Tensor Renormalization Group Approach to Entanglement Entropy

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in collaboration with

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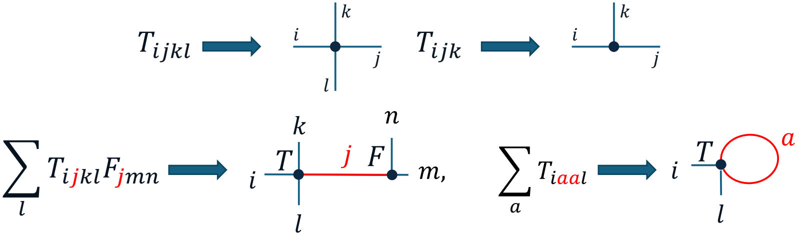
Introduction - Tensor network

- Partition functions and expectation values of physical quantities can be represented as a tensor network.

$$Z = \int \mathcal{D}\phi e^{-S[\phi]} = \sum_{\dots,a,b,c,d,e,f,g,\dots} \dots T_{abcd}T_{efag}\dots$$

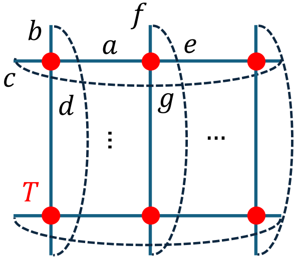
- Tensor diagram:



Introduction - Tensor network

- Partition function represented as a tensor diagram:

$$Z = \int \mathcal{D}\phi e^{-S[\phi]} = \sum_{\dots,a,b,c,d,e,f,g,\dots} \dots T_{abcd} T_{efag} \dots =$$

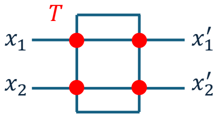


Introduction - Tensor renormalization group

- Coarse-graining of tensor networks and reducing computational cost.
- Various TRG algorithms
  - Tensor renormalization group [Levin-Nave, 2007]
  - Higher-order tensor renormalization group (HOTRG) [Xie et al., 2012]
  - Tensor network renormalization [Evenbly-Vidal, 2015]
  - Anisotropic TRG [Adachi-Okubo-Todo, 2019]
  - Triad TRG [Kadoh-Nakayama, 2019]

Introduction - Higher-order tensor renormalization group (HOTRG)

1. Obtain isometry U and $U^\dagger$.



$$= \sum_{k=1}^{N^2} U_{(x_1x_2)k}\, \lambda_k\, V^T_{k(x'_1x'_2)} \sim \sum_{k=1}^{D} U_{(x_1x_2)k}\, \lambda_k\, V^T_{k(x'_1x'_2)}$$

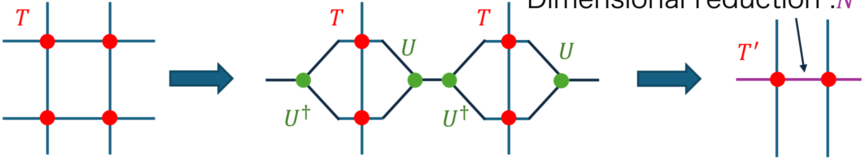
Dimension of — is N

Truncation of the sum at D

D determines the accuracy of the analysis.

2. Contract the tensor and isometry.

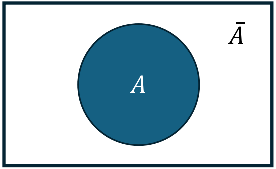
Dimensional reduction :$N^2 \to D$



Introduction - entanglement entropy

- A measure of the degree of quantum entanglement between two subregions in a bipartite quantum system.

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A$$



  where $\rho_A = \mathrm{Tr}_{\bar{A}}\rho$

- Applications in various fields
  - Particle physics: quantum correction to the entropy of black holes.
  - Condensed matter physics: probe of the quantum phase transitions.
  - Quantum information: quantification of entanglement in quantum states.

$\rightarrow$ A general computation method of the EE is needed.

Introduction - numerical analysis of the EE

- Monte Carlo method: EE through entropic C-function.
  e.g. 4d SU(3) gauge theory [Itou-Nagata-Nakagawa-Nakamura-Zakharov, 2015]

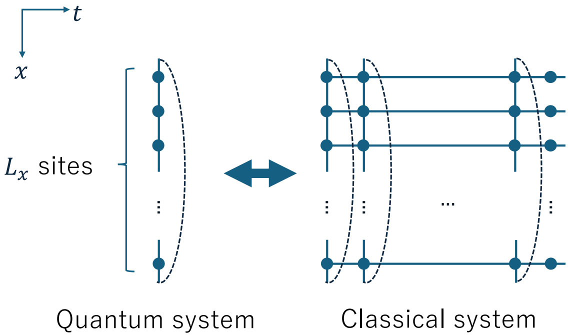
- Tensor network: direct computaion of the EE, without sign problem.
  e.g. $(1+1)$d $O(3)$ non-linear sigma model [Kuramashi-Luo, 2023]
  (Kuramashi san told us the detail in his talk on last Tue.)

- Previous tensor network analyses are limited to the EE of half-space subregion.

We propose a general TRG method of computing the entanglement entropy for any subregion size.

EE of a 1D quantum system with periodic boundary condition

- 1D quantum system at zero temperature with spatial size $L_x = 2^m$.
  $\leftrightarrow$ $(1+1)$D classical system with spatial size L_x and infinite temporal size.

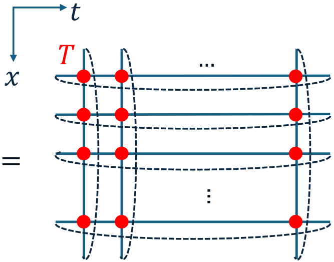


We consider the corresponding (1+1)D classical system instead.

Tensor networks in the 2D classical system

- Partition function Z as a tensor network:

$$Z = \mathrm{Tr}\rho =$$

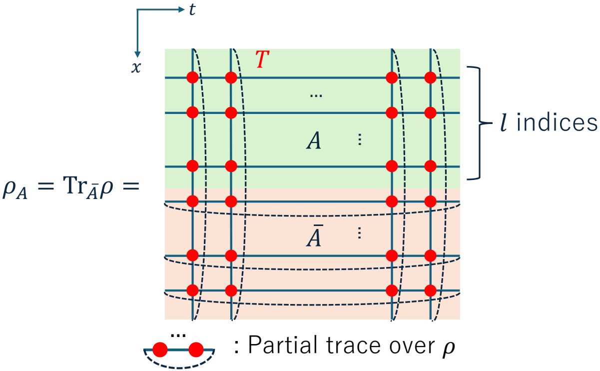


 : Periodic boundary condition  : Trace over ρ

- The number of tensors is $L_x \times L_t$ where $L_x = 2^m$ and $L_x \ll L_t$.
  The detail of the tensor T depends on the model.

Tensor networks in the 2D classical system

- Reduced density matrix ρ_A of subregion A with spatial size l:



- l spatial indices within subregion $A \to$ index of ρ_A
  $\to \rho_A$ is a $D^l \times D^l$ matrix.

Tensor networks in the 2D classical system

- EE can be computed from ρ_A

$$S_A = -\mathrm{tr}\rho_A \log \rho_A = -\sum_a \lambda_a \log \lambda_a,$$

where λ_a is an eigenvalue of ρ_A.

Problems:

- Large number of tensor contractions for computing ρ_A.
  $\rightarrow$ use HOTRG for reducing computational cost.
- Huge matrix size of ρ_A.
  $\rightarrow$ our new approach: reducing the ρ_A to the smaller effective one ρ'_A:

$$\rho_A \rightarrow \rho'_A$$

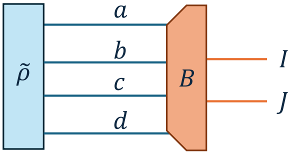
where ρ'_A is a $D^h \times D^h$ matrix, and $h \ll l$.

Our approach to the reduced densoty matrix

- Our approach: reducing ρ_A to smaller effecticve matrix ρ'_A.

  $\rho'_A =$ (coarse-grained density matrix $\tilde{\rho}$) $\times$ (tree-like tensor B made of isometries)

$$(\rho'_A)_{IJ} = \tilde{\rho}_{abcd} B_{abcdIJ} =$$



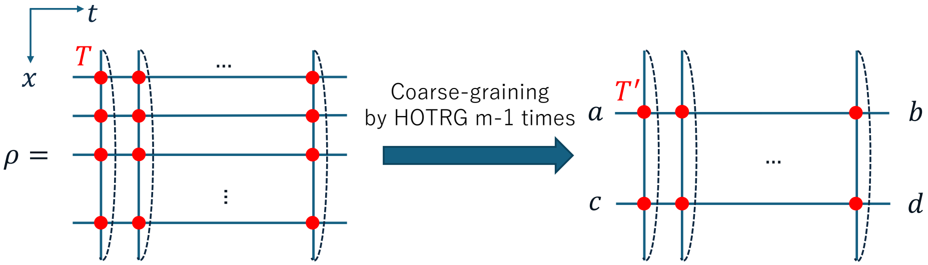
where $I, J = 1, 2, \ldots, D^h$, $l = \sum_n 2^n \cdot C_n$ $(C_n = 0, 1)$, $h = \sum_n C_n$.

$\rightarrow$ matrix size is reduced from $D^l \times D^l$ to $D^h \times D^h$

$$h = 1 \text{ for } l = 2^p$$
$$h = 2 \text{ for } l = 2^p + 2^q$$
$$\vdots$$

Our approach to reduced density matrix

- Coarse-grained density matrix $\tilde{\rho}$

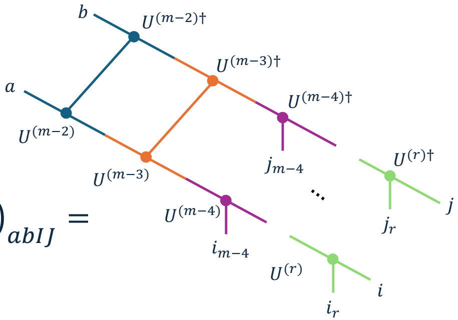


Isometries $U^{(k-1)}$ obtained through k-th HOTRG procedure in spatial direction $(k = 1, 2, \ldots, m-1)$ are ingredients of tensor B.

Our approach to reduced density matrix

- Tensor B
  $=$(product of isometries $U^{(n)}$ and $U^{(n)\dagger}$) $\times$ (Kronecker deltas).
- Product of isometries

$$\left(U^{(m-2)}\, U^{(m-2)\dagger}\, U^{(m-3)}\, U^{(m-3)\dagger} \cdots \right)_{abIJ} =$$

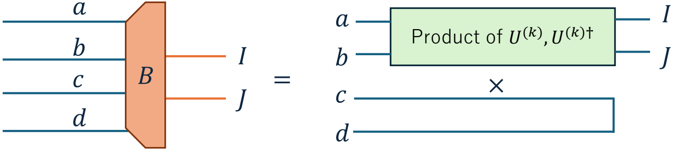


$$I = (i_{m-4}, \cdots, i), \qquad J = (j_{m-4}, \cdots, j)$$

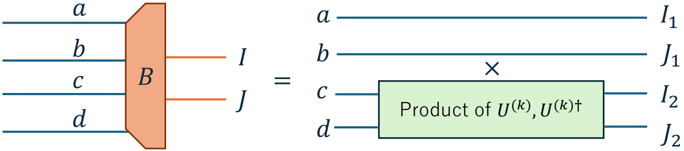
The detail of the product of isometries $U^{(k)}, U^{(k)\dagger}$ depends on the size of subregion A.

Our approach to reduced density matrix

- For $l \leq L_x/2$: $B_{abcdIJ} = \left(U^{(m-1)}U^{(m-1)\dagger}U^{(m-2)}U^{(m-2)\dagger}\cdots\right)_{abIJ}\delta_{cd}$



- For $l > L_x/2$: $B_{abcdIJ} = \left(U^{(m-1)}U^{(m-1)\dagger}U^{(m-2)}U^{(m-2)\dagger}\cdots\right)_{cdI_1J_1}\delta_{aI_2}\delta_{bJ_2}$
  where $I = (I_1, I_2),\ J = (J_1, J_2)$.



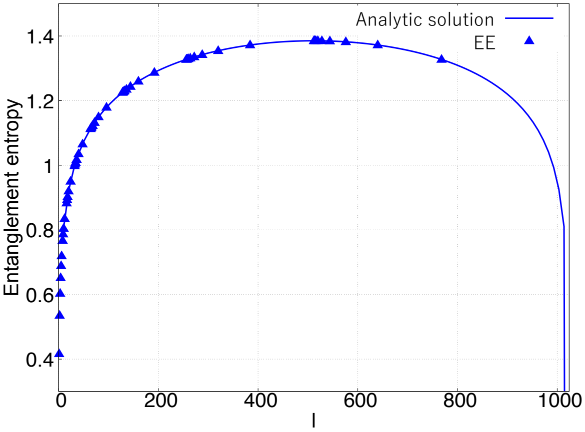
Numerical analysis of EE

- 2d classical Ising model

$$H = -J \sum_{<x,y>} s_x s_y$$

- Parameters
  - Spatial size $L_x = 1024$, Temporal size $L_t = 16 \times 1024$
  - D is fixed to 96.
  - Periodic boundary conditions in both spatial and temporal directions.
- Subregion size dependence of the EE at T_c is studied.

Subregion size dependence of EE for fixed $L_x = 1024$ at $T = T_c$



- EE for subregion size $l = 2^m$ and $l = 2^m + 2^p \ (0 \leq m < 10, p < m)$.
- Analytic solution for EE

$$S \sim \frac{c}{3} \log \left(\sin \frac{l\pi}{L_x} \right)$$

- Fitting result for $7 \leq l \leq 1024$:

$$c = 0.49997(8)$$

$\rightarrow$ The analysis reproduces known results and demonstrates the validity of our method.

Conclusions

Main Results

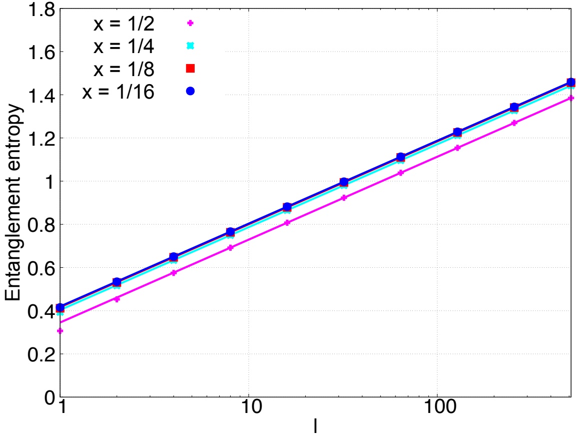
- We developed a general method to compute the EE of any size subregion.
- We studied the EE of 1d quantum Ising model using our method and reproduced the known result.

Future directions

- Application to the higher dimensional $(d > 2)$ field theories.
  - Efficient TRG algorithm
  - Parallelization of the algorithm
- Correspondence to holography.
  - Ryu-Takayanagi formula

Backup slides

EE for fixed $x = l/L_x,\ T = T_c = 2/\log(1+\sqrt{2})$



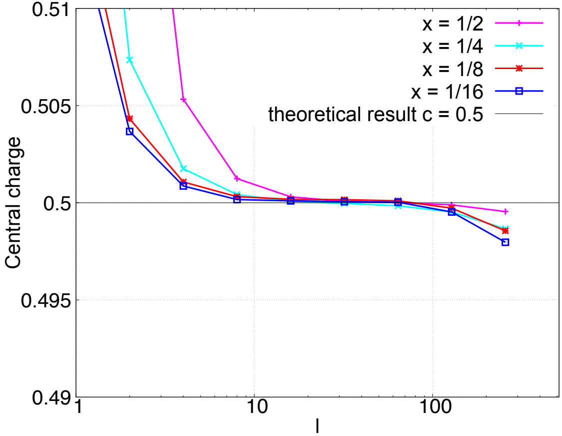
- Varying spatial size of the system L_x and subregion size l such that $x = l/L_x$ is fixed.
- We can see the theoretical solution

$$S_A(l,x) = \frac{c}{3}\log l + k(x)$$

is reproduced.

c: central charge of the theory

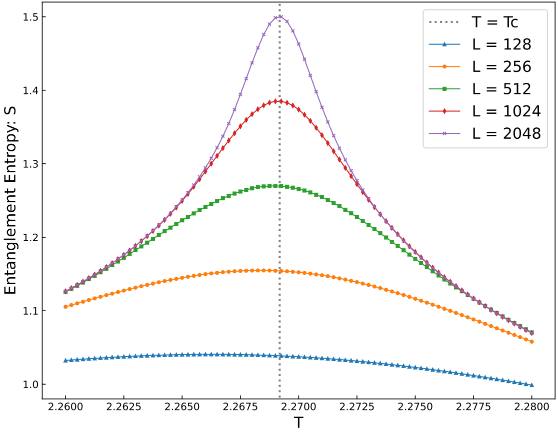
Central charge for fixed $x = l/L_x$ at $T = T_c$



Average for $16 \leq l \leq 128$:

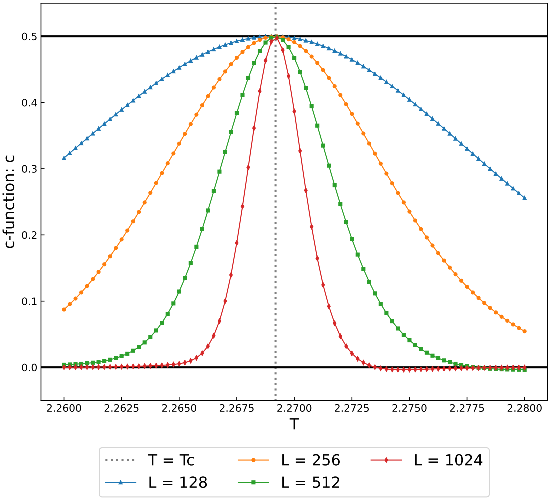
x	central charge
$1/2$	0.5001(2)
$1/4$	0.5000(4)
$1/8$	0.5001(4)
$1/16$	0.5001(5)

Temperature dependence of EE for $l = L_x/2$



- Peak at critical temperature
- Sharper peak for larger system size

Temperature dependence of EE for $l = L_x/2$



- Analytical result $c = 0.5$ is reproduced at critical temperature.
- Sharper peak for larger system size.