Spectroscopy with the Tensor Renormalization Group Method

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arXiv:2404.15666v1

Accepted by Phys. Rev. D

LATTICE 2024

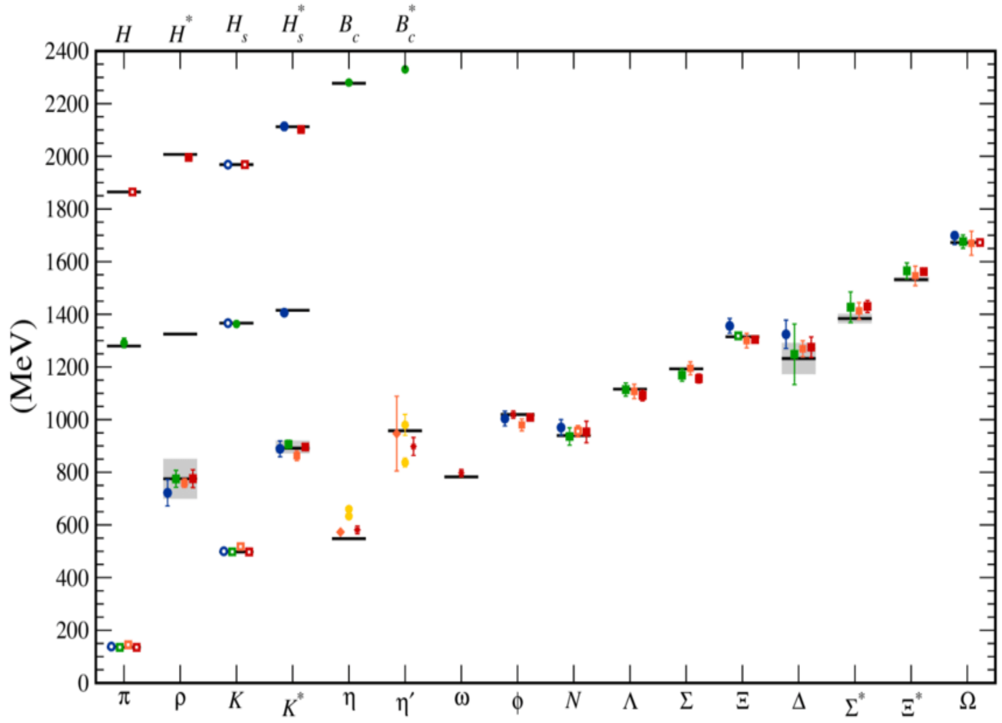
University of Liverpool

Hadron Spectroscopy in Lattice QCD by Monte Carlo

J^{PC}, flavor, ...

$$\hat{H}_{QCD}|n,q\rangle = E_{n,q}|n,q\rangle$$

Energy Eigenstate
with quantum
number q



2013 snowmass report

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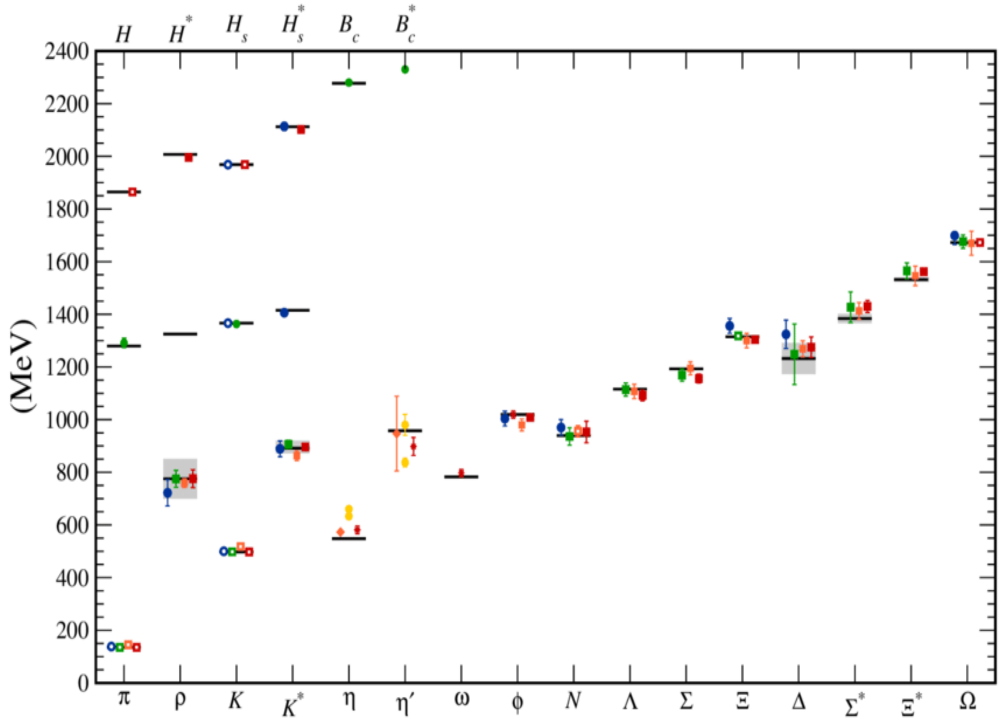

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2- point function (Euclidean time)

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Ground State

Energy gap $\omega_{n,q}$

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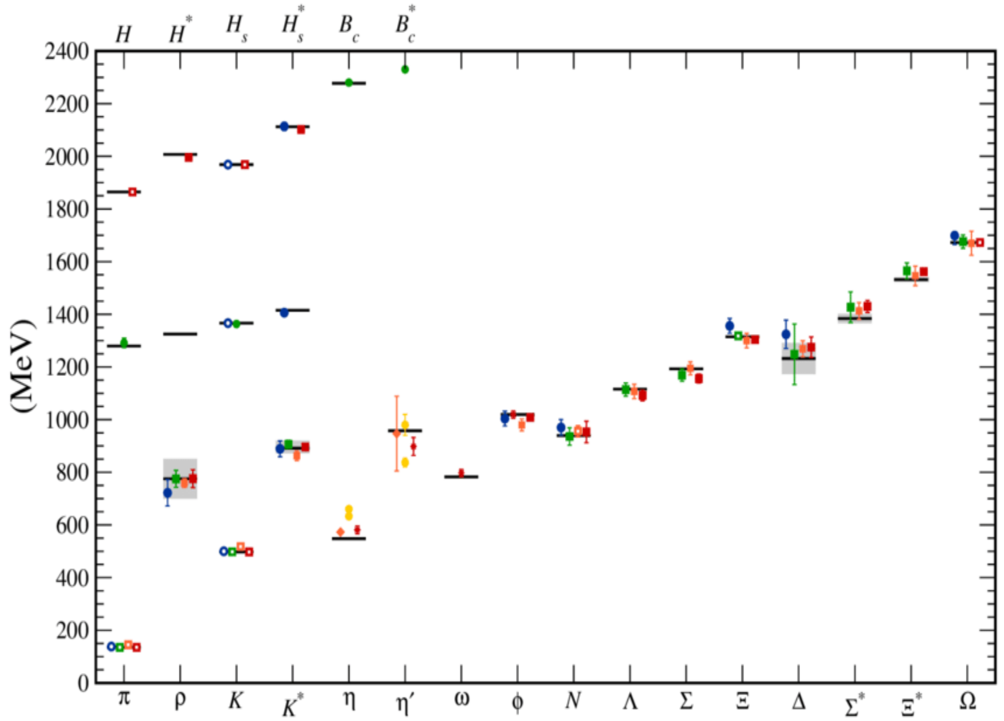

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Ground State

Problem of Monte Carlo:

- Need sufficiently large time extend and time separation to extract $\omega_{n,q} = E_{n,q} - E_\Omega$.
- Need large statistic to extract excited states.

Spectroscopy by Tensor Network + Transfer Matrix

Metods of spectroscopy by Tensor Network:

1. By Hamiltonian Formalism [E. Itou, A. Matsumoto, Y. Tanizaki, JHEP11(2023)231]
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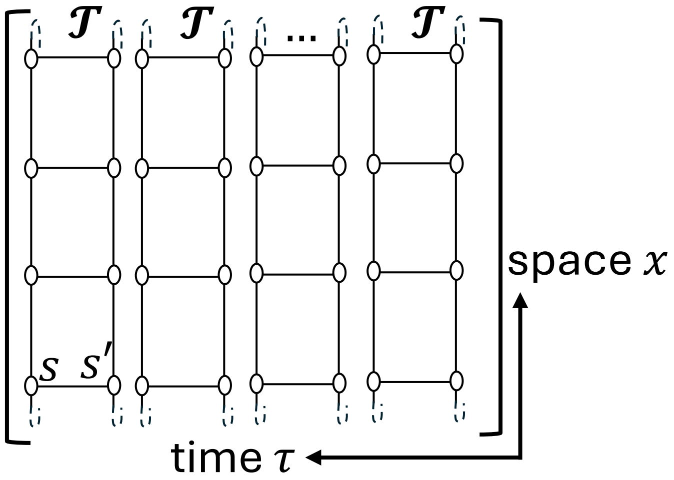

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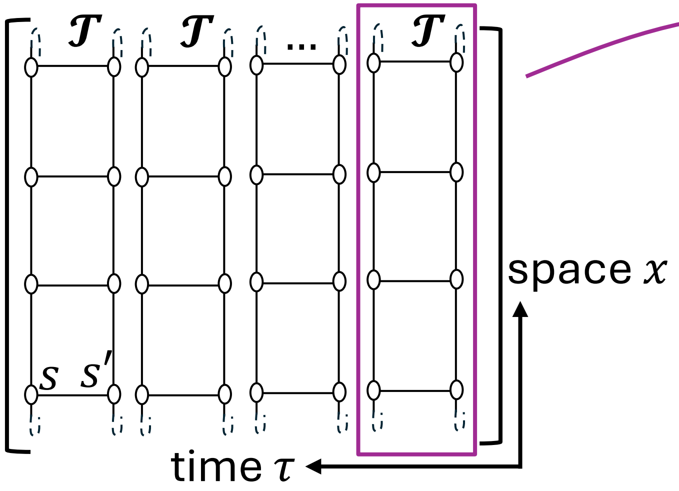

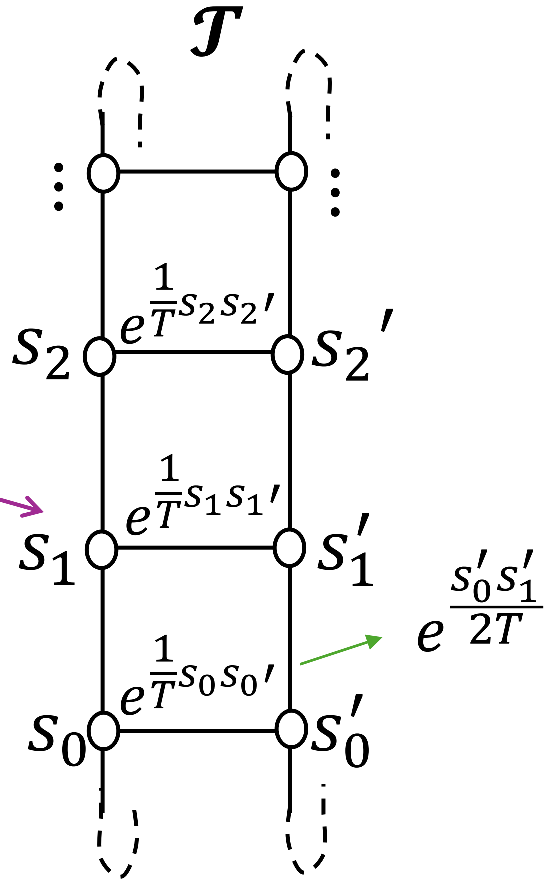

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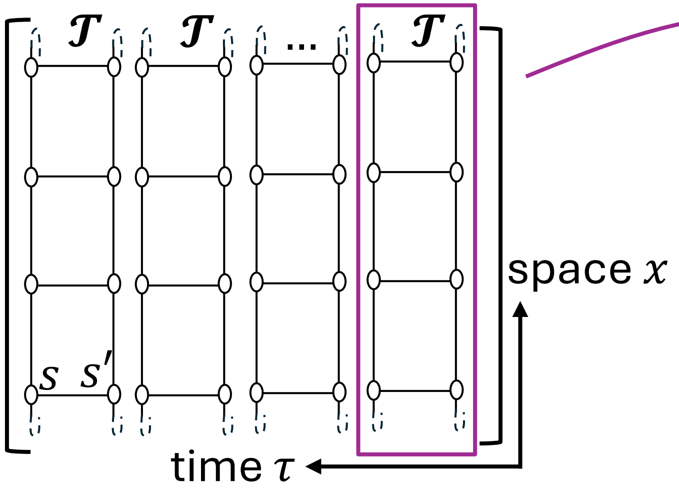

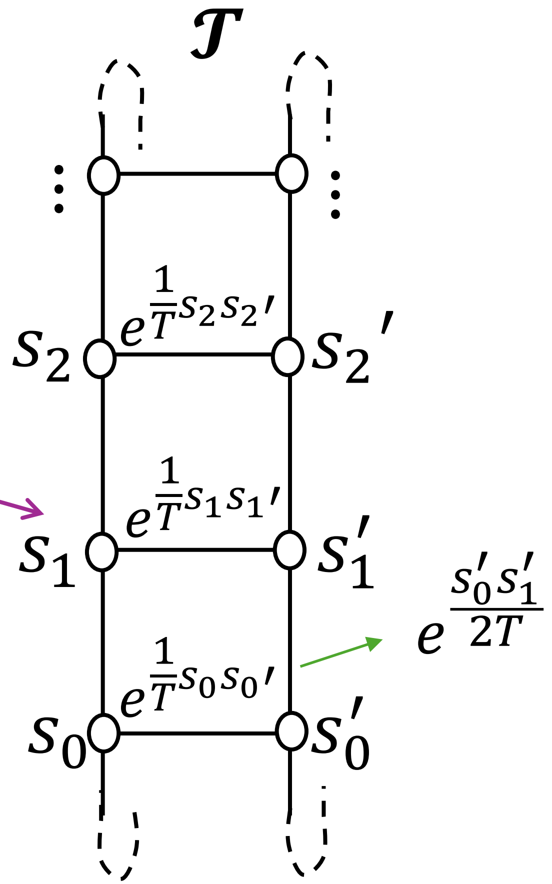

$$\mathcal{T}_{ss'} = \sum_a U_{sa} \lambda_a U^{\dagger}_{as'} = U_{sa} e^{-E_a} U^{\dagger}_{as'}$$

Eigenvectors

Eigenvalues

Energy of state $|a\rangle$

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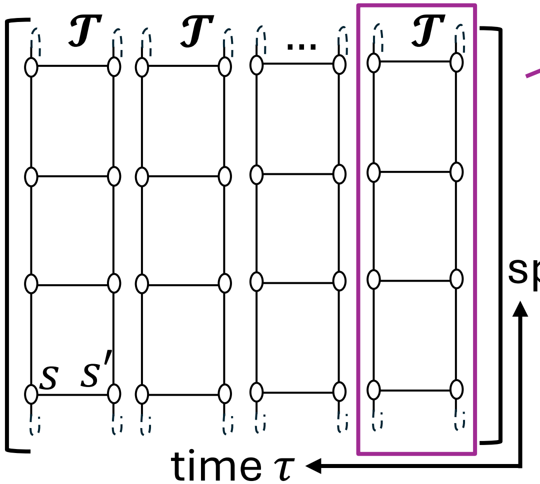

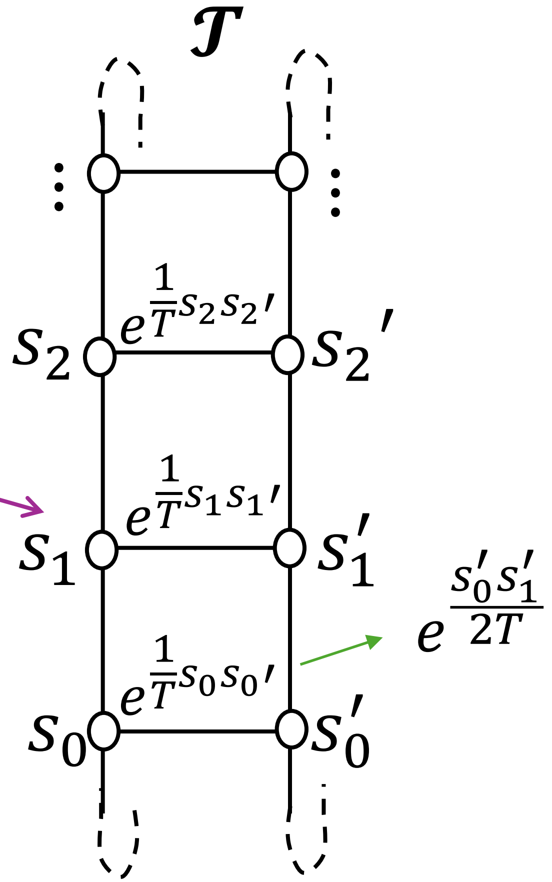

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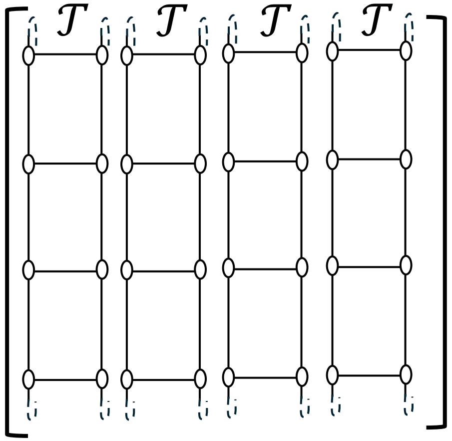
Energy Gaps ω_a:

$$\omega_a = E_a - E_0 = \log\frac{\lambda_0}{\lambda_a}$$

Computing Transfer Matrix Spectrum by Tensor Network

$Z = Tr$



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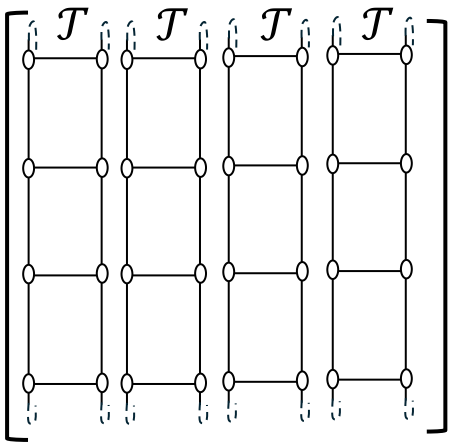

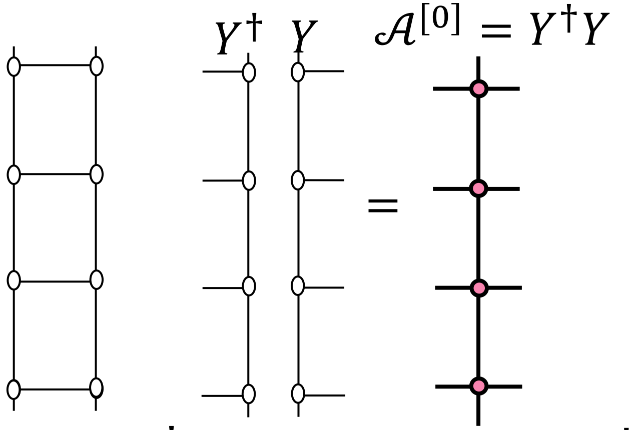

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$\dim \mathcal{T} = 2^{L_x}$

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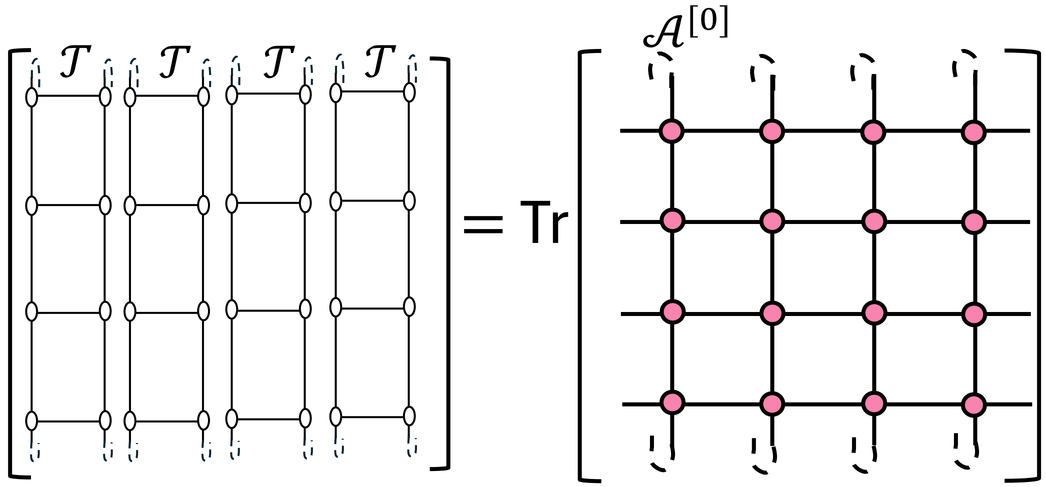
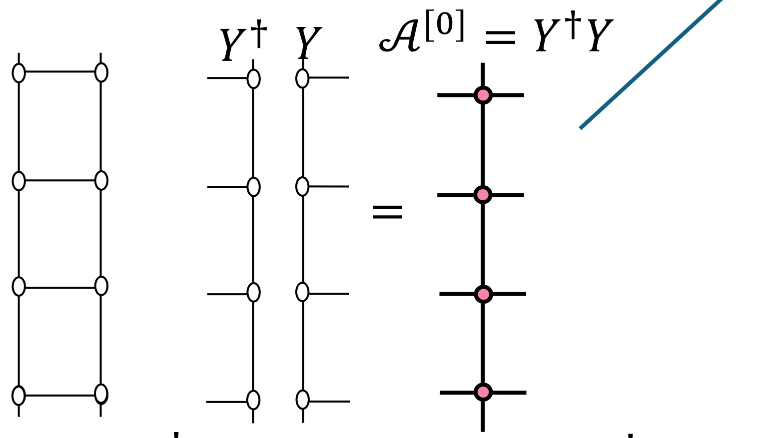
$Z = Tr$  $= \mathrm{Tr}$



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Coarse-Graining by HOTRG

(higher order tensor renormalization group)

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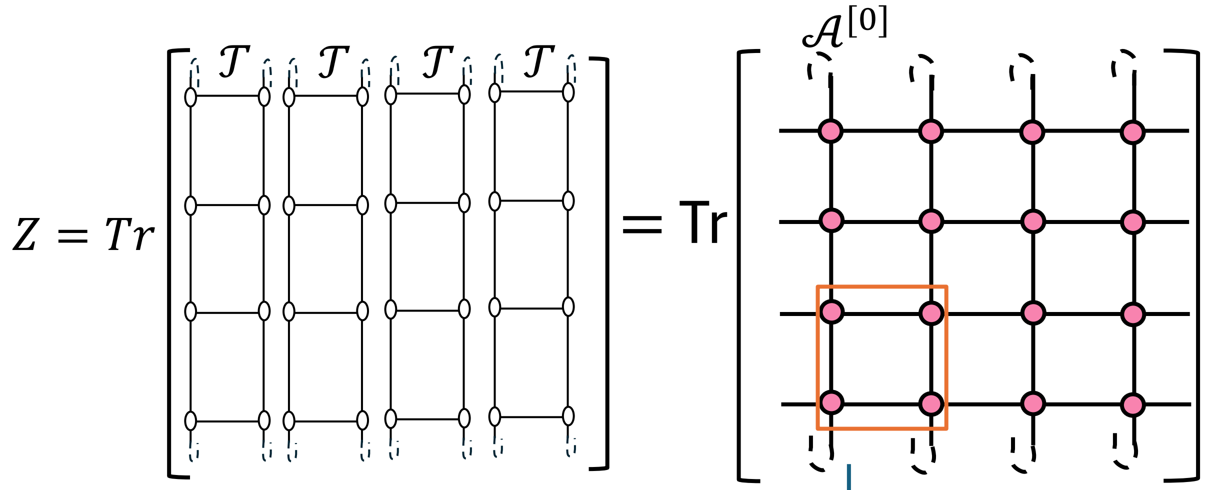

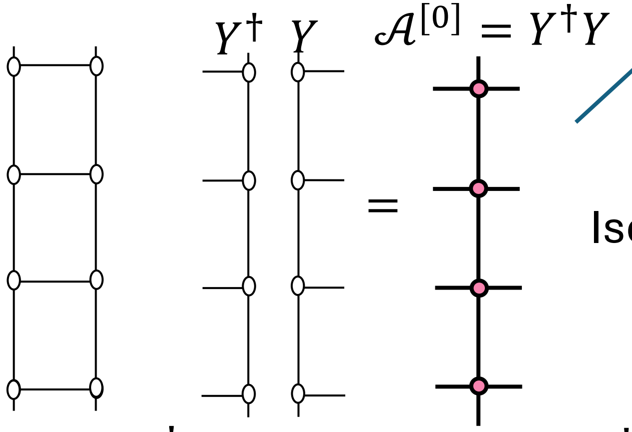

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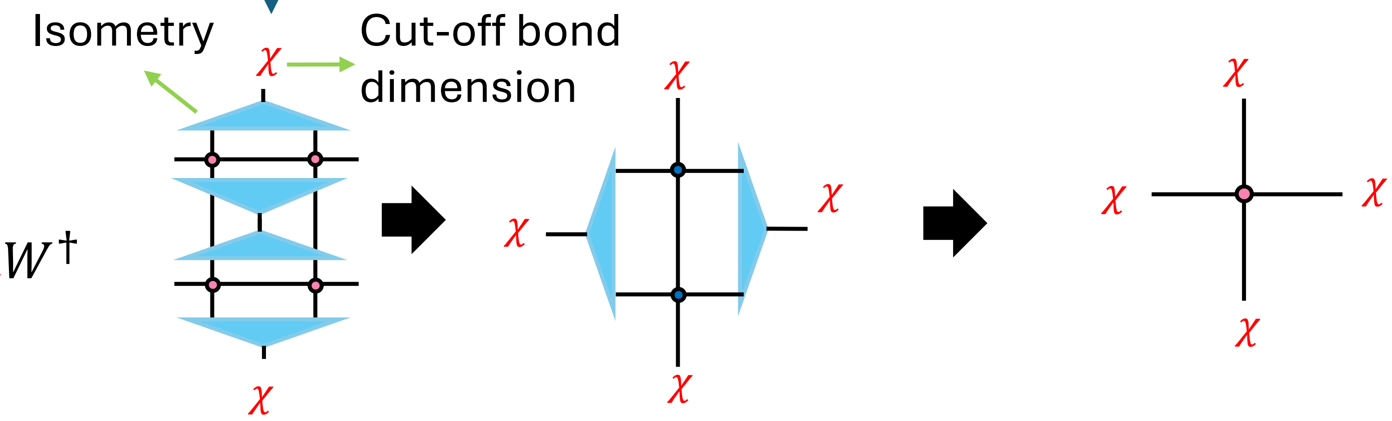

[Z. Y. Xie, et. al., Phys. Rev. B. 86 [2012]

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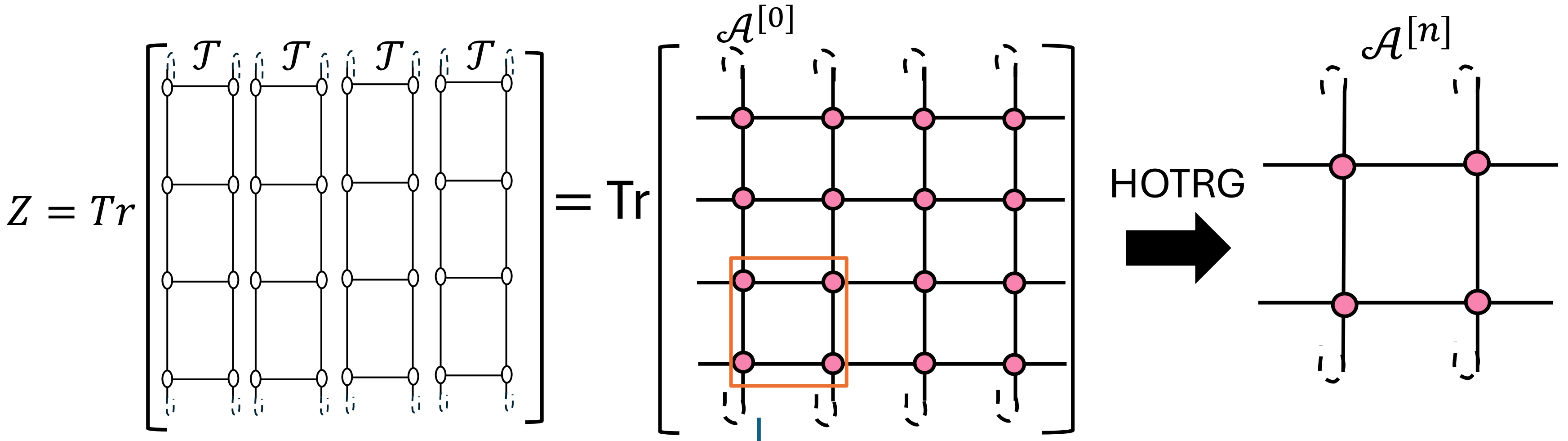

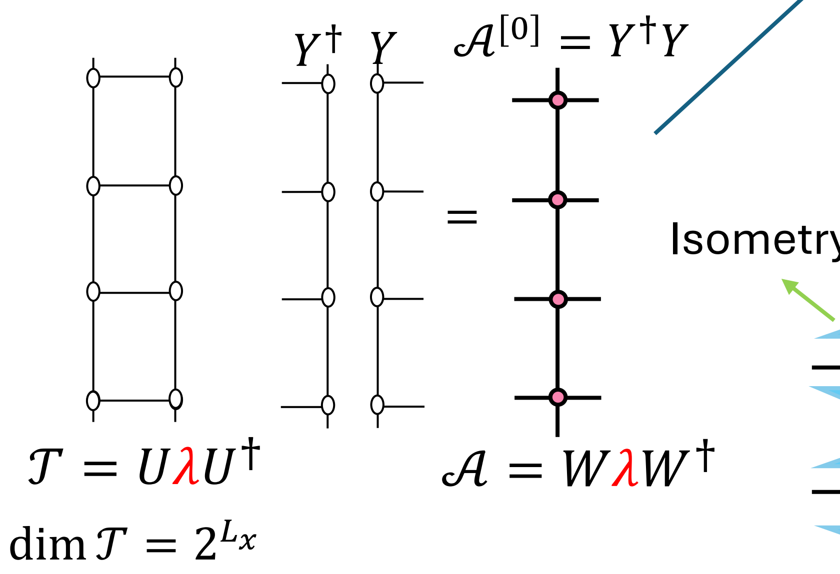

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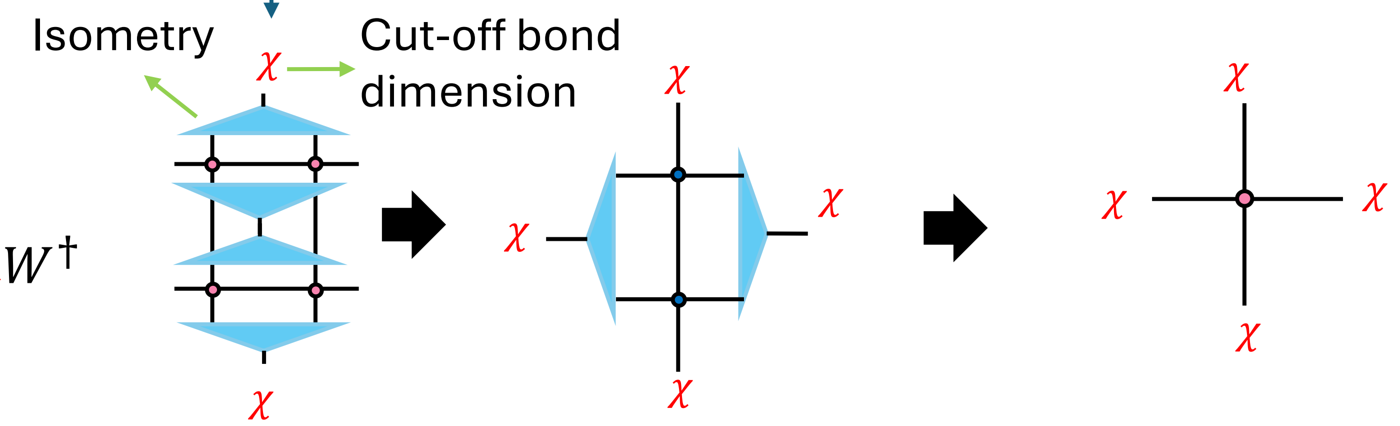

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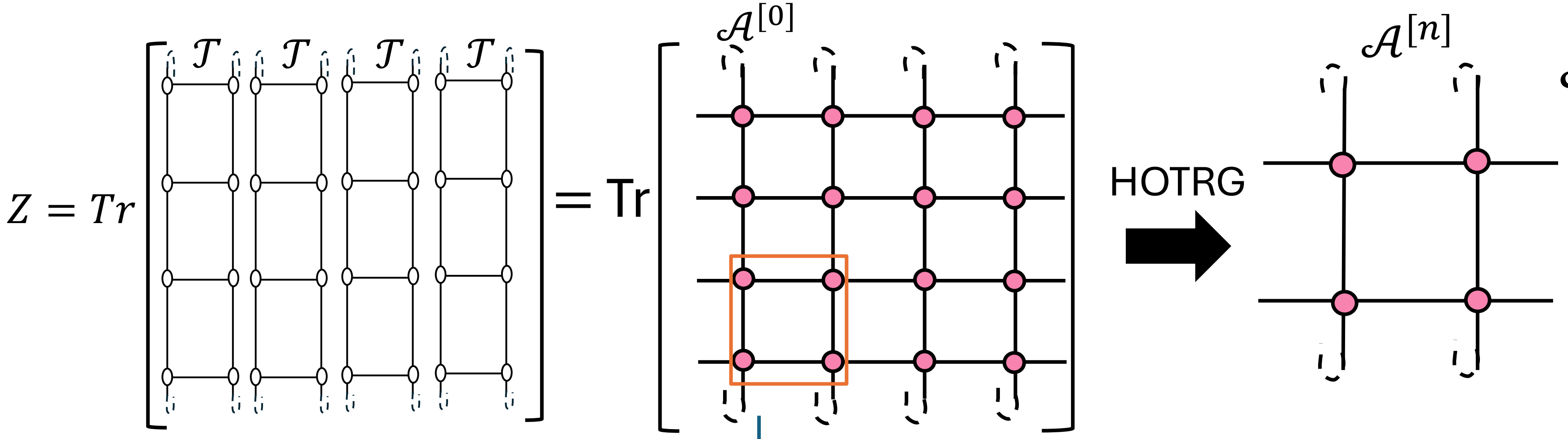
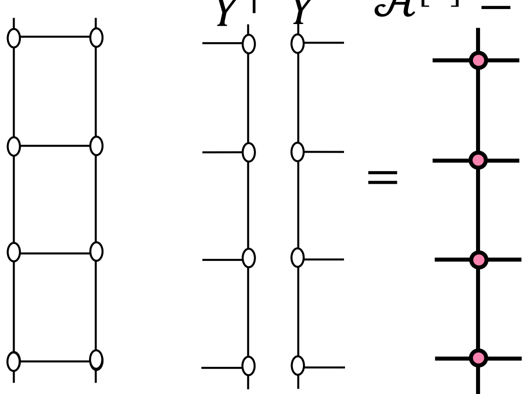


Diagonalize

$$\mathcal{A}^{[n]} = W^{[n]}\lambda^{[n]}W^{[n]\dagger}$$

$\mathcal{A}^{[0]} = Y^{\dagger}Y$



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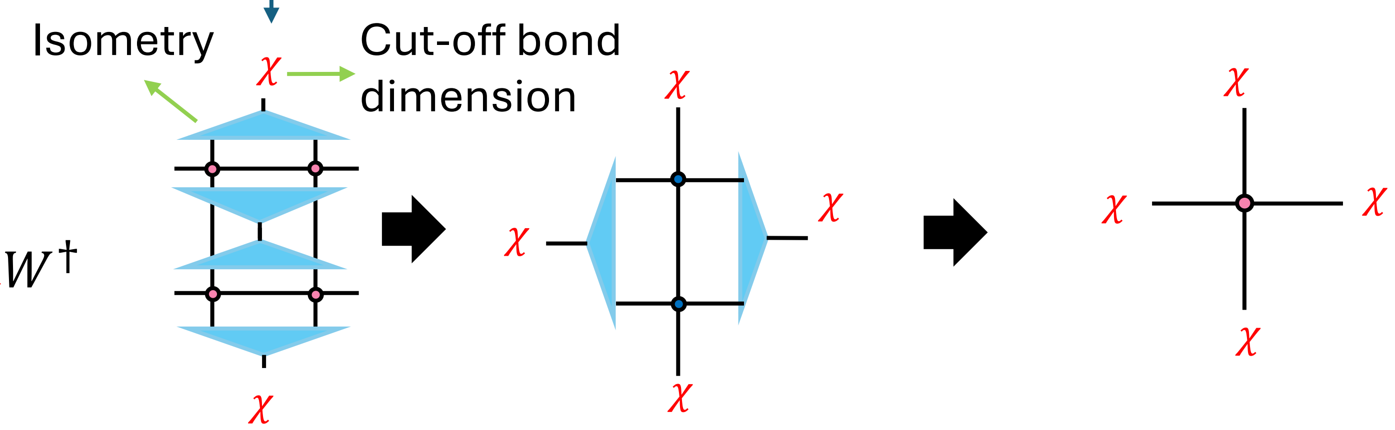

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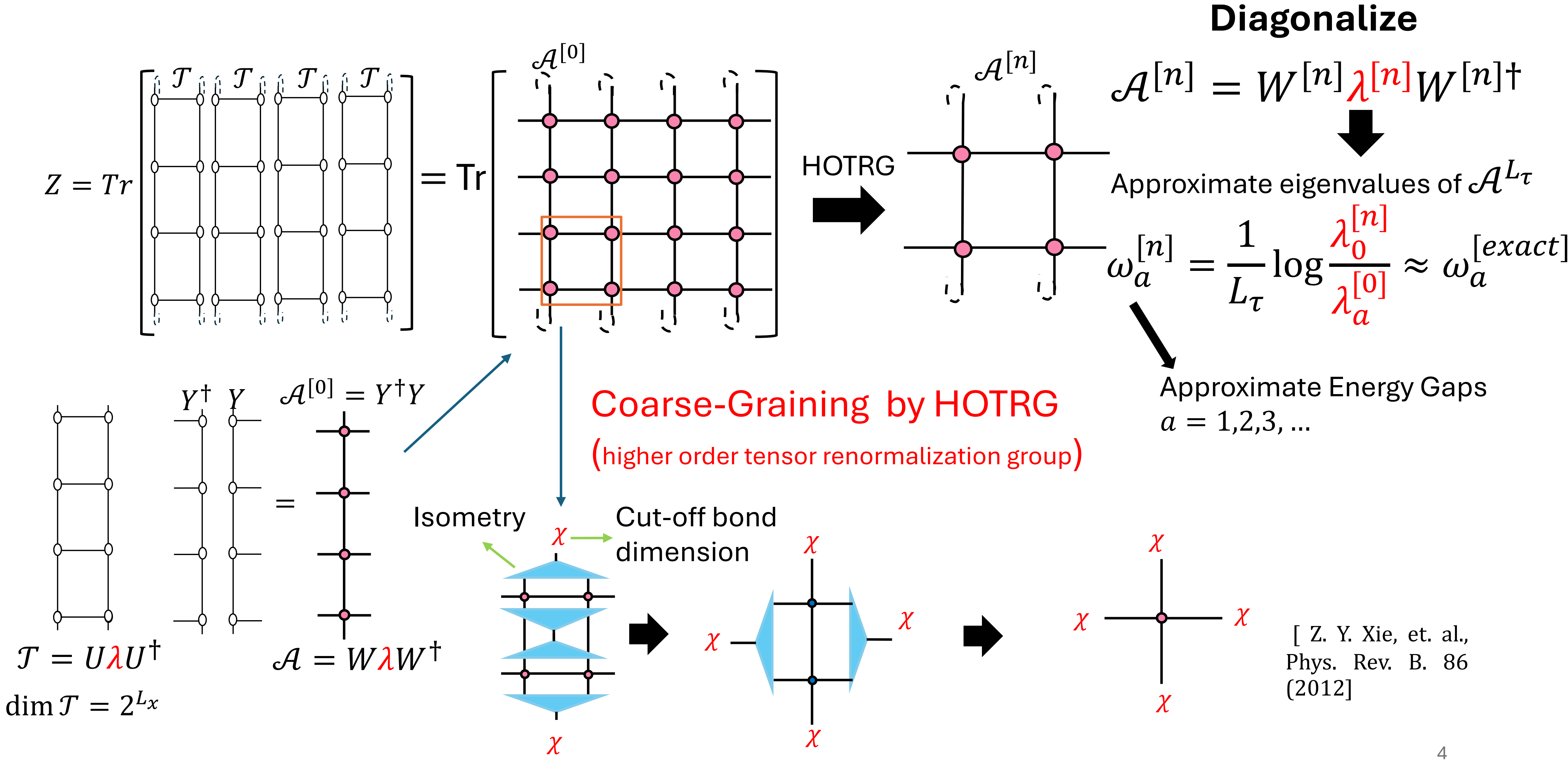

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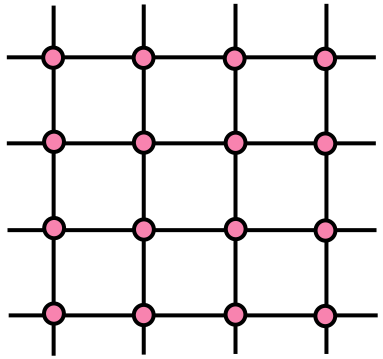

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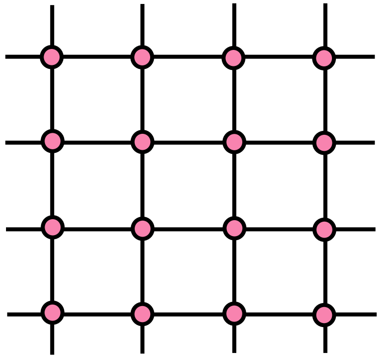

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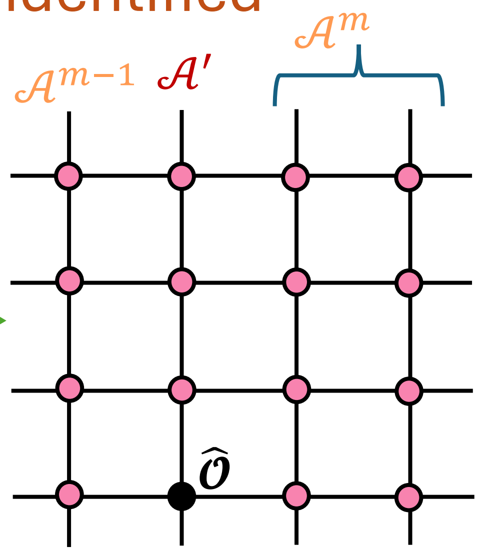

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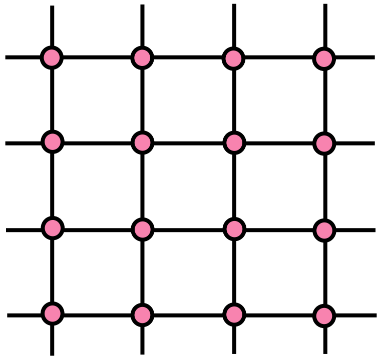

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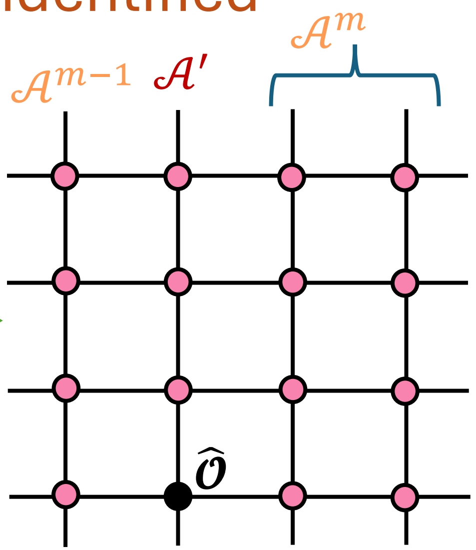
After Coarse graining by HOTRG



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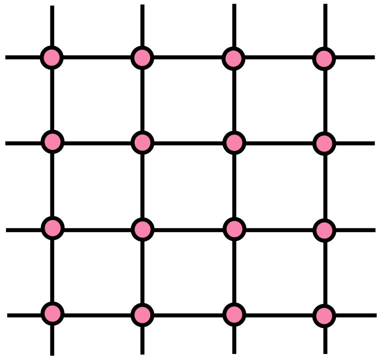

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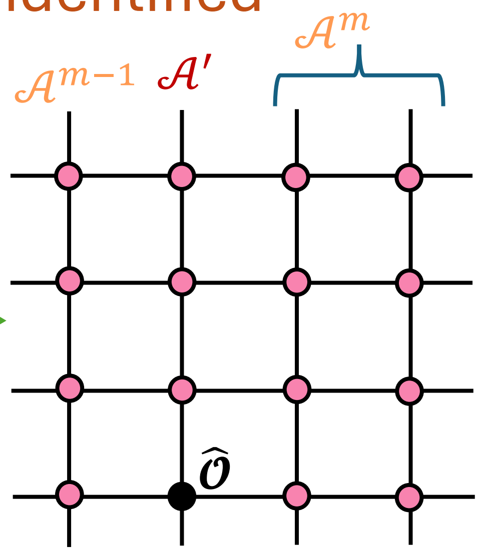

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$$\approx \left(\left(\lambda^{[n]}\right)^{-\frac{1}{L_\tau}(m-\frac{1}{2})} \mathrm{W}^{[n]\dagger} \mathcal{A}'^{[n]} W^{[n]} \left(\lambda^{[n]}\right)^{-\frac{1}{L_\tau}(m+\frac{1}{2})}\right)_{ba} = \boldsymbol{B}_{\boldsymbol{ba}}^{[\boldsymbol{hotrg}]}$$

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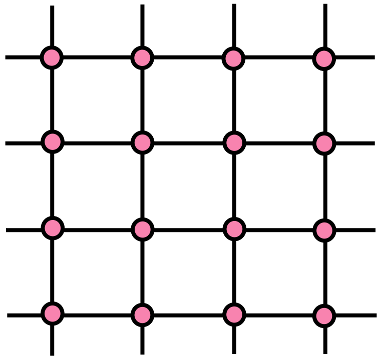

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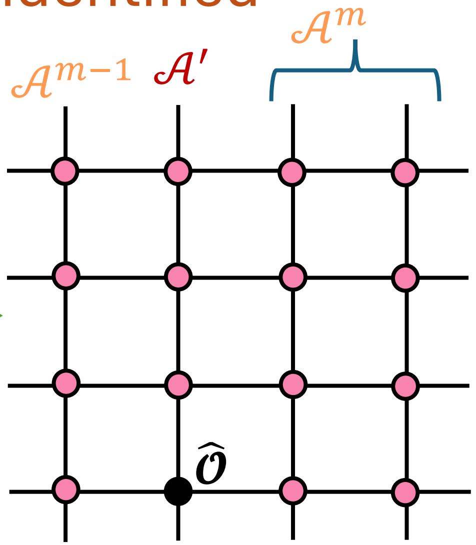

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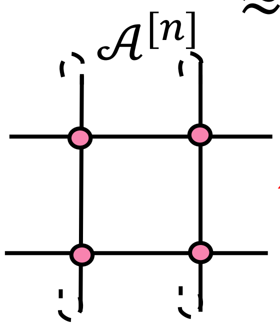

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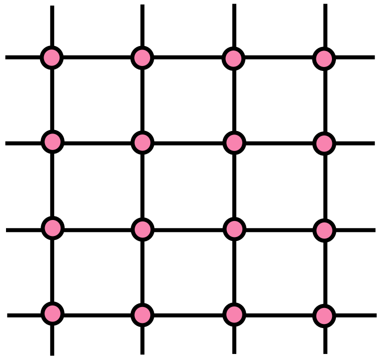
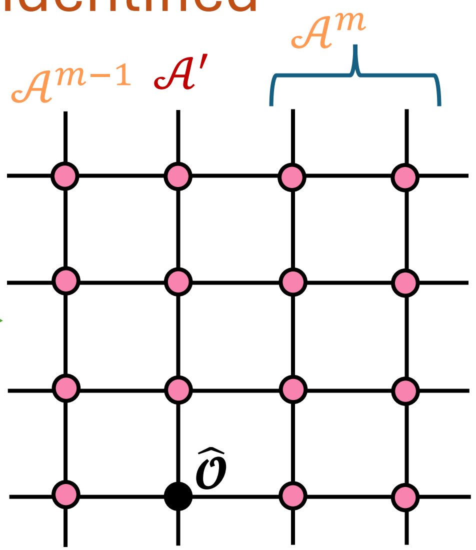
$$\langle \boldsymbol{b}|\hat{\mathcal{O}}|\boldsymbol{a}\rangle = \left(\lambda^{-(m-\frac{1}{2})} W^\dagger \mathcal{A}^{m-1}\mathcal{A}'\mathcal{A}^m W \lambda^{-(m+\frac{1}{2})}\right)_{ba}$$

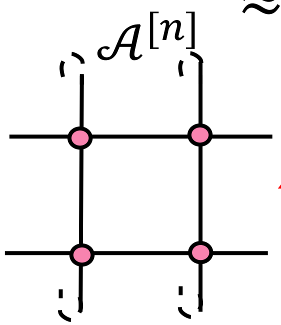


$\mathcal{A} = W\lambda W^\dagger$

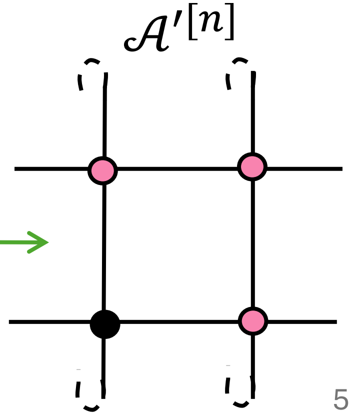


After Coarse graining by HOTRG

$$\approx \left(\left(\lambda^{[n]}\right)^{-\frac{1}{L_\tau}(m-\frac{1}{2})} \mathrm{W}^{[n]\dagger} \mathcal{A}'^{[n]} W^{[n]} \left(\lambda^{[n]}\right)^{-\frac{1}{L_\tau}(m+\frac{1}{2})} \right)_{ba} = \boldsymbol{B}_{\boldsymbol{ba}}^{[\boldsymbol{hotrg}]}$$



$$\mathcal{A}^{[n]} = W^{[n]}\lambda^{[n]}W^{[n]\dagger}$$



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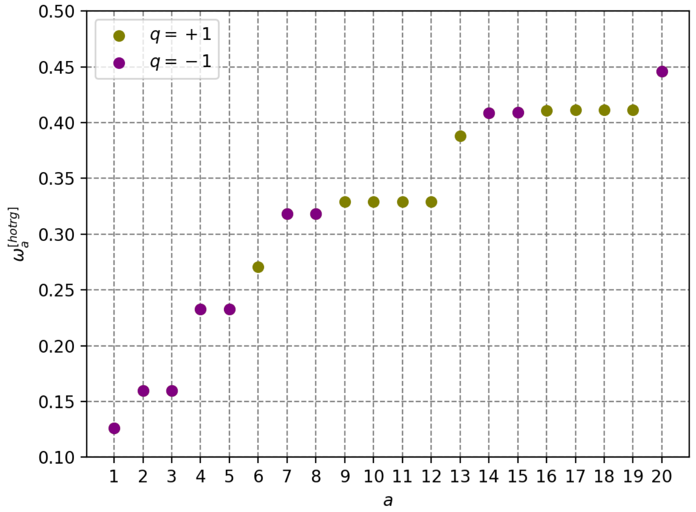

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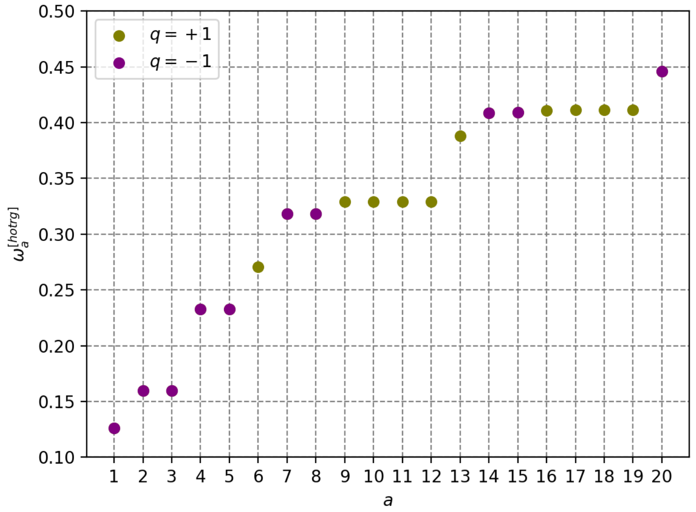

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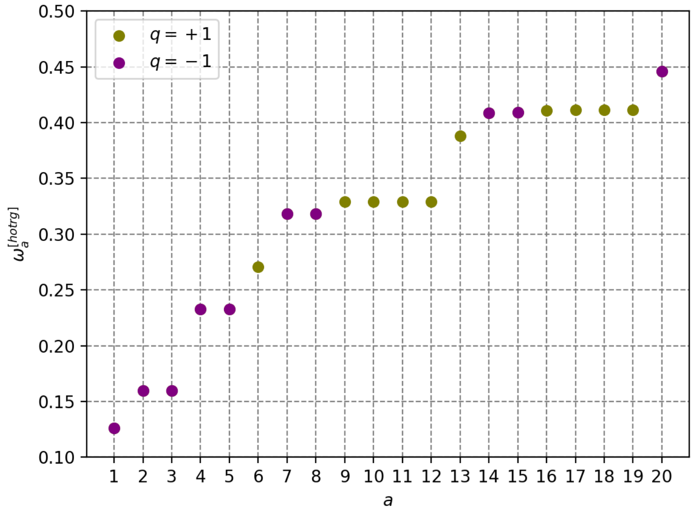

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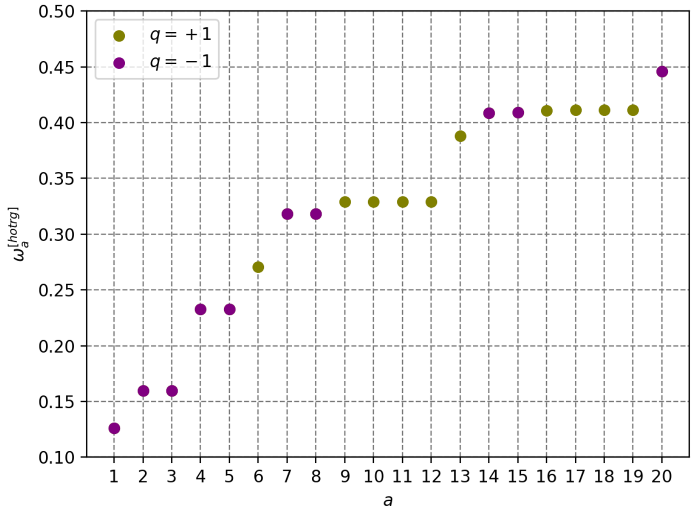

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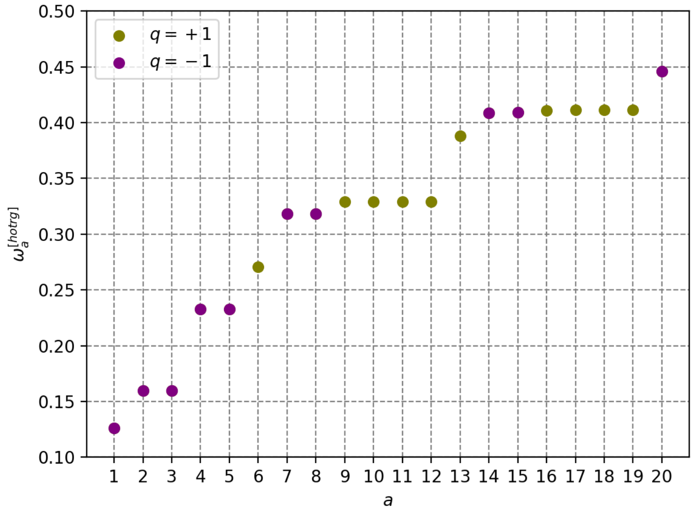

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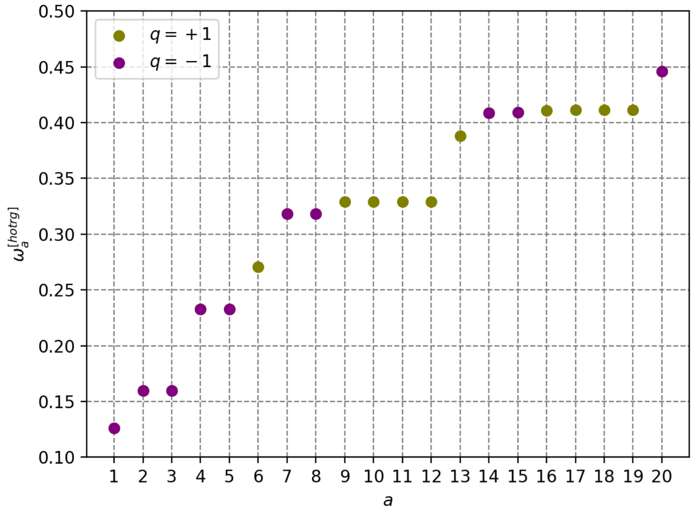

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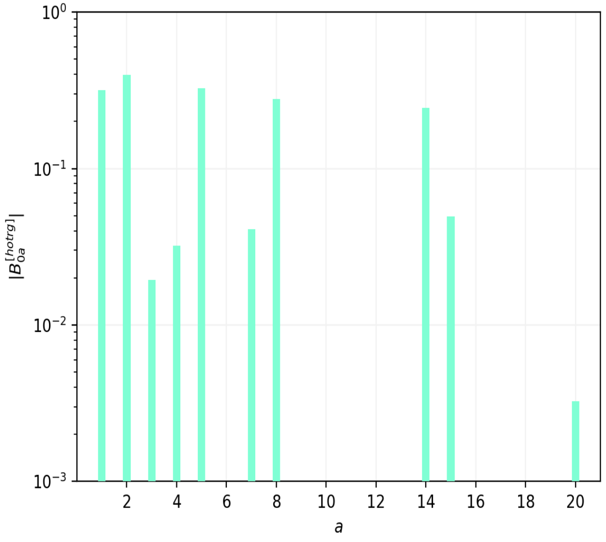

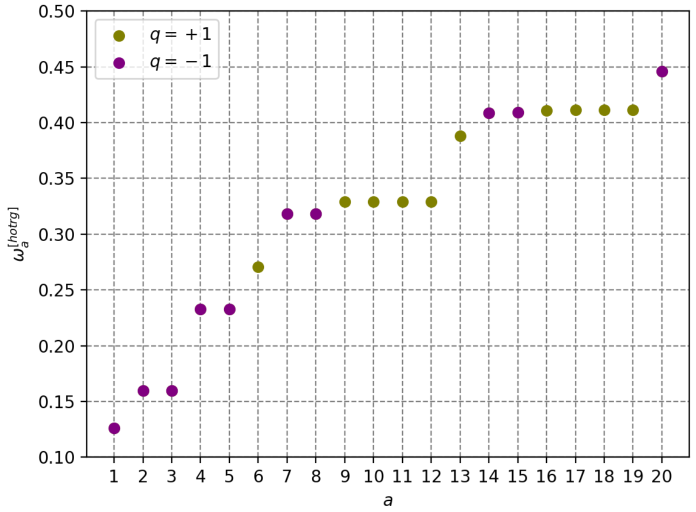

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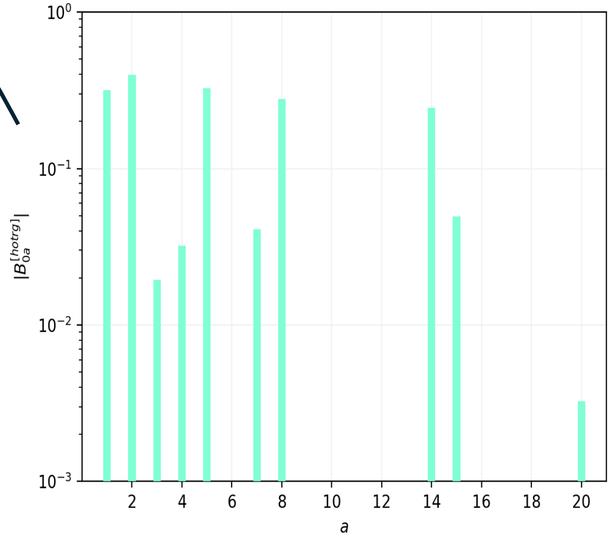
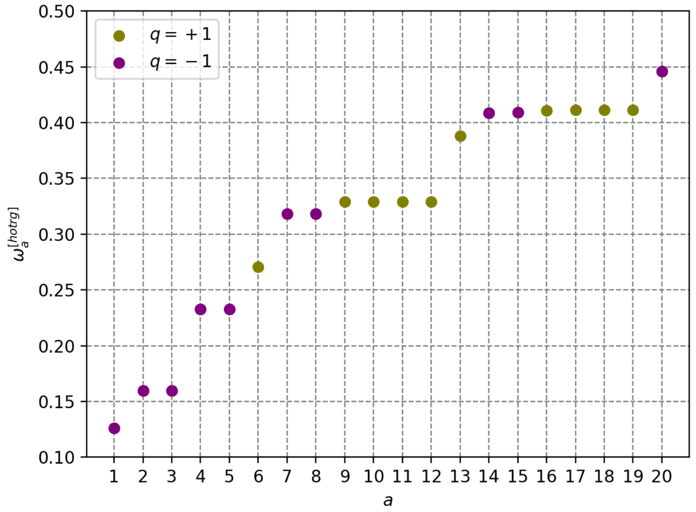
19	0.4100181	+	0.4114574	+	0.003510
20	0.4457831	−	0.4461242	−	0.000765

[Kaufman, Phys. Rev. 76, (1949)]

Energy Gaps from TN+TM





Matrix elements to judge quantum number:

$$|\langle\Omega|s|a\rangle| \approx |B_{0a}^{[hotrg]}|$$

$q_\Omega = 1$ $\quad q_s = -1$

$$\left|B_{0a}^{[hotrg]}\right| \neq 0 \Rightarrow q_\Omega q_s q_a = 1$$

$$\Updownarrow$$

$$\boldsymbol{q_a = -1}$$

Energy and Quantum numbers

$\boldsymbol{T = 2.44, L = 64, \chi = 80}$

a	$\omega^{[\text{exact}]}$	q_a	$\omega^{[\text{hotrg}]}$	q_a	$\delta\omega$
1	0.1262302	−	0.1262307	−	0.000004
2	0.1597880	−	0.1597889	−	0.000006
3	0.1597880	−	0.1597911	−	0.000020
4	0.2326853	−	0.2327046	−	0.000083
5	0.2326853	−	0.2327095	−	0.000104
6	0.2708016	+	0.2708359	+	0.000127
7	0.3181546	−	0.3183329	−	0.000560
8	0.3181546	−	0.3183705	−	0.000679
9	0.3290037	+	0.3291180	+	0.000347
10	0.3290037	+	0.3291425	+	0.000422
11	0.3290037	+	0.3291456	+	0.000431
12	0.3290037	+	0.3293794	+	0.001142
13	0.3872058	+	0.3878486	+	0.001660
14	0.4073042	−	0.4083937	−	0.002675
15	0.4073042	−	0.4090231	−	0.004220
16	0.4100181	+	0.4109090	+	0.002173
17	0.4100181	+	0.4112006	+	0.002884
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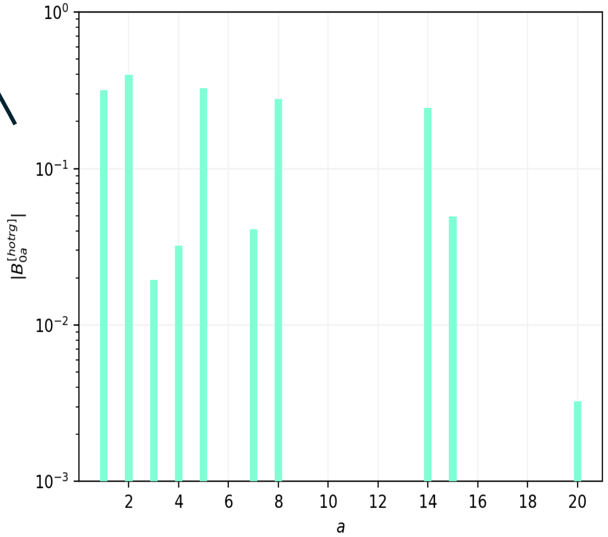

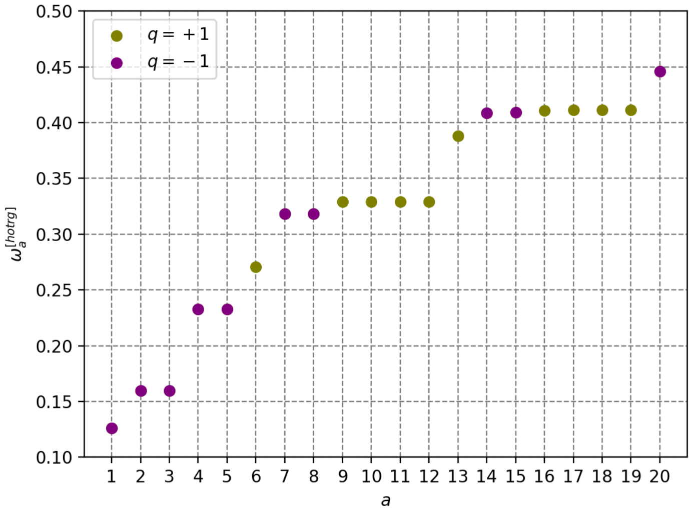

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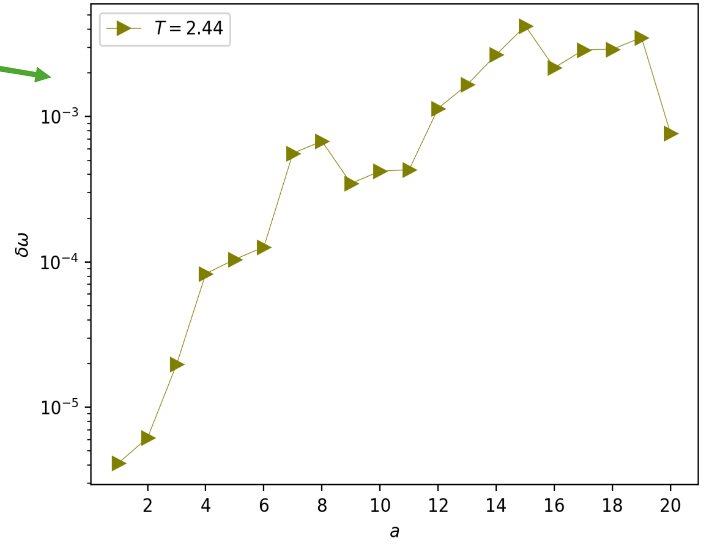

$$\delta\omega_a = \frac{\left|\omega_a^{[hotrg]} - \omega_a^{[exact]}\right|}{|\omega_a^{[exact]}|}$$

1-particle state energy of Ising Model ($q = -1$ sector)

$$|\langle\Omega|\hat{\phi}(p)|a\rangle| = |\langle\Omega|\frac{1}{L_x}\sum_{x=0}^{L_x-1} s_x e^{-ipx}|a\rangle|$$

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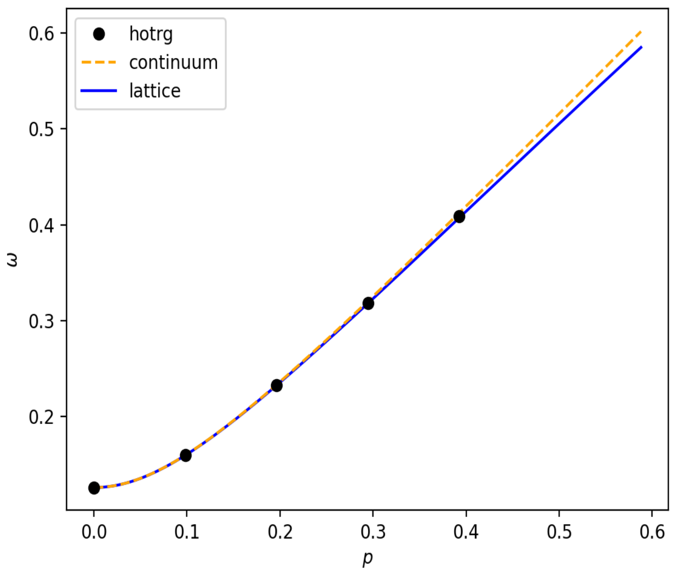

Dispersion Relation

Continuum:
$$\omega = \sqrt{m^2 + p^2}$$

Lattice:
$$\omega = \cosh^{-1}(1 - \cos p + \cosh m)$$

2-particle states energy of Ising Model

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Operator for identifying 2-particle state

$$|\langle\Omega|\widehat{\mathcal{O}}_2(P,p)|a\rangle| = |\langle\Omega|\,\frac{1}{L_x^2}\sum_{x,y=0}^{L_x-1} s(x)s(y)e^{-ip_1x}e^{-ip_2y}\,|a\rangle$$

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$p_1 = \frac{2\pi n_1}{L_x}$ $\quad$ $p_2 = \frac{2\pi n_2}{L_x}$

$P = p_1 + p_2$ total momentum

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infinite volume limit
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$m = 0.12621870$

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Lüscher's formula,

$$e^{i2\delta(k)} = e^{-ikL_x}$$

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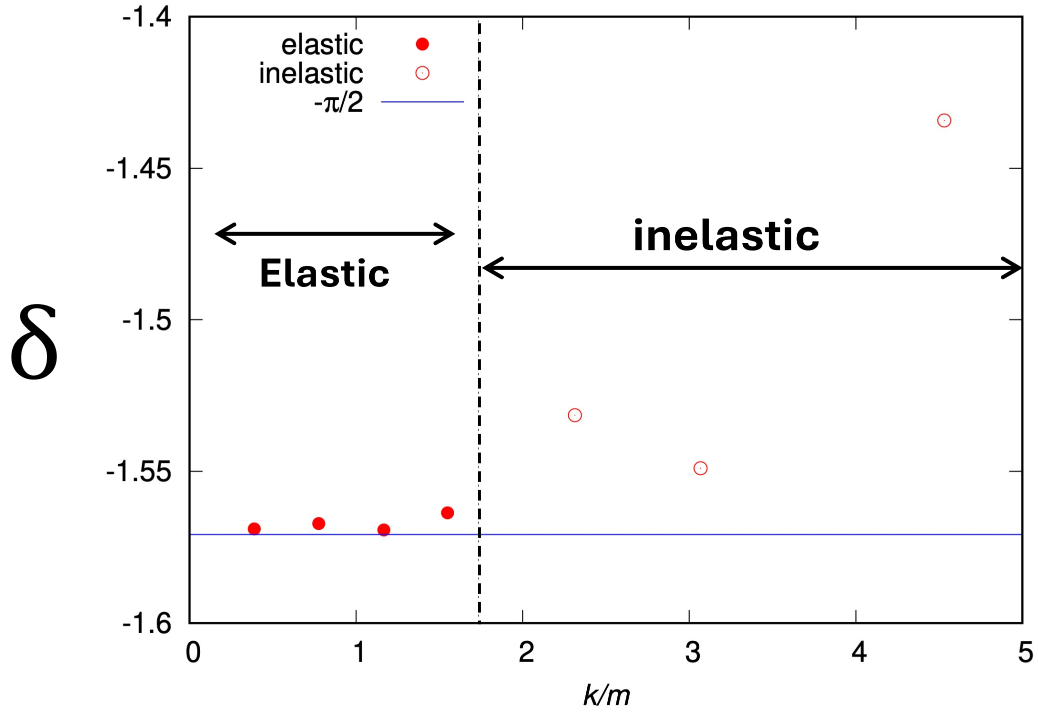
Elastic region
$2m \leq \omega < 4m$



$\delta_{ising} = -\frac{\pi}{2}$ [C. R. Gattringer, C. B. Lang, 1993]

Summary and future plan

- By using our scheme, the energy spectrum is obtained from eigenvalues of tensor network
- The the quantum number is judged from the matrix elements of a proper spin operator
- The momentum of one-particle state energy can be identified
- The two-particle state energy with total momentum zero can be identified
- By Lüscher's formula, the scattering phase shift is obtained from two-particle state energy whose total momentum is zero
- Future works : application to 2d scalar fields, phase shift from non-rest frame, etc

Thank you

Appendix

Identification of Quantum Numbers

Quantum number can be identified from Matrix elements of operator $\hat{\mathcal{O}}_q$

i.e. $\mathrm{B_{ba}} = \langle b|\hat{\mathcal{O}}_q|a\rangle$

Reason:

System with Discrete Symmetry

Ex: (1+1)d Ising Model, Sym over Z_2, $q = \pm 1$

Let $\widehat{D}$ be a discrete transformation operator.

Discrete transformation of operator $\hat{X}$ is

$$\widehat{D}\hat{X}\widehat{D}^{-1} = q_X\hat{X}$$

$$\widehat{D}|a\rangle = q_a|a\rangle$$

$$\begin{aligned}\langle \mathrm{b}|\hat{X}|a\rangle &= \langle \mathrm{b}|\widehat{D}^{-1}\widehat{D}X\widehat{D}^{-1}\widehat{D}|a\rangle \\ &= q_b q_X\, q_a\langle b|\hat{X}|a\rangle\end{aligned}$$

This gives us selection rule:

$$\boldsymbol{\langle \mathbf{b}|X|a\rangle \neq 0 \;\Rightarrow\; q_b q_X q_a = 1}$$

q_X Assumed to be known

Choose $\langle b|$ as $\langle\Omega|$ where $q_\Omega = 1$

Then q_a can be identified

Identification of Energy spectrum ω_a based on Quantum Numbers

Identification can be done by computing matrix element of interpolating operator $\hat{\mathcal{O}}_q$ i.e.

$\mathrm{B_{ba}} = \langle b|\hat{\mathcal{O}}_q|a\rangle$.

Reason:

<u>For system with Continious symmetry</u>

- Let $\hat{Q}$ be a conserved charge of continious symmetry and $\left[\hat{Q}\,,\hat{H}\right] = 0$
- If Quantum number of an operator $\hat{X}$ is q_X then

$$\left[\hat{Q}\,,\hat{X}\right] = q_X\hat{X}$$

Assume $|\Omega\rangle$ has no charge $\hat{Q}\,|\Omega\rangle = 0$,

$$\hat{Q}\hat{X}\,|\Omega\rangle = q_X\hat{X}\,|\Omega\rangle$$

For energy eigenstate $|a\rangle, |b\rangle$

$$\langle b|(\hat{Q}\hat{X} - \hat{X}\hat{Q})|a\rangle = \langle b|q_X\hat{X}\,|a\rangle$$

$$(q_a - q_b - q_X)\langle b|\hat{X}\,|a\rangle = 0$$

Selection Rule:

If $\left\langle b\middle|\hat{X}\middle|a\right\rangle \neq 0$, then $(q_a - q_b - q_X) = 0$

If $\langle b| = \langle\Omega|$ then $q_a = q_X$

Tensor Network Representation for $\langle b|\hat{\mathcal{O}}_q|a\rangle$

$$\langle b|\hat{\mathcal{O}}_q|a\rangle = \left(U^\dagger\, \hat{\mathcal{O}}_q U\right)_{ba}$$

$\mathcal{T} = U\lambda U^\dagger$

$m = L_\tau/2$ ($\hat{\mathcal{O}}_q$ is in the middle of square lattice)

$$= \left(U^\dagger \mathcal{T}^{-m}\mathcal{T}^{m}\hat{\mathcal{O}}_q\mathcal{T}^{(m+1)}\mathcal{T}^{-(m+1)}U\right)_{ba}$$

Using $\mathcal{T}\,\mathcal{T}^{-1} = I$

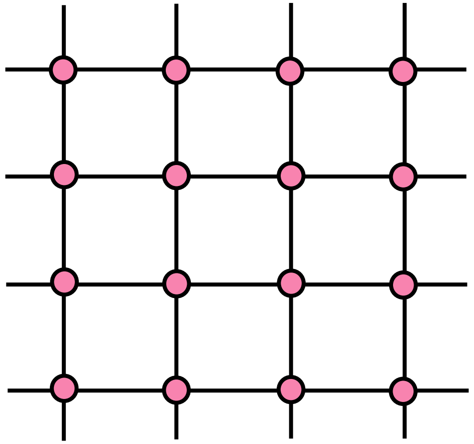
$U\lambda^{-m}U^\dagger \quad (YY^\dagger)^m \quad (YY^\dagger)^{m+1} \quad U\lambda^{-(m+1)}U^\dagger$

$$= \left(\lambda^{-m}U^\dagger Y\left(Y^\dagger Y\right)^{m-1} Y^\dagger\hat{\mathcal{O}}_q Y \left(Y^\dagger Y\right)^{m} Y^\dagger U\lambda^{-(m+1)}\right)_{ba}$$

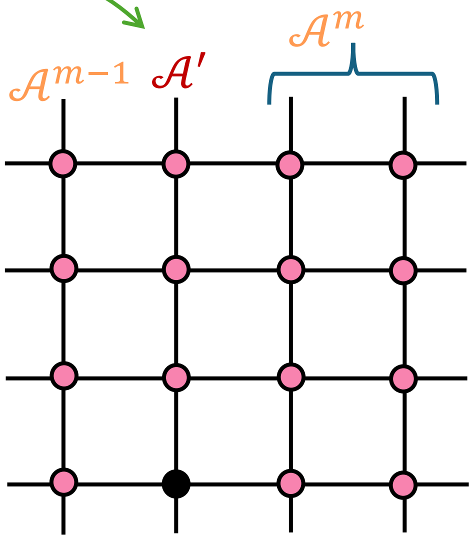
$U\sqrt{\lambda}W^\dagger \quad \mathcal{A} \quad \mathcal{A}' \quad \mathcal{A} \quad U^\dagger\sqrt{\lambda}W$

$$= \left(\lambda^{-(m-\frac{1}{2})}W^\dagger \mathcal{A}^{m-1}\mathcal{A}'\mathcal{A}^{m}W\lambda^{-(m+\frac{1}{2})}\right)_{ba}$$

$\mathcal{A} = W\lambda W^\dagger$



All terms are obtained from Tensor Network

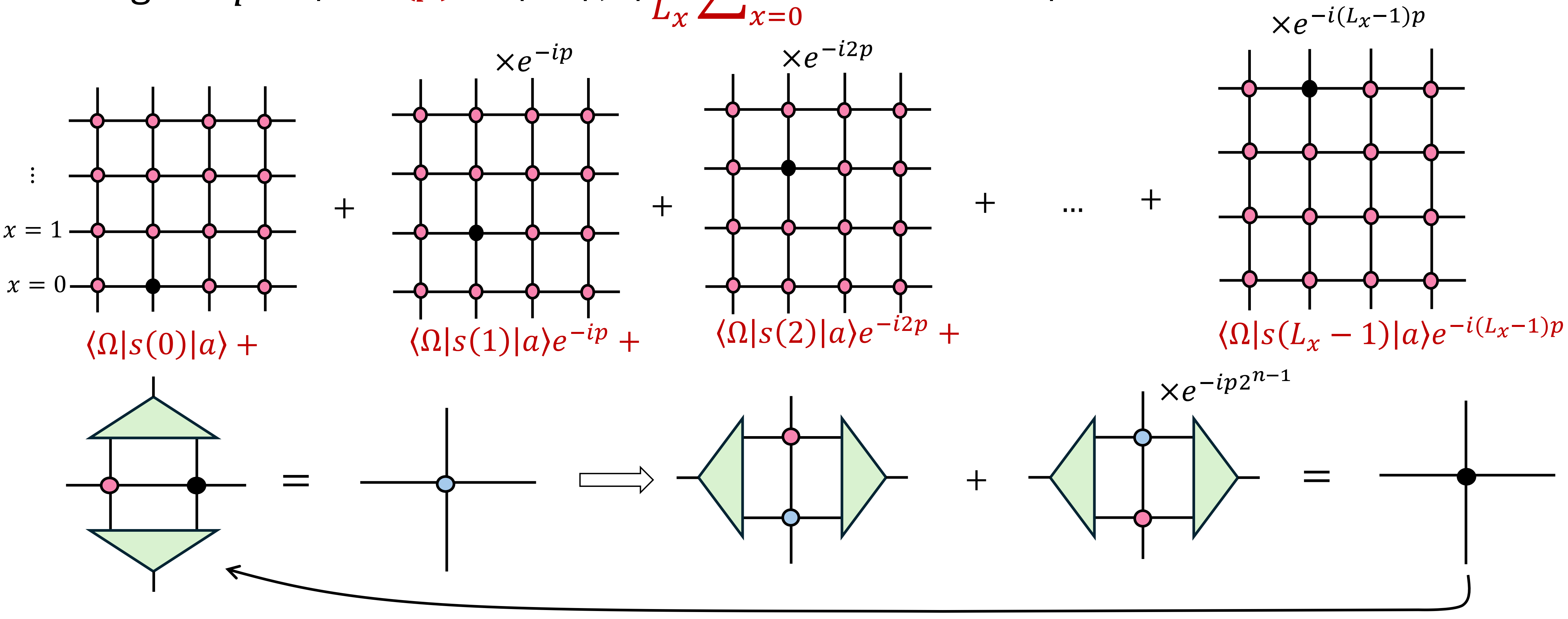


Impurity Tensor Network

Tensor Network Representation for Momentum of 1 −particle state

$\hat{O}_q(p) = s(p) \Longrightarrow$ Momentum operator for single particle state

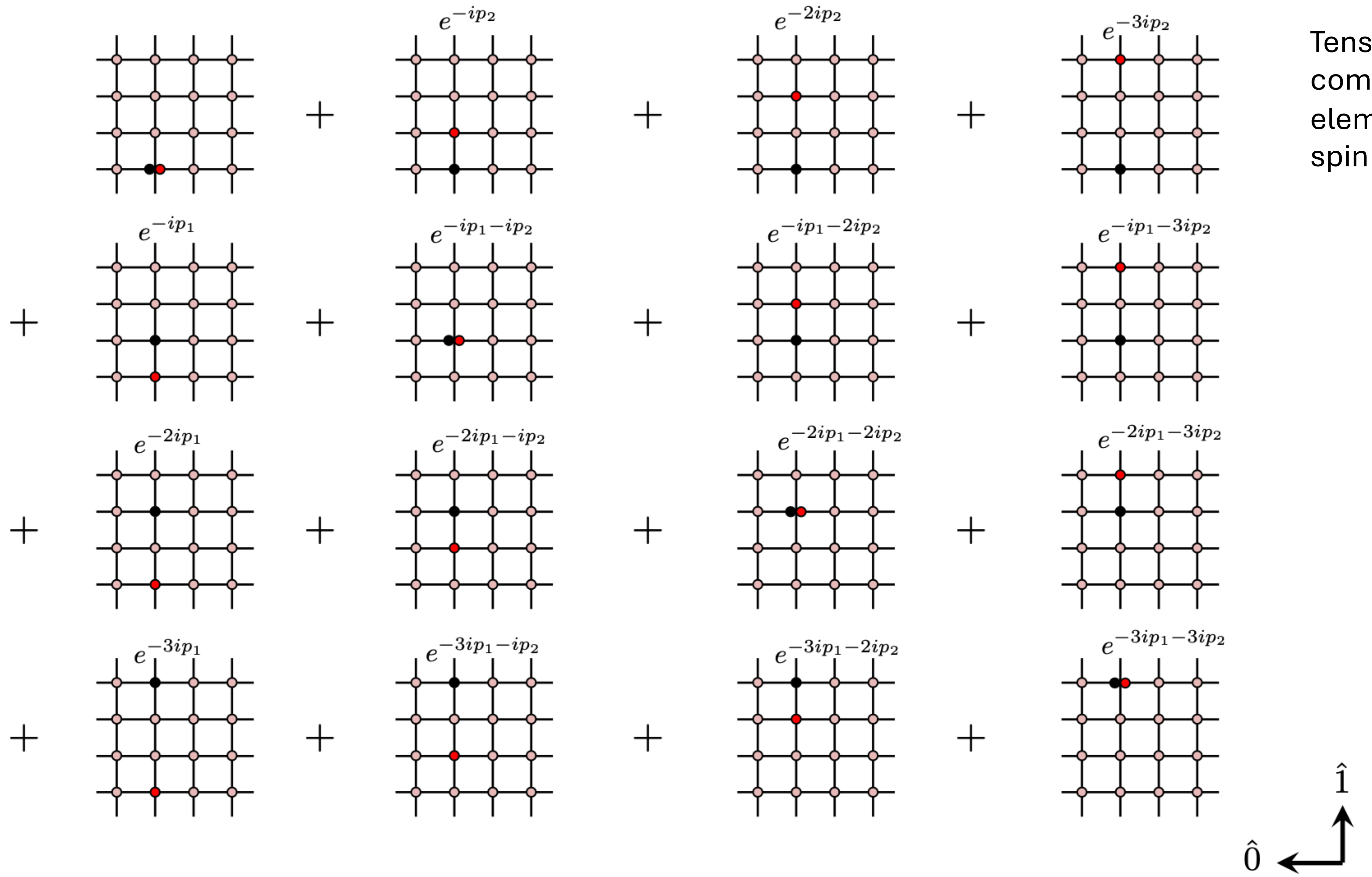
For a given p $\quad |\langle\Omega|s(p)\,|a\rangle| = |\langle\Omega|\frac{1}{L_x}\sum_{x=0}^{L_x-1} s(x)e^{-ipx}\;|a\rangle| \neq 0$ ➡ p is momentum of $|a\rangle$



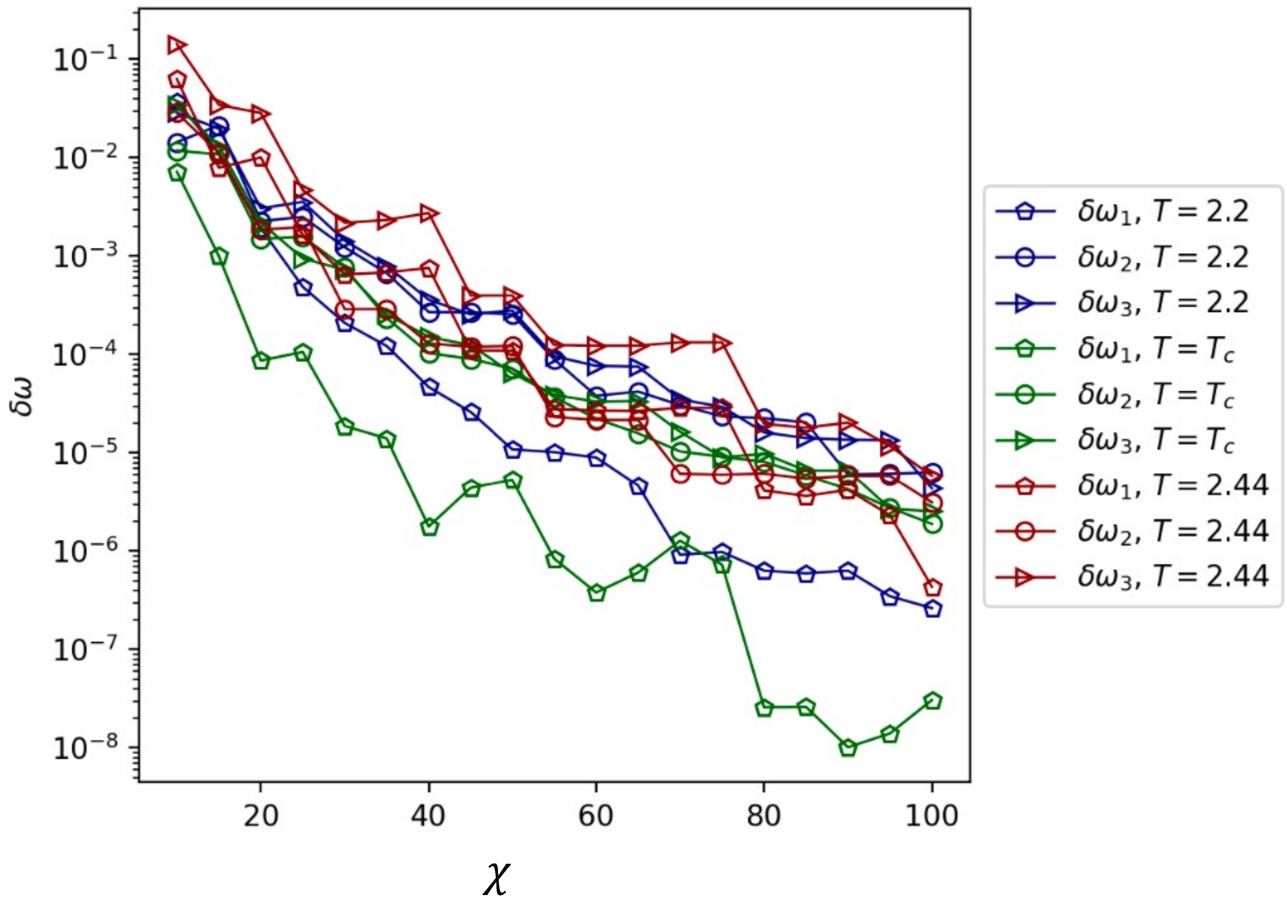
coarse-graining $\langle\boldsymbol{\Omega}|\hat{\boldsymbol{O}}_q(\boldsymbol{p})|\boldsymbol{a}\rangle$ **.**

[S. Morita, N. Kawashima, 2019]

Tensor Network to compute matrix elements of double spin operator



Relative Error Over χ

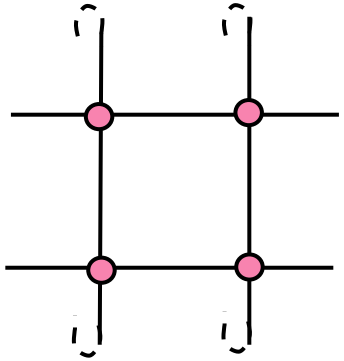


- The error is getting smaller when χ is increased.
- Error near T_c is smaller compared to $T > T_c$ and $T < T_c$
- $\chi = 80$ is large enough to get relatively small error for eigenstate up to $a = 20$ and its computational cost is still manageable.

Energy Spectrum

After HOTRG

$\mathcal{A}^{[n]}$

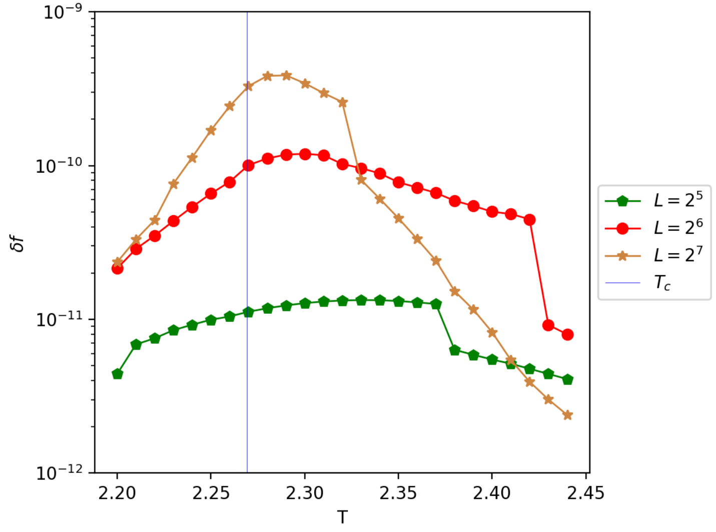


$$\mathcal{A}^{[n]} \xrightarrow{\text{Diagonalize}} \mathcal{A}^{[n]} = W^{[n]}\lambda^{[n]}W^{[n]\dagger}$$

$$\lambda^{[n]} = e^{E^{[n]}}$$

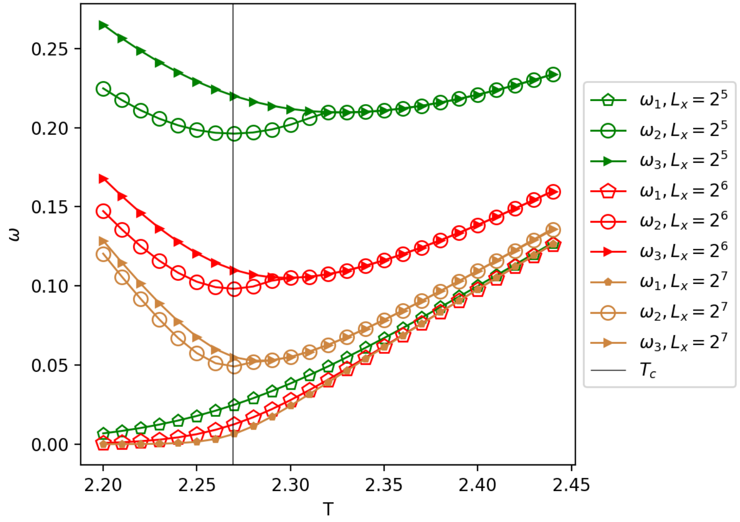
$$\omega_a^{[n]} = \frac{1}{L_\tau}\log\frac{\lambda_0^{[n]}}{\lambda_a^{[n]}} \approx \omega_a \Rightarrow$$ Original energy gap before HOTRG

$$f_V = -T\frac{\log Z_V}{V}$$

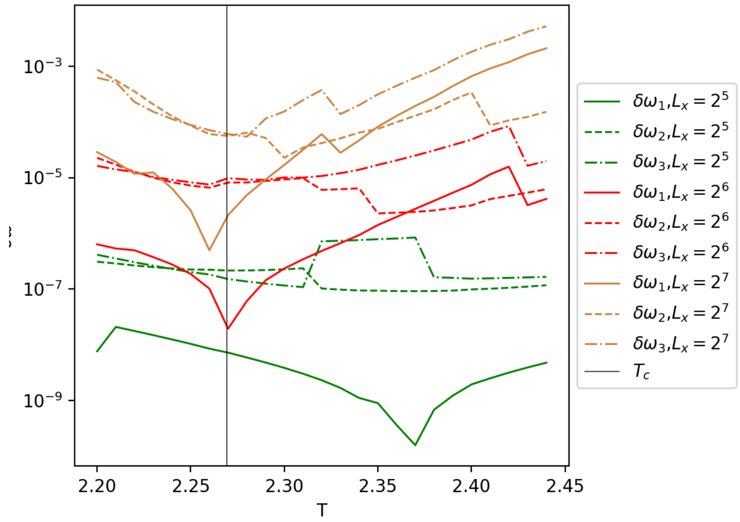


Relative Error of Free Energy

$$\delta f = \frac{\left|\delta f_V^{[hotrg]} - \delta f_V^{[exact]}\right|}{|\delta f_V^{[exact]}|}$$



Energy Gap over Temperature, $\chi = 80$



Relative Error of Energy Gap

$$\delta\omega_a = \frac{\left|\delta\omega_a^{[hotrg]} - \delta\omega_a^{[exact]}\right|}{|\delta\omega_a^{[exact]}|}$$

$f_V^{[exact]}, \omega_a^{[exact]}$ [Kaufman, Phys. Rev. 76, (1949)]

Transfer Matrix of Ising Model

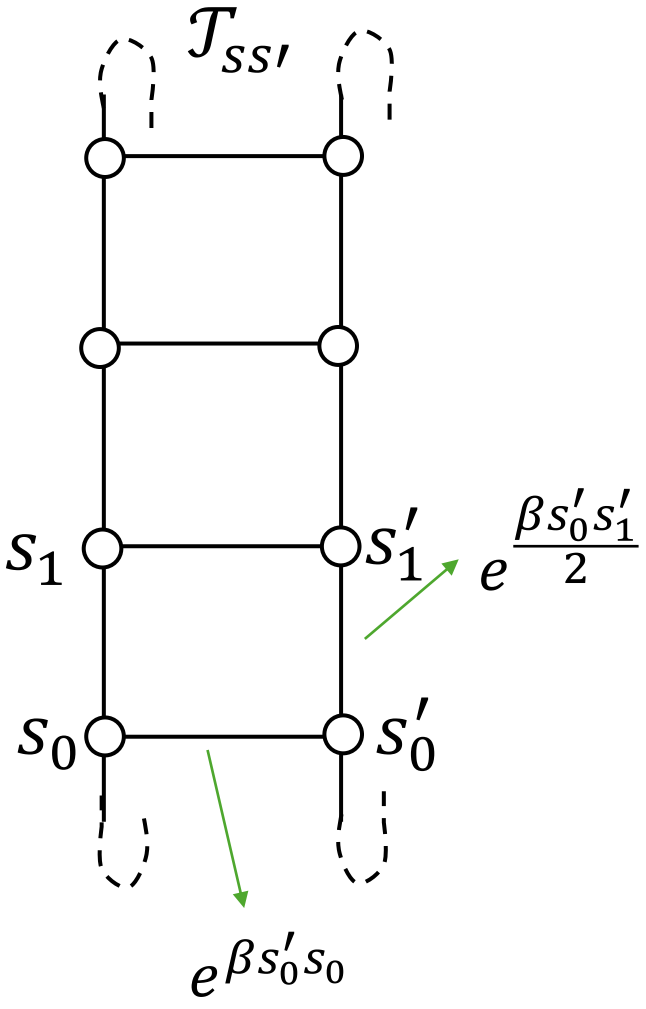
$$Z = \sum_{\{s\}} e^{\beta \sum_{r \in \Gamma} \sum_{\mu=1}^{2} s(r+\hat{\mu}) s(r)} = Tr[\mathcal{T}^{L_\tau}]$$

The transfer matrix of Ising Model is given by

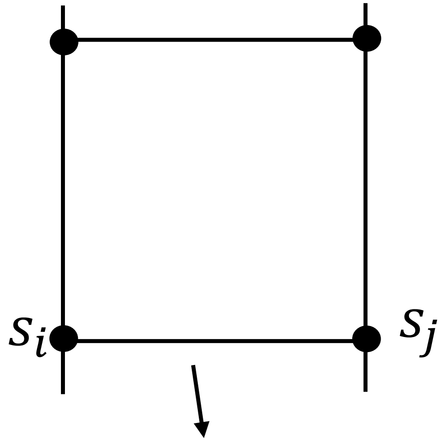
$$\mathcal{T}_{s's} = \left(\prod_{x=0}^{L_x - 1} e^{\beta s(\tau+1,x) s(\tau,x)} \right) \times \left(\prod_{x=0}^{L_x-1} e^{\frac{\beta}{2} s(\tau+1,x+1) s(\tau+1,x)} e^{\frac{\beta}{2} s(\tau,x+1) s(\tau,x)} \right)$$

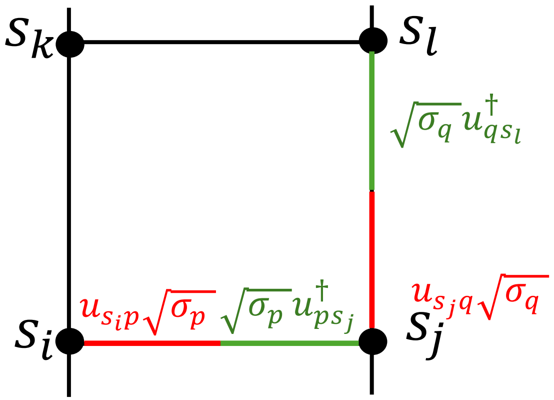
The spin configuration on Euclidean time slice at $\tau + 1$ and τ is

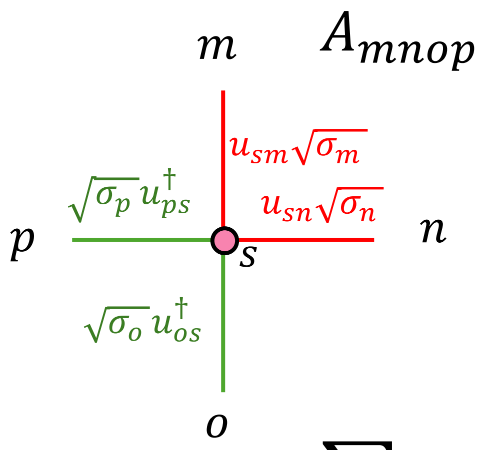
$$s' = \{s(\tau + 1, x) | x = 0,1,2, \dots, L_x - 1\}$$
$$s \ = \{s(\tau \quad , x) | x = 0,1,2, \dots, L_x - 1\}$$



Initial Tensor Network







$$A_{mnop} = \sqrt{\sigma_m \sigma_n \sigma_o \sigma_p} \sum_s u_{sm} u_{sn} u^\dagger_{os} u^\dagger_{ps}$$

$$e^{\beta s_i s_j} = u_{s_i k}\sqrt{\sigma_k}\,\sqrt{\sigma_k}\, u^\dagger_{k s_j}, \quad s = \pm 1$$

$$u_{s_i k} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\sigma_k = \begin{pmatrix} 2\cosh\beta & 0 \\ 0 & 2\sinh\beta \end{pmatrix}$$

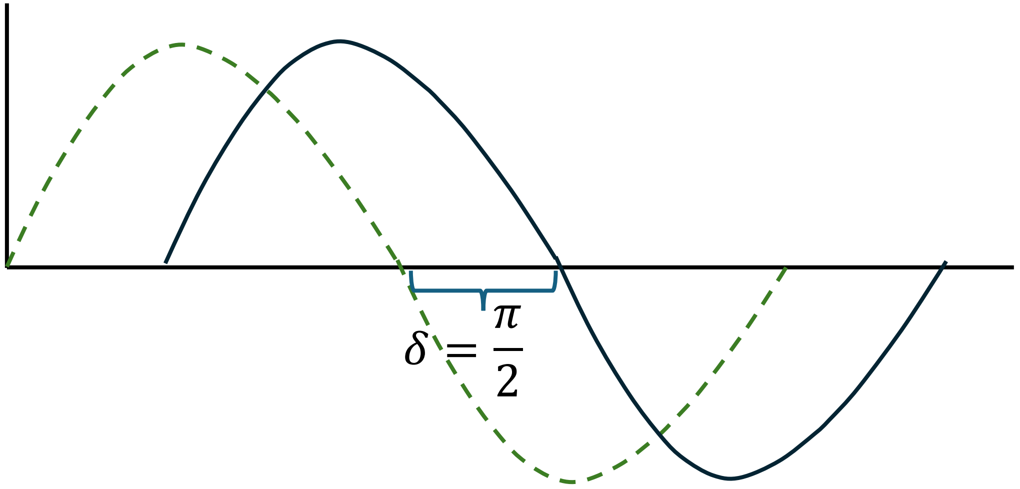
Scattering Phase Shift

In scattering theory, the S −matrix can be written as

$$S = \frac{outgoing\ wave\ function}{incoming\ wave\ function} = e^{2i\delta}$$

phase shift

- In Ising Model, $\delta_{ising} = -\frac{\pi}{2}$ or equivalently $S = -1$
- $\delta < 0$ means system has repulsive potential
- And phase $-\pi/2$ means outgoing wave function is being pulled out by $\pi/2$.



Scattering Phase Shift

L_x	a	$\omega_a^{[\text{hotrg}]}$
8	4	0.814585
	19	2.133922
16	4	0.465348
	18	1.171480
32	4	0.319553
	14	0.636356
64	6	0.270836
	13	0.387849

$$\omega = 2\sqrt{k^2 + m^2}$$

Relative momentum

infinite volume limit
exact rest mass
$m = 0.12621870$

Elastic region : $2m \leq \omega < 4m$
m is rest mass