# Tensor renormalization group study of (1+1)-dimensional U(1) gauge-Higgs model at $\theta = \pi$ with Lüscher's admissibility condition

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**Based on SA-Kuramashi, arXiv:2407.10409**

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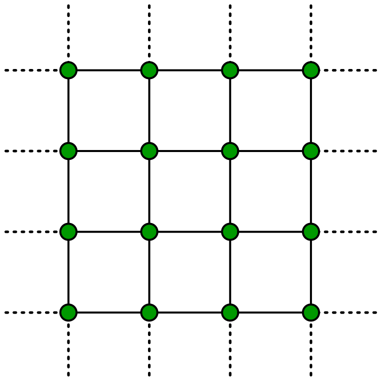
# TRG approach

✔ Tensor renormalization group (TRG) approximately contracts a given TN based on the idea of real-space renormalization group

- No sign problem
- **The computational cost scales logarithmically w. r. t. system size**
- **Direct evaluation of the Grassmann integrals**
- **Direct evaluation of the path integral**



✔ **Applicability to the higher-dimensional systems**

- If the system is translationally invariant on a lattice, we can easily apply the TRG to contract the TN
- In higher dimensions, TRG approach is usually less computationally expensive than the variational TN approach
- Higher-dimensional TRG computations must be informative to develop and improve various higher-dimensional TN algorithms

Cf. TRG for 3D SU(2) PCM, SA-Jha-Unmuth–Yockey, arXiv:2406.10081

# Grassmann TRG approach

✔ TRG can directly deal with the Grassmann path integral w/o pseudo-fermion

Gu-Verstraete-Wen, arXiv.1004.2563, Gu, PRB88(2013)115139, Shimizu-Kuramashi, PRD90(2014)014508

✔ **Introduction to the Grassmann TRG**

SA-Meurice-Sakai,
Journal of Physics: Condensed Matter 36 (2024) 343002

✔ **A Python package by A. Yosprakob**

Yosprakob, SciPostPhys. Codebases 20 (2023)
Cf. Talk by Yosprakob (7/30)

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Topical Review

## Tensor renormalization group for fermions

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**Abstract**

We review the basic ideas of the tensor renormalization group method and show how they can be applied for lattice field theory models involving relativistic fermions and Grassmann variables in arbitrary dimensions. We discuss recent progress for entanglement filtering, loop optimization, bond-weighting techniques and matrix product decompositions for Grassmann tensor networks. The new methods are tested with two-dimensional Wilson–Majorana fermions and multi-flavor Gross–Neveu models. We show that the methods can also be applied to the fermionic Hubbard model in 1+1 and 2+1 dimensions.

Keywords: tensor networks, lattice gauge theory, relativistic lattice fermions, Fermi Hubbard model, Grassmann path integrals, sign problems

**Contents**



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# Lüscher's admissibility condition

Lüscher, NPB549(1999)295-334

✔ The gauge action reads

$$\beta S_g = \beta \sum_{n,\mu>\nu} \frac{1-\mathrm{Re}P_{\mu\nu}(n)}{1-\|1-P_{\mu\nu}(n)\|/\epsilon} \qquad \text{if} \qquad \|1-P_{\mu\nu}(n)\| < \epsilon$$

and $\beta S_g = \infty$, otherwise

✔ The gauge fields are separated into disconnected subspaces, corresponding to topological charge

✔ In the MC simulation, the topological change is substantially suppressed

Fukaya-Onogi, PRD68(2003)074503

✔ With a $\theta$ term, the MC simulation is extremely difficult due to **the complex action problem and the topological freezing**

# Why don't we take the advantage of TRG?

✔ TRG allows us to compute the path integral w/o suffering from the sign problem and **the full contributions from every topological sector should be automatically included**

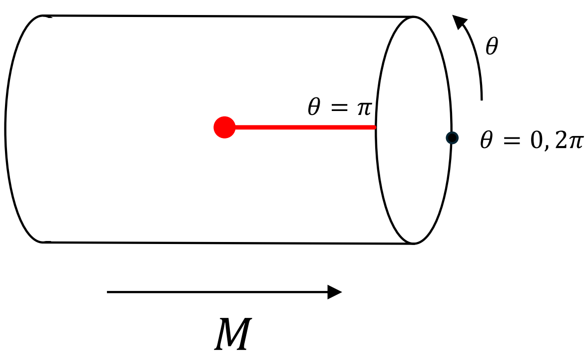
✔ As an example, we consider the U(1) gauge-Higgs model w/ a $\theta$ term in (1+1)D

$$S = \beta S_g + S_h + S_\Theta$$

$$S_h = -\sum_n [\sum_\nu \{\phi^*(n) U_\nu \phi(n+\hat{\nu}) + \phi^*(n+\hat{\nu}) U_\nu^* \phi(n)\} + M|\phi(n)|^2 + \lambda|\phi(n)|^4]$$

$$S_\Theta = -\frac{\mathrm{i}\theta}{2\pi} \sum_n \log P_{12}(n)$$

✔ At $\theta = \pi$, the first-order transition takes place w/ $M > M_\mathrm{c}$ and the critical behavior at $M = M_\mathrm{c}$ is in the 2D Ising universality class




Gattringer+, NPB935(2018)344-364
Komargodski+, SciPost Phys. 6(2019)003

# TN formulation

✔ Parametrizing $U_\nu(n) = \mathrm{e}^{\mathrm{i}\vartheta_\nu(n)}$ and $\phi(n) = r(n)\mathrm{e}^{\mathrm{i}\varphi(n)}$, we choose the unitary gauge which eliminates $\varphi(n)$ from the action

✔ The path integral $Z$ is discretized by Gauss quadrature rules

Kuramashi-Yoshimura, JHEP04(2020)089, Kadoh+, JHEP02(2020)161

$$\int_{-\pi}^{\pi} \frac{\mathrm{d}\vartheta_\nu}{2\pi} f(\vartheta_\nu) \simeq \sum_{\widetilde{\vartheta_\nu}\in D_g} w_{\widetilde{\vartheta_\nu}} f\left(\widetilde{\vartheta_\nu}\right), \qquad \int_0^\infty \mathrm{d}r\, g(r) \simeq \sum_{\tilde{r}\in D_h} w_{\tilde{r}}\, g(\tilde{r})$$

✔ The accuracy of the discretized path integral is controlled by # of sampling points in $D_g$ and $D_h$: $Z \simeq Z\left(K_g, K_h\right)$

✔ The tensor network representation for $Z\left(K_g, K_h\right)$ is straightforwardly derived

✔ We use the Bond-weighted TRG (BTRG) algorithm to evaluate $Z\left(K_g, K_h\right)$

Adachi+, PRB105(2022)L060402

# Pure gauge theory 1/2

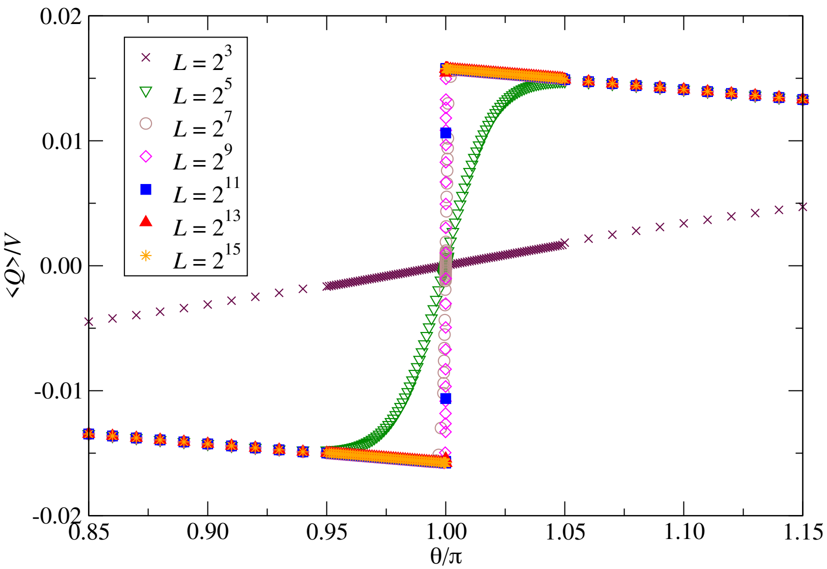
w/ $\beta = 3, \epsilon = 1, D = K_g = 30$

✔ A Clear signal of the first-order transition in the topological charge

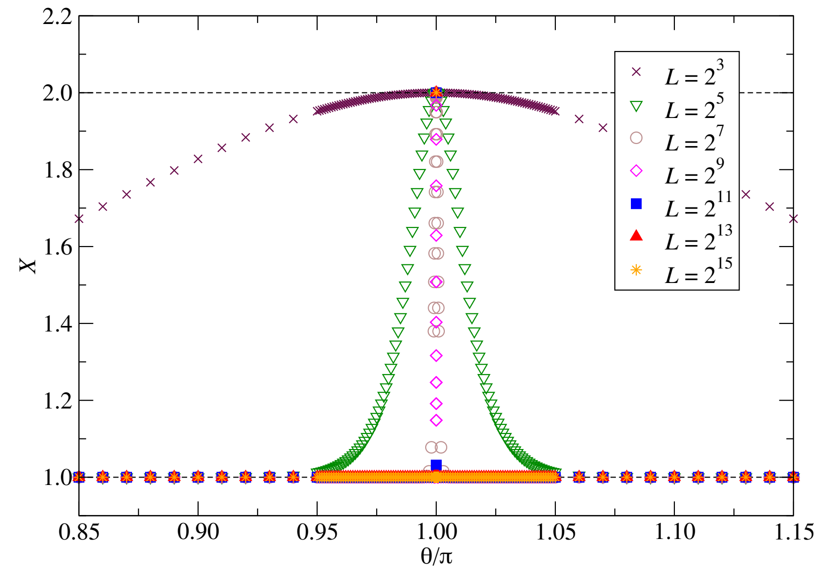
✔ The two-fold ground state degeneracy at $\theta = \pi$ is also observed

Gu-Wen, PRB80(2009)155131

Topological charge density




Ground-state degeneracy



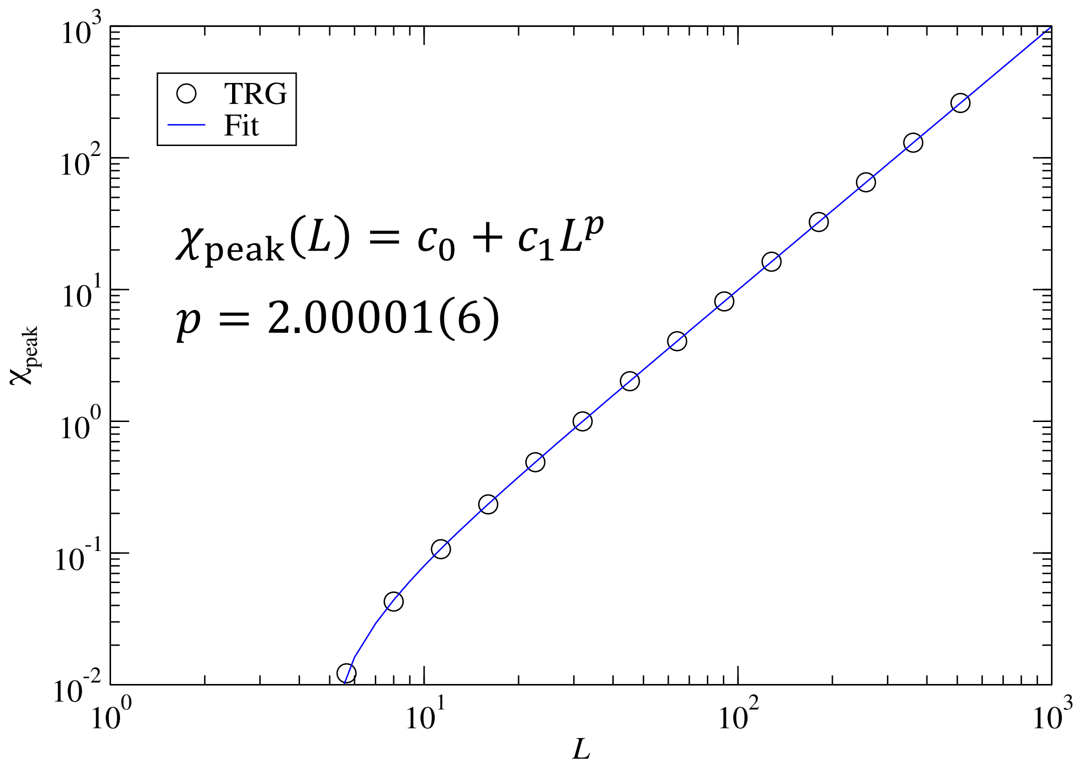
# Pure gauge theory 2/2

w/ $\beta = 3, \epsilon = 1, D = K_g = 30$

✔ The peak height of the topological susceptibility is proportional to the volume

✔ **TRG is successfully dealing with the Lüscher gauge action**

Peak height of topological susceptibility



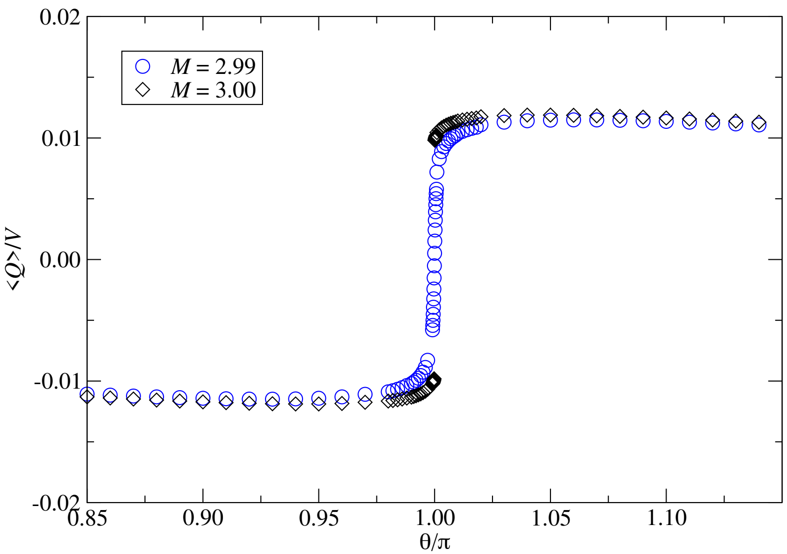
# The gauge-Higgs model

w/ $\beta = 3, \lambda = 0.5, \epsilon = 1, K_g = K_h = 20, D = 160$

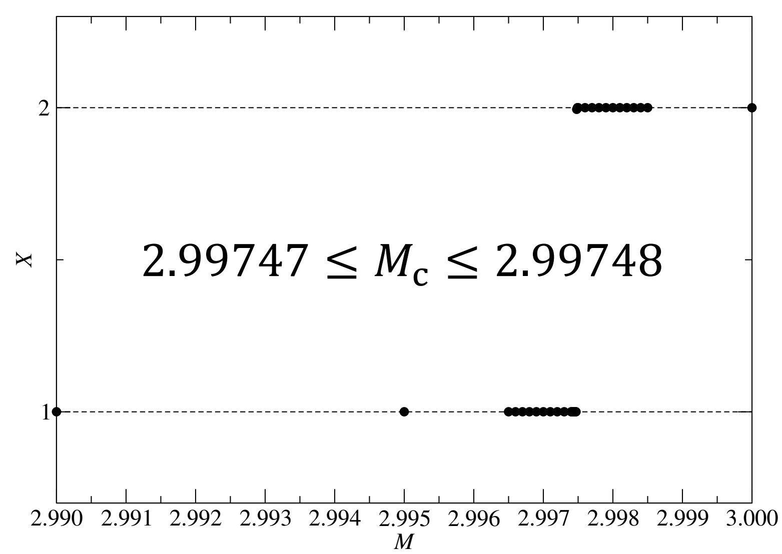
✔ Discontinuity in the topological charge is vanishing by decreasing the mass $M$

✔ Computing the ground-state degeneracy, we can bound the critical mass $M_c$

Topological charge density




Ground-state degeneracy



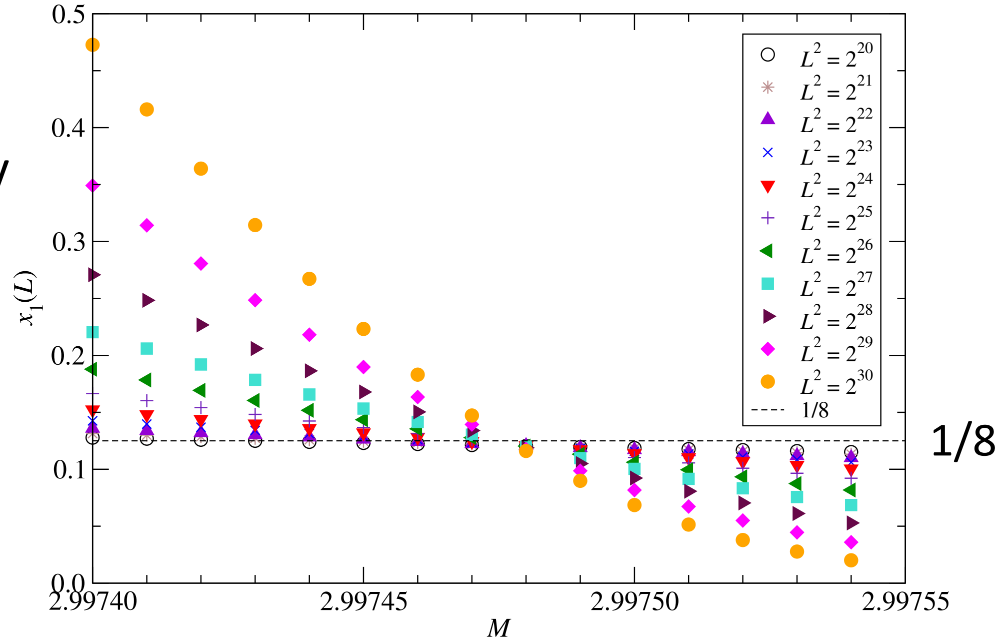
# Identification of the universality class

w/ $\beta = 3, \lambda = 0.5, \epsilon = 1, K_g = K_h = 20, D = 160$

- ✔ Transfer matrix $T$ is easily obtained from the TN representation

Gu-Wen, PRB80(2009)155131

- ✔ Ratio of the largest eigenvalue of $T$ to smaller one: $x_n(L) = \frac{1}{2\pi}\ln\frac{\lambda_0(L)}{\lambda_n(L)}$
- ✔ These are nothing but the **scaling dimensions** when the system is sufficiently large and at criticality
- ✔ The volume independence in $x_1(L)$ is observed w/ $x_1(L) = 1/8$, which agrees with the 2D Ising universality class



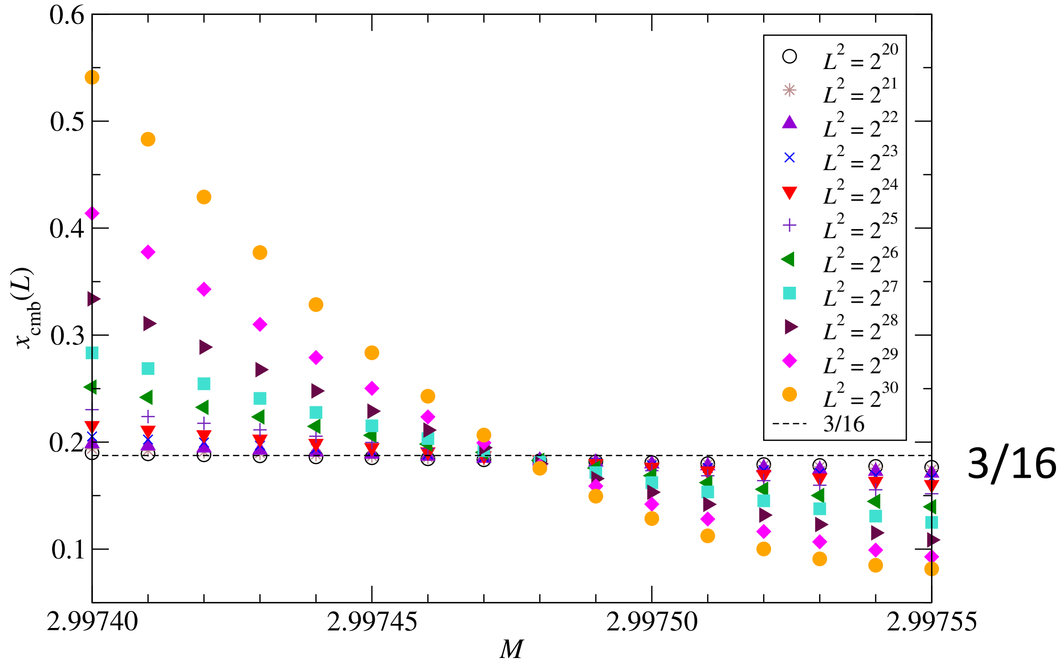
# Tensor-network-based level spectroscopy

✔ Assuming the 2D Ising universality class, we employ the level spectroscopy to determine the critical mass $M_c$ from scaling dimensions intersections

**Ueda-Oshikawa, PRB108(2023)024413**

**Cf. Next talk by Fathiyya**

✔ We particularly use the intersections of $x_{\mathrm{cmb}} = x_1 + x_2/16$ to remove the effect of the leading irrelevant perturbation



# Critical point and central charge

w/ $\beta = 3, \lambda = 0.5, \epsilon = 1, K_g = K_h = 20, D = 160$

✔ The resulting critical mass is $\boldsymbol{M_c = 2.9974765(14)}$

✔ This is consistent not only with the previous bound from the ground-state degeneracy, but also comparable with the previous MC result based on dual representation employing the Villain-type gauge action: $\boldsymbol{M_c = 2.989(2)}$

Gattringer+, NPB935(2018)344-364

✔ Investigating the finite-size correction for the free energy, the central charge is obtained as $\boldsymbol{c = 0.50(7)}$, in agreement with the 2D Ising universality class

✔ The algorithmic-parameter dependence of $M_c$ seems well suppressed

| $K_g$ | $K_h$ | $\chi$ | $D$ | $M_c$ |
|---|---|---|---|---|
| 24 | 20 | 8 | 192 | 2.9982886(1) |
| 22 | 20 | 8 | 176 | 2.9998263(13) |
| 20 | 20 | 8 | 160 | 2.9974765(14) |
| 24 | 10 | 6 | 144 | 2.9929635(1) |
| 22 | 10 | 6 | 132 | 2.9945222(9) |
| 20 | 10 | 7 | 140 | 2.9921698(6) |

$\chi$ is another algorithmic parameter to compress the initial bond dimension from $K_g K_h$ to $K_g \chi$

# Summary and outlook

- ✔ The critical behavior in the (1+1)D gauge-Higgs model with a $\theta$ term has been investigated by the TRG, **employing the Lüscher gauge action**
- ✔ The 2D Ising universality class is confirmed at $\theta = \pi$, as expected
- ✔ All numerical results show that the TRG is a promising approach to investigate the lattice gauge theories with Lüscher's admissibility condition

# Appendices

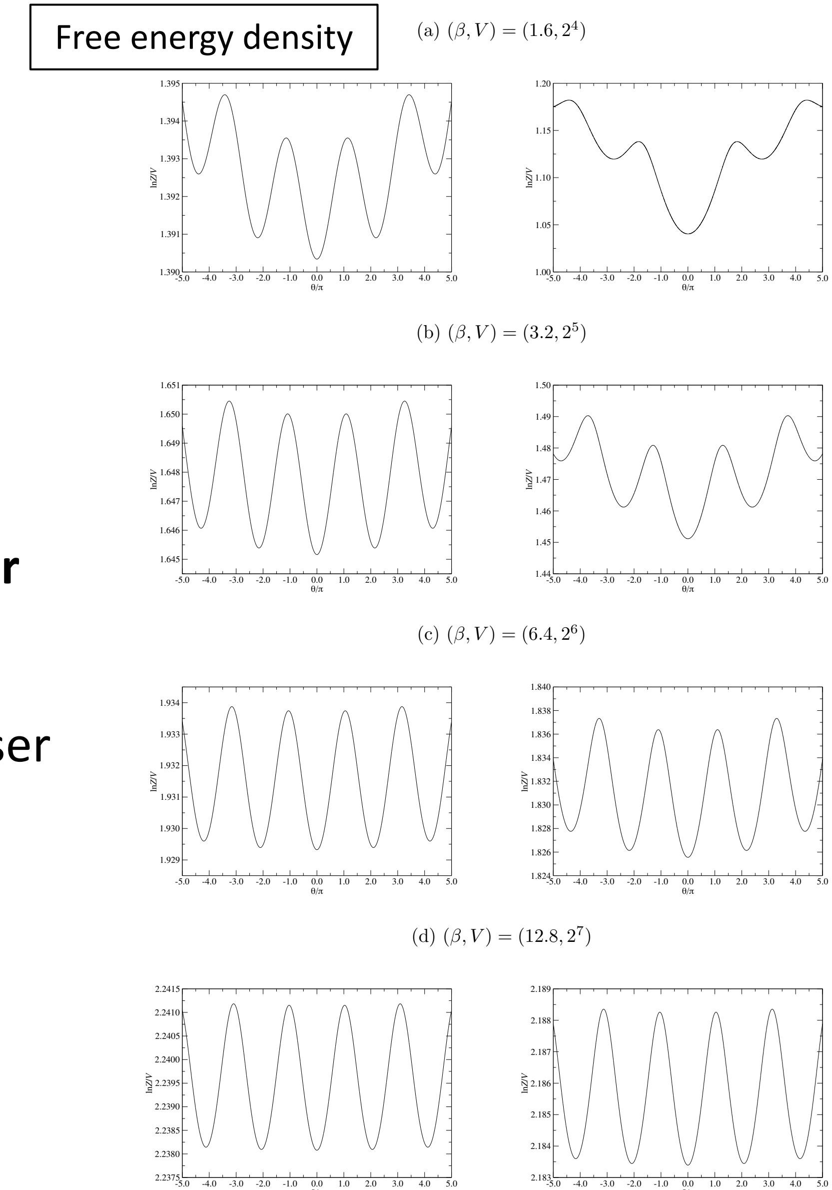
# Comparison w/ the standard Wilson action 1/2

- ✔ The field-theoretical def. for the topological term:

$$S_\Theta = -\frac{\mathrm{i}\theta}{2\pi}\sum_n \mathrm{Im}\,P_{12}(n)$$

- ✔ The $2\pi$ periodicity appears in the continuum limit
- ✔ **The Lüscher action shows the faster convergence than the Wilson one**
- ✔ The peak of the susceptibility is closer to $\theta/\pi = 1$

| $\beta$ | Lüscher ($\epsilon = 1$) | Wilson |
|---|---|---|
| 1.6 | 1.11932(3) | 1.67903(2) |
| 3.2 | 1.08112(2) | 1.26026(1) |
| 6.4 | 1.05111(4) | 1.09604(7) |
| 12.8 | 1.03070(3) | 1.04296(1) |



# Comparison w/ the standard Wilson action 2/2

- ✔ The field-theoretical def. for the topological term:

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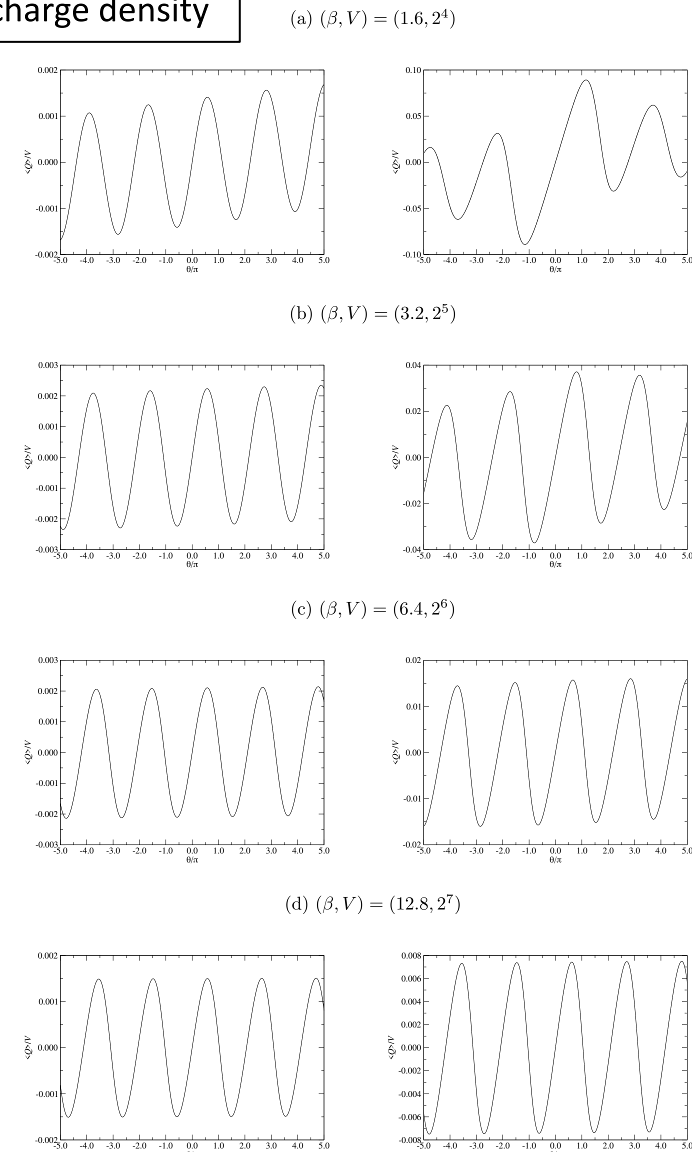


# Tensor-network-based level spectroscopy

✔ Assuming the 2D Ising universality class, we employ a level spectroscopy to determine the critical point $T_c$

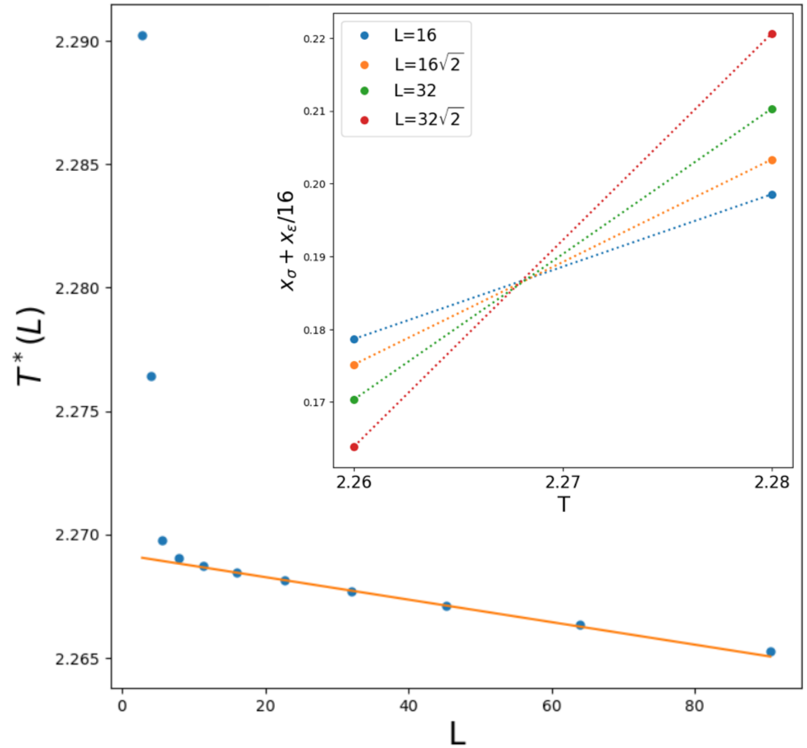
**Ueda-Oshikawa, PRB108(2023)024413**

(i) Choose two mass parameter $T^{(\pm)}$ such that $T^{(-)} \leq T_{\mathrm{c}} \leq T^{(+)}$

(ii) At these two points, compute $x_{\mathrm{cmb}}(L) = x_1(L) + x_2(L)/16$. This combination removes the effect from the leading irrelevant perturbation associated with the scaling dimension 4

(iii) Perform liner interpolations of $x_{\mathrm{cmb}}(L) - 3/16$ btw $T^{(-)}$ and $T^{(+)}$ at each system size and find a crossing point $T^*(L)$ of two lines with the system sizes $L$ and $\sqrt{2}L$

(iv) Fit $T^*(L)$ by $T^*(L) = T_{\mathrm{c}} + aL$, and we finally obtain the critical point $T_{\mathrm{c}}$



# TN representation 1/2

✔ The discretized path integral is described by a four-leg local tensor $T$:

$$Z(K_g, K_h) = \mathrm{tTr}\left[\prod_n T_n\right]$$

$$(T_n)_{xyx'y'} = T^{(g)}_{x_g y_g x'_g y'_g} T^{(\theta)}_{x_g y_g x'_g y'_g} T^{(h)}_{x_h y_h x'_g x'_h y'_g y'_h}$$

$$T^{(g)}_{x_g y_g x'_g y'_g} = \begin{cases} \frac{\sqrt{w_{x_g} w_{y_g} w_{x'_g} w_{y'_g}}}{2^2} \exp\left[-\beta \frac{1-\cos\pi\left(y'_g + x_g - y_g - x'_g\right)}{1-\left[1-\cos\pi\left(y'_g + x_g - y_g - x'_g\right)\right]/\epsilon}\right] & \text{if admissible} \\ 0 & \text{otherwise} \end{cases},$$

$$T^{(\theta)}_{x_g y_g x'_g y'_g} = \exp\left(\frac{\mathrm{i}\theta}{2\pi}\ln\left[\mathrm{e}^{\mathrm{i}\pi\left(y'_g + x_g - y_g - x'_g\right)}\right]\right)$$

# TN representation 2/2

✔ Compression for the hopping term:

$$H_{\tilde{\ell}(n)\tilde{\theta}_\nu(n)\tilde{\ell}(n+\hat{\nu})}$$
$$= \frac{\sqrt[4]{w_{\tilde{\ell}(n)} w_{\tilde{\ell}(n+\hat{\nu})}} \mathrm{e}^{(\tilde{\ell}(n)+\tilde{\ell}(n+\hat{\nu}))/4}}{\sqrt{2}}$$
$$\times \exp\left[2\sqrt{\tilde{\ell}(n)\tilde{\ell}(n+\hat{\nu})}\cos\pi\tilde{\theta}_\nu(n) - \frac{M}{4}\left(\tilde{\ell}(n)+\tilde{\ell}(n+\hat{\nu})\right) - \frac{\lambda}{4}\left(\tilde{\ell}(n)^2+\tilde{\ell}(n+\hat{\nu})^2\right)\right]$$

$$H_{\tilde{\ell}(n)\tilde{\vartheta}_\nu(n)\tilde{\ell}(n+\hat{\nu})} \simeq \sum_{\alpha=1}^{\chi} A_{\tilde{\ell}(n)\tilde{\vartheta}_\nu(n)\alpha} B_{\tilde{\ell}(n+\hat{\nu})\alpha}$$

$$T^{(h)}_{x_h y_h x'_g x'_h y'_g y'_h} = \sum_{\tilde{\ell}(n)} A_{\tilde{\ell}(n)y'_g x_h} A_{\tilde{\ell}(n)x'_g y_h} B_{\tilde{\ell}(n)x'_h} B_{\tilde{\ell}(n)y'_h}$$