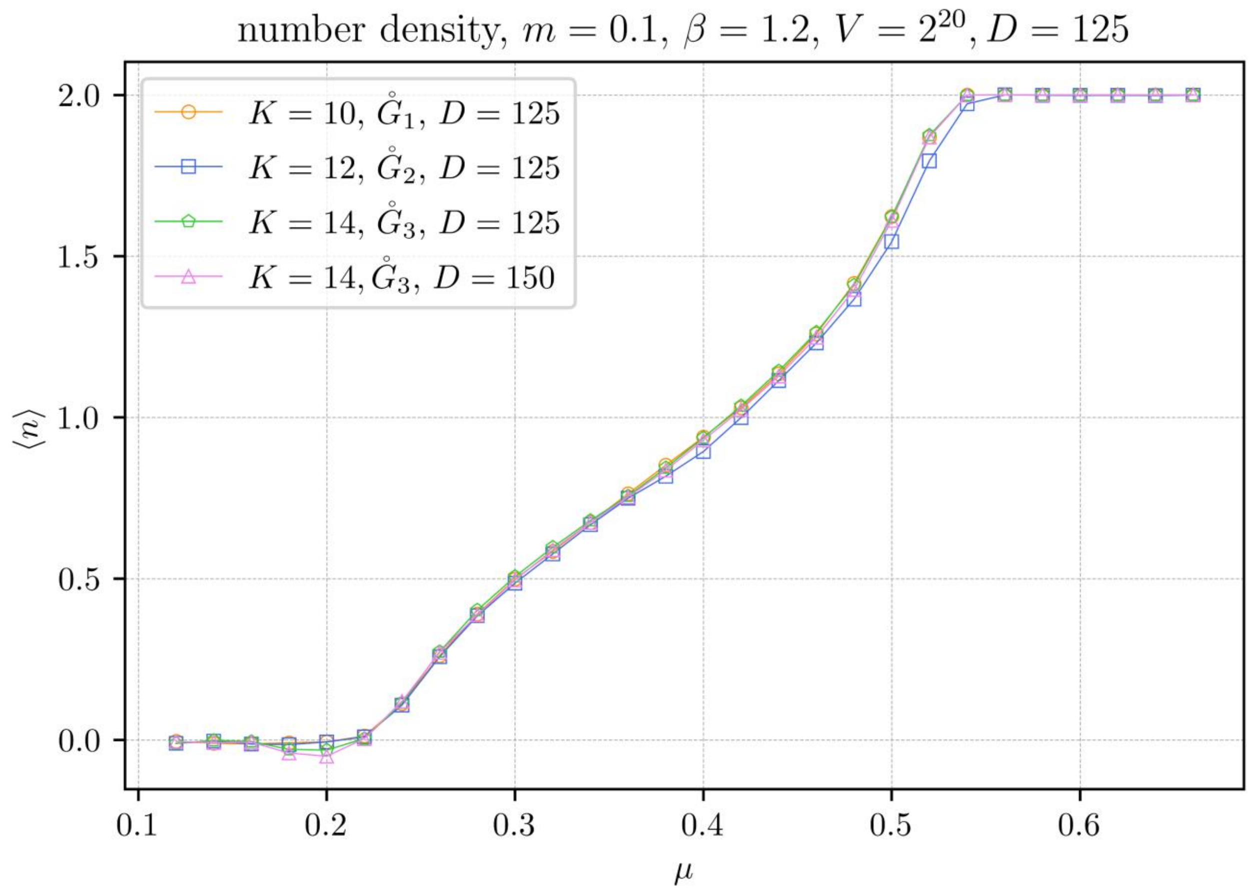
$$\int \mathrm{d}U\ f(U) \simeq \frac{1}{K} \sum_{i=1}^{K} f(U_i)$$

$$G_{ijkl} \equiv \frac{1}{K^2} \mathrm{e}^{(\beta/N)\,\mathrm{Re\,Tr}\left(U_i U_j^\dagger U_k^\dagger U_l\right)}$$

Backup: algorithmic parameter dependence

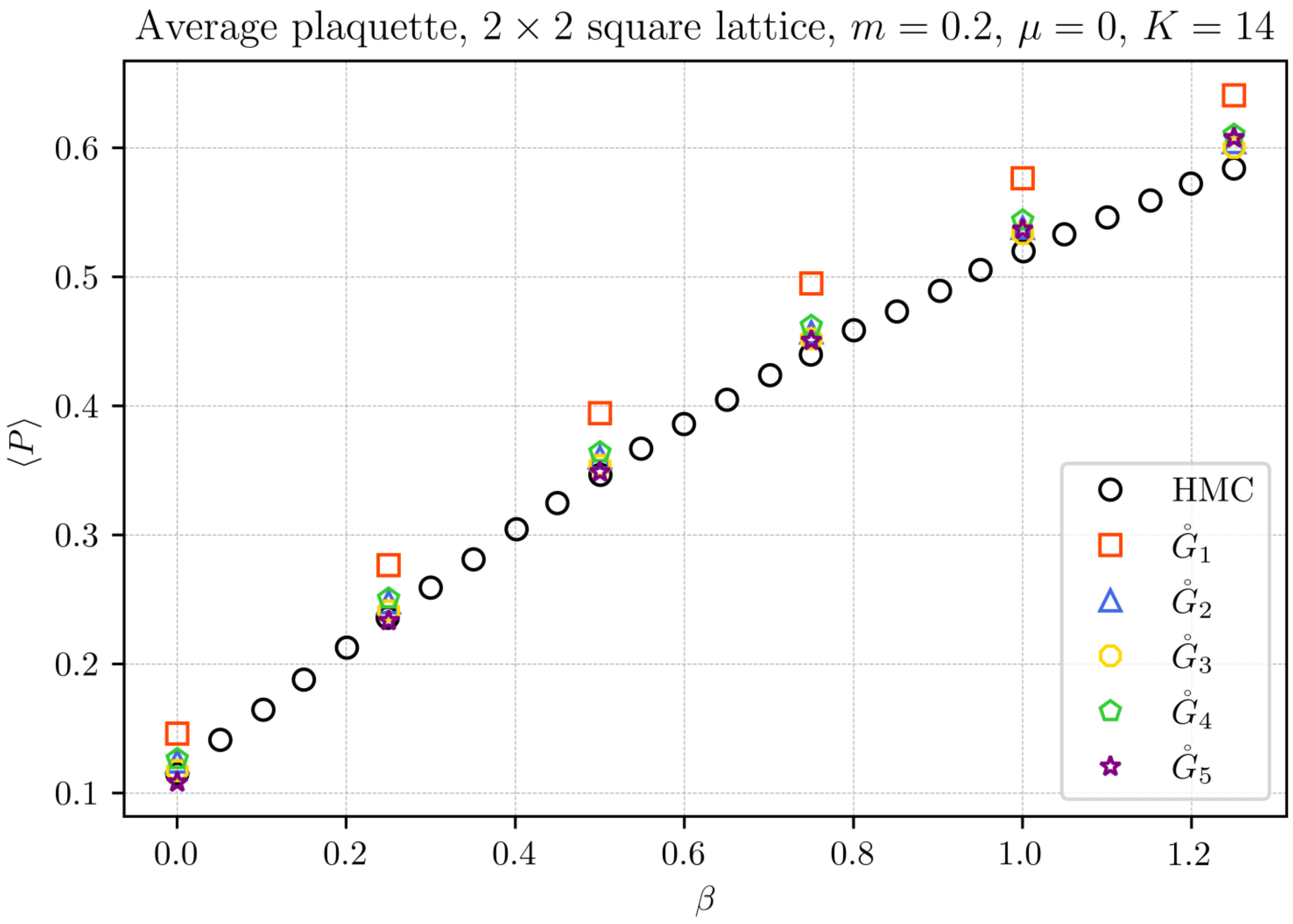
K=14





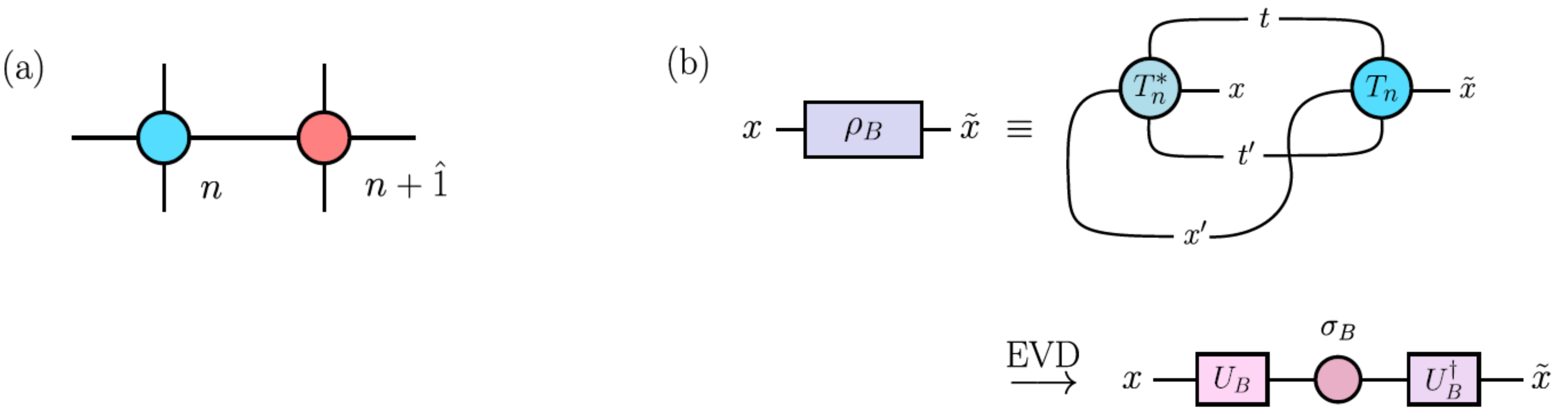
Backup: Results on a 2×2 lattice by exact contraction

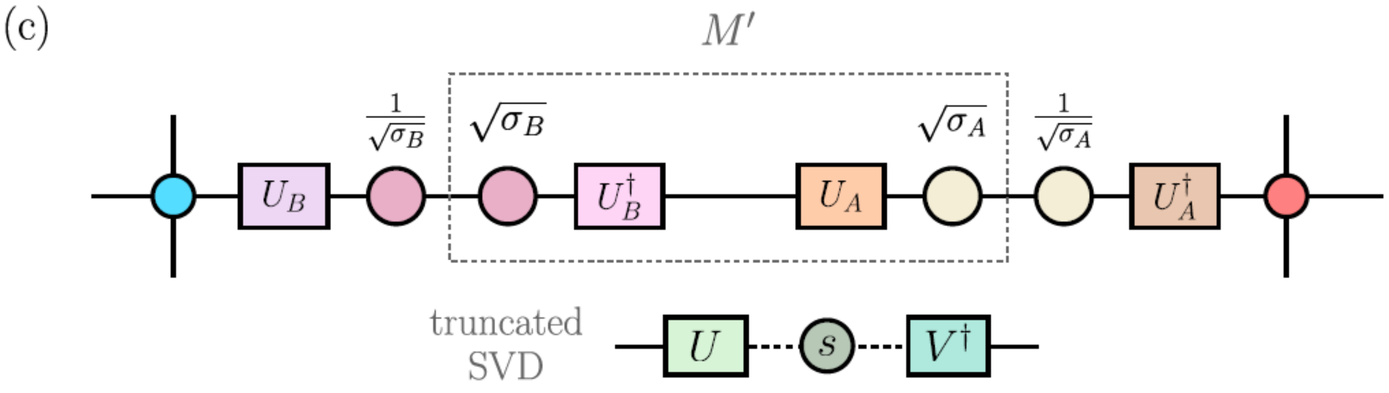
K=14



[G. Gagliardi & W. Unger, PRD 101, 034509 (2020)]

Backup: scheme of squeezer construction





D' (the bond dimension of the compressed tensor) is equal to the number of singular values kept in this step

$$\frac{\sum_{i=1}^{D'} s_i^2}{\sum_{i=1}^{2^{2N}K} s_i^2} \geq r = 0.9999$$

Backup: scheme of squeezer construction