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2310.17539 + 24MM.XXXXX

# GENERALIZED BKT TRANSITIONS AND PERSISTENT ORDER ON THE LATTICE

Aleksey Cherman

UMN

Theo Jacobson

pd at UCLA

Srimoyee Sen

Iowa State

Seth Buesing

Macalester

Shi Chen

pd at UMN

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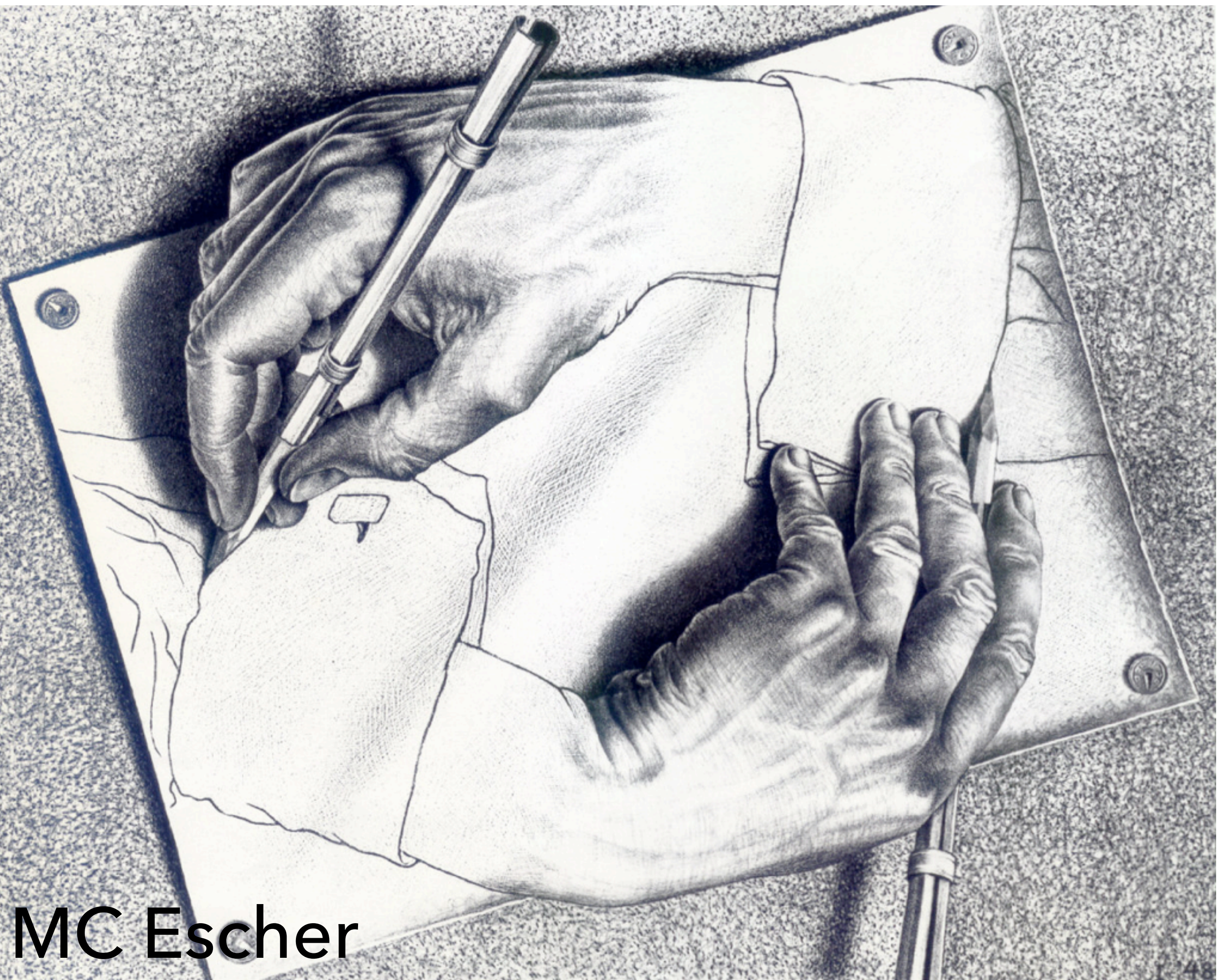
# 24MM.XXXX

# MOTIVATION: EYES ON THE PRIZE

EB, Cherman, Jacobson 2310.17539
see Aleksey Cherman, 14:35
also: S. Sen, 12:44 Monday
H. Singh, 11:15 Tuesday
T. Onogi, 12:15 Tuesday
S. Onoda, 11:15 Friday
S. Catterall, 14:55 Friday
… and more!

## An honest 2D chiral gauge theory on the lattice!

| Continuum | | $N_f$=1 QED | | $N_f$=2 QED | | 3450 Chiral |
|---|---|---|---|---|---|---|
| $\mathscr{L} = \frac{1}{4e^2} f_{\mu\nu}^2$ | → | $\frac{\beta}{2}(da-2\pi r)_p^2$ | → | $\frac{\beta}{2}(da-2\pi r)_p^2$ | → | $\frac{\beta}{2}(da-2\pi r)_p^2$ |
| $+\frac{1}{8\pi}\partial_\mu\varphi\partial^\mu\varphi$ | → | $+\frac{\kappa}{2}(d\varphi-2\pi n)_\ell^2$ | → | $+\frac{\kappa}{2}(d\varphi^j-2\pi n^j)_{\tilde{\ell}}^2$ | → | $+\frac{\kappa}{2}(d\varphi^j-2\pi n^j - Q_A^j a_f)_{\tilde{\ell}}^2$ |
| $+\frac{iQ}{2\pi}\varphi\epsilon^{\mu\nu}\partial_\mu a_\nu$ | → | $+\frac{iQ}{2\pi}\varphi_{\star p}(da-2\pi r)_p$ | → | $+\frac{i}{2\pi}Q^j\varphi^j_{\star p}(da-2\pi r)_p$ | → | $+\frac{i}{2\pi}Q_V^j\varphi^j_{\star p}(da-2\pi r)_p$ |
| | | $-iQa_\ell n_{\star\ell}$ | → | $-iQ^j n^j_{\star\ell}a_\ell$ | → | $-iQ_V^j n^j_{\star\ell}a_\ell$ |
| | | $-i\chi_s(dn)_{\star s}$ | → | $+in^j_{\star\ell}(d\chi^j)_\ell$ | → | $+in^j_{\star\ell}(d\chi^j)_\ell$ |
| | | | | | | $-ir_{f(\star s)}Q_A^j\chi_s^j$ |


MC Escher

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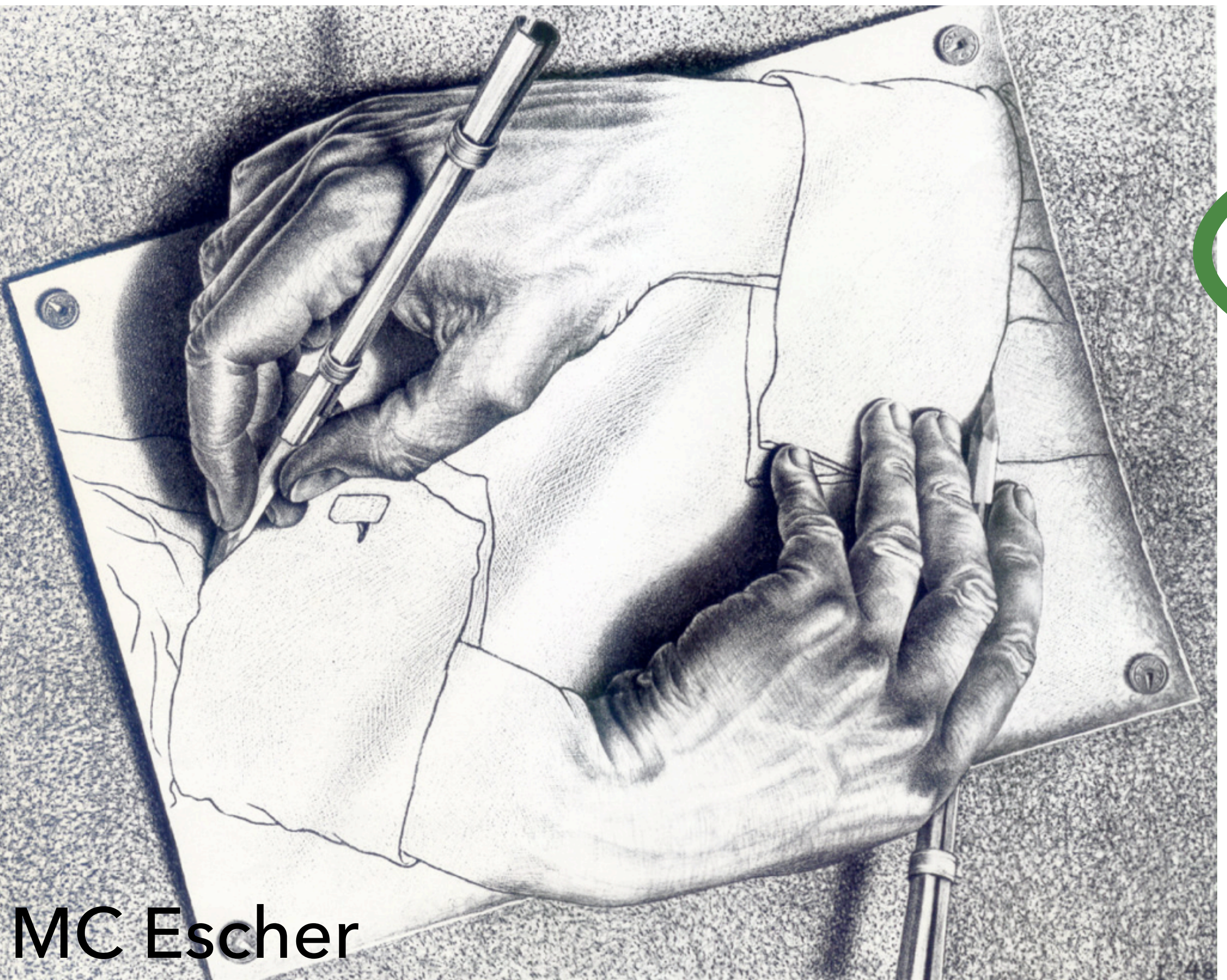

MC Escher

In this talk: one compact boson using the modified Villain formulation without U(1) gauge field [turns out to be interesting by itself!]

# COMPACT BOSON HAS A MIXED 'T HOOFT ANOMALY

EB, Cherman, Jacobson 2310.17539
see Aleksey Cherman, 14:35

- Compact boson in 1+1D results from bosonizing a free fermion
- Much simpler than gauge theory
- In fact we can study a whole class of models which interpolate between XY and the compact boson
- Lattice formulations maintain 't Hooft anomalies, even with finite DOFs
- Can we implement a lattice action that gets this right?

Some literature on mixed 't Hooft anomalies in bosonic systems

hep-th:

Sulejmanpasic, Shao, Seiberg, Lam, Fazzi, Gorantla, Gattringer, Cheng…

cond-mat:

Lieb+Shutz+Mattis, Kitaev, Kapustin+Thorngren, …

hep-lat:

Catterall et al., Singh et al.

# CODE AVAILABILITY

https://github.com/evanberkowitz/supervillain/

https://supervillain.readthedocs.io/

# COMPACT BOSON + XY

# THE 2D COMPACT BOSON

$$\mathcal{Z} = \int \mathcal{D}\varphi \; e^{-\int d^2x \, \frac{1}{8\pi}(d\varphi)^2}$$

$\varphi \sim \varphi + 2\pi$ (the model isn't a simple gaussian)

$$J^s_\mu = \frac{i}{4\pi}\partial_\mu \varphi$$

shift symmetry

$\varphi \to \varphi + \varepsilon$

conserved by φ EOM

$$J^w_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu}\partial^\nu \varphi$$

winding symmetry

topological, not Noetherian

conserved as long as partials commute

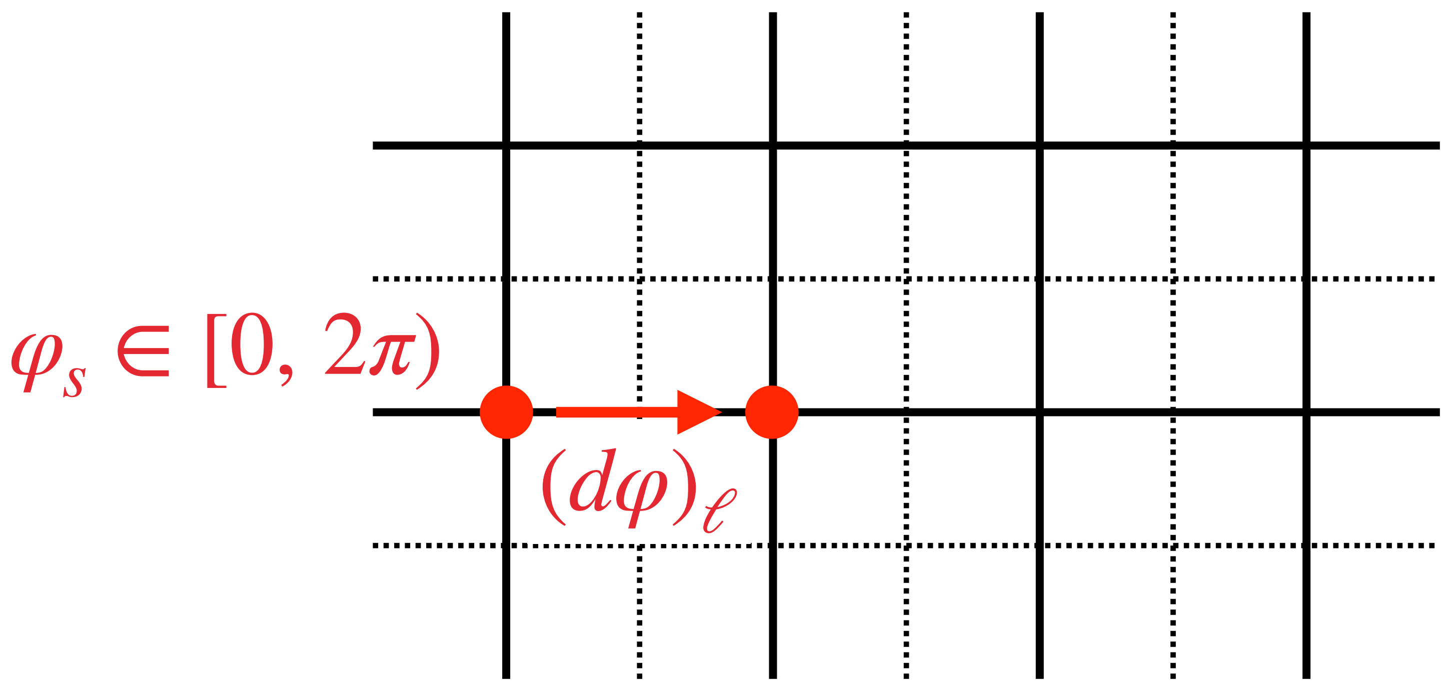
NO VORTICES

These U(1)s have a mixed 't Hooft anomaly!

# XY A LA VILLAIN

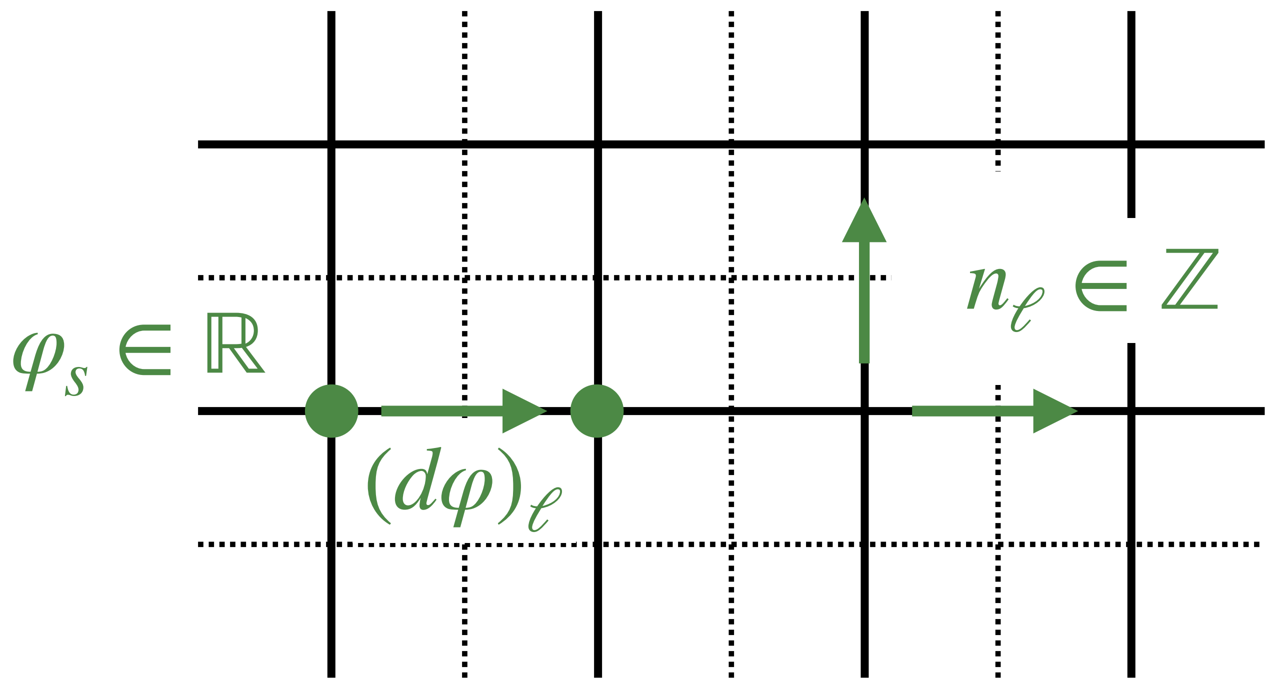
J. Villain (1975) J. Phys France 36 6 581-590

$$S = \frac{\kappa}{2}\sum_{\ell}\left[1 - \cos(d\varphi)_\ell\right]$$




$$S = \frac{\kappa}{2}\sum_{\ell}\left[(d\varphi)_\ell - 2\pi n_\ell\right]^2$$

(quadratic!)




$\varphi \to \varphi + c$ global shift

$\varphi_s \to \varphi_s + 2\pi$ local shift

$\varphi \to \varphi + c$ global shift

$\varphi_s \to \varphi_s + 2\pi q_s$
$n_\ell \to n_\ell + (dq)_\ell$
discrete gauge redundancy ($q_\ell \in \mathbb{Z}$)

$U(1) \cong \mathbb{R}/2\pi\mathbb{Z}$

Dynamical vortices (no winding symmetry), BKT, gapped phase for small κ.

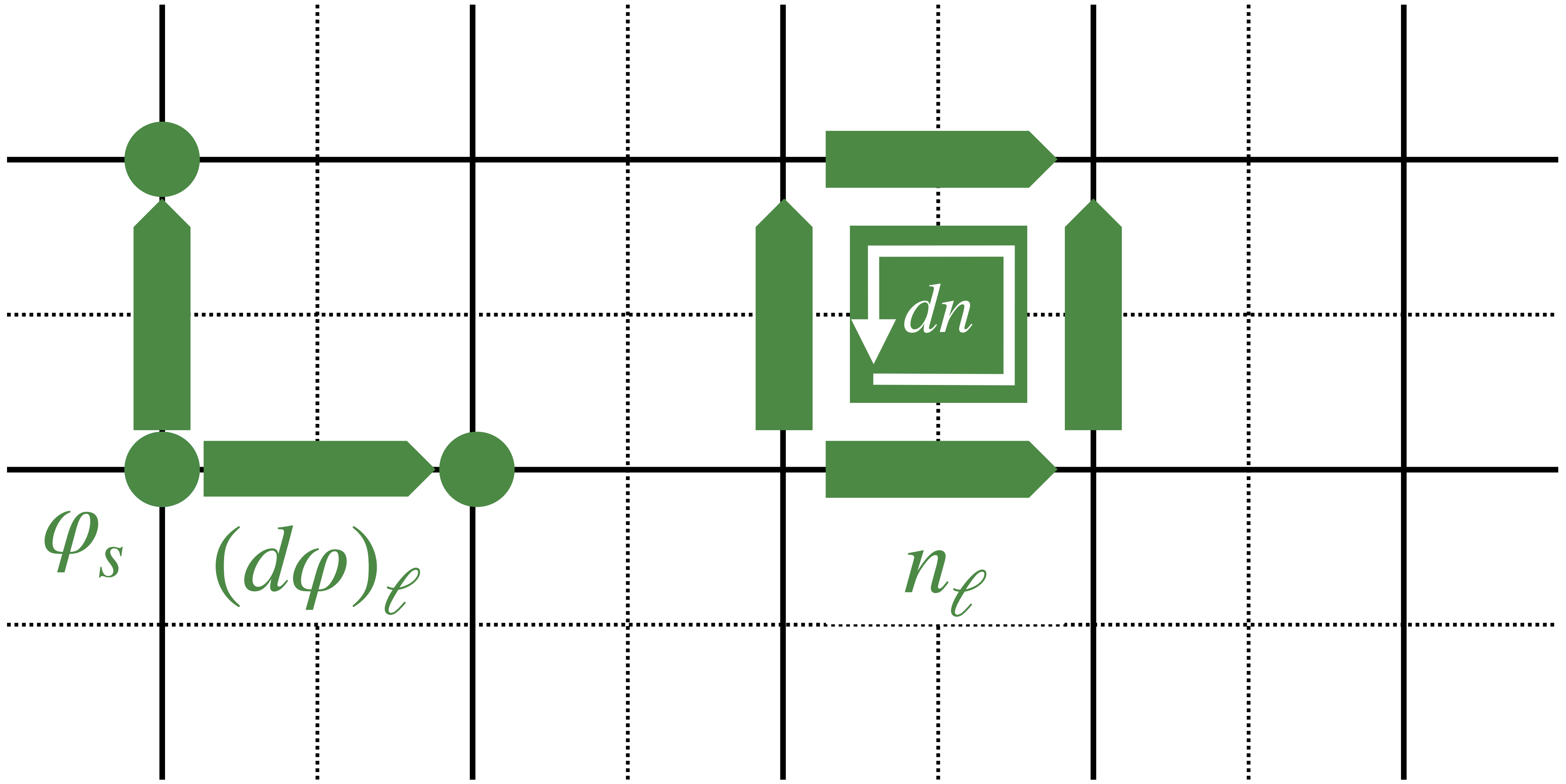
# XY WITH INTEGER WINDING

Review: Sulejmanpasic + Gattringer (2019)
Nucl. Phys. B 943 114616 1901.02637

$$w_A = \frac{1}{2\pi}\oint_{\partial A} \omega = \frac{1}{2\pi}\int_A d\omega$$

$$S = \frac{\kappa}{2}\sum_{\ell}\left[(d\varphi)_\ell - 2\pi n_\ell\right]^2$$

$$\frac{1}{2\pi}\sum_{\ell\in\partial A}(d\varphi - 2\pi n)_\ell = \frac{1}{2\pi}\sum_{p\in A}\left[\underbrace{d(d\varphi}_{d^2=0} - 2\pi n)\right]_p = -\sum_{p\in A}(dn)_p \in \mathbb{Z}$$



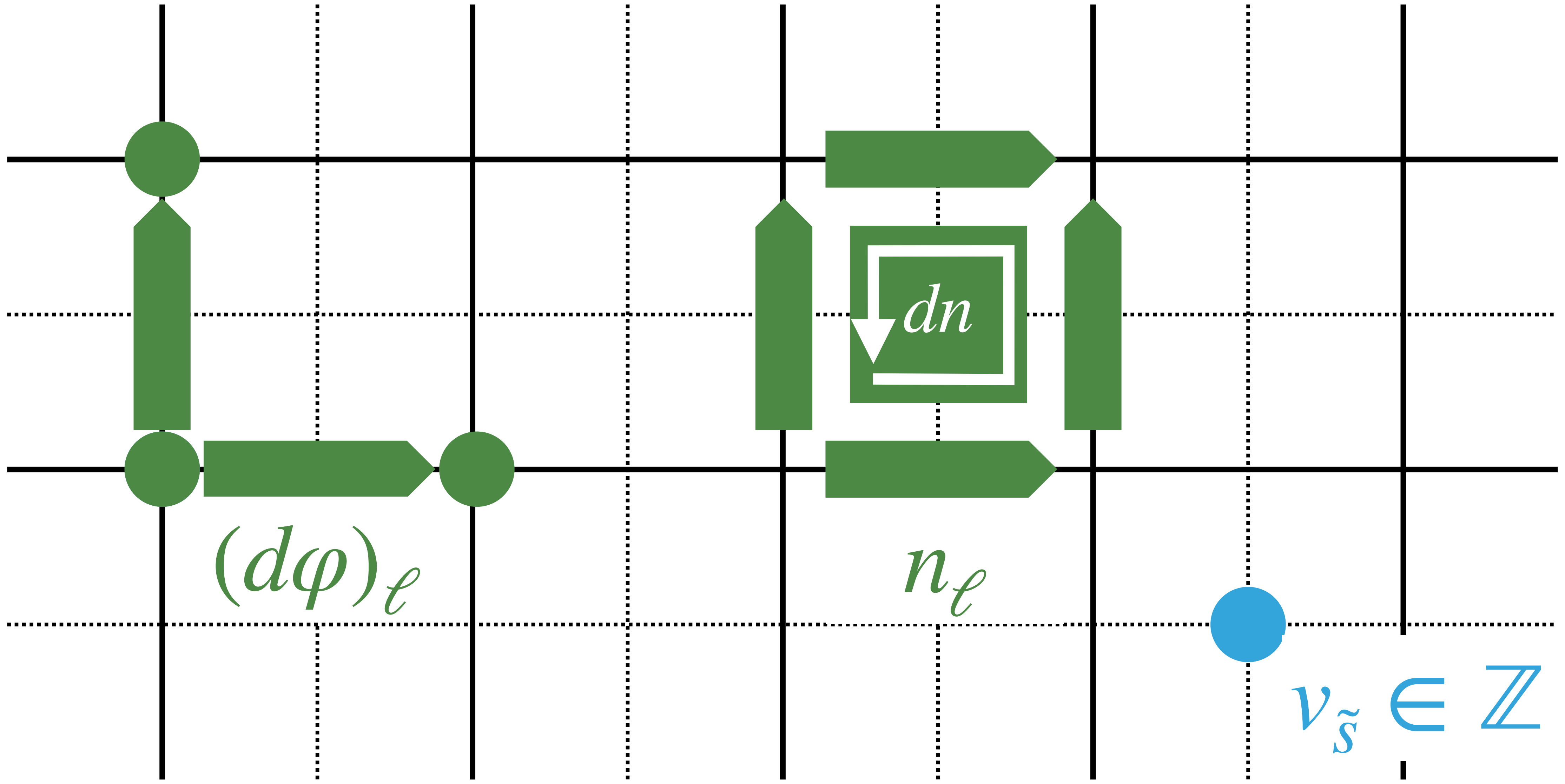

$dn \neq 0$

Dynamical vortices,
BKT,
gapped phase for small κ.

# MODIFIED VILLAIN



$$S = \frac{\kappa}{2}\sum_{\ell}\left[(d\varphi)_\ell - 2\pi n_\ell\right]^2 + i\sum_p \frac{2\pi v_{\star p}}{W}(dn)_p$$




Path integrate over a Lagrange multiplier field

$\mathbb{Z}_W$ winding symmetry

$v \to v + 2\pi z/W$ $\quad e^{2\pi i v_{\tilde{s}}/W}$ vortex operator

Exactly conserved current from v EOM

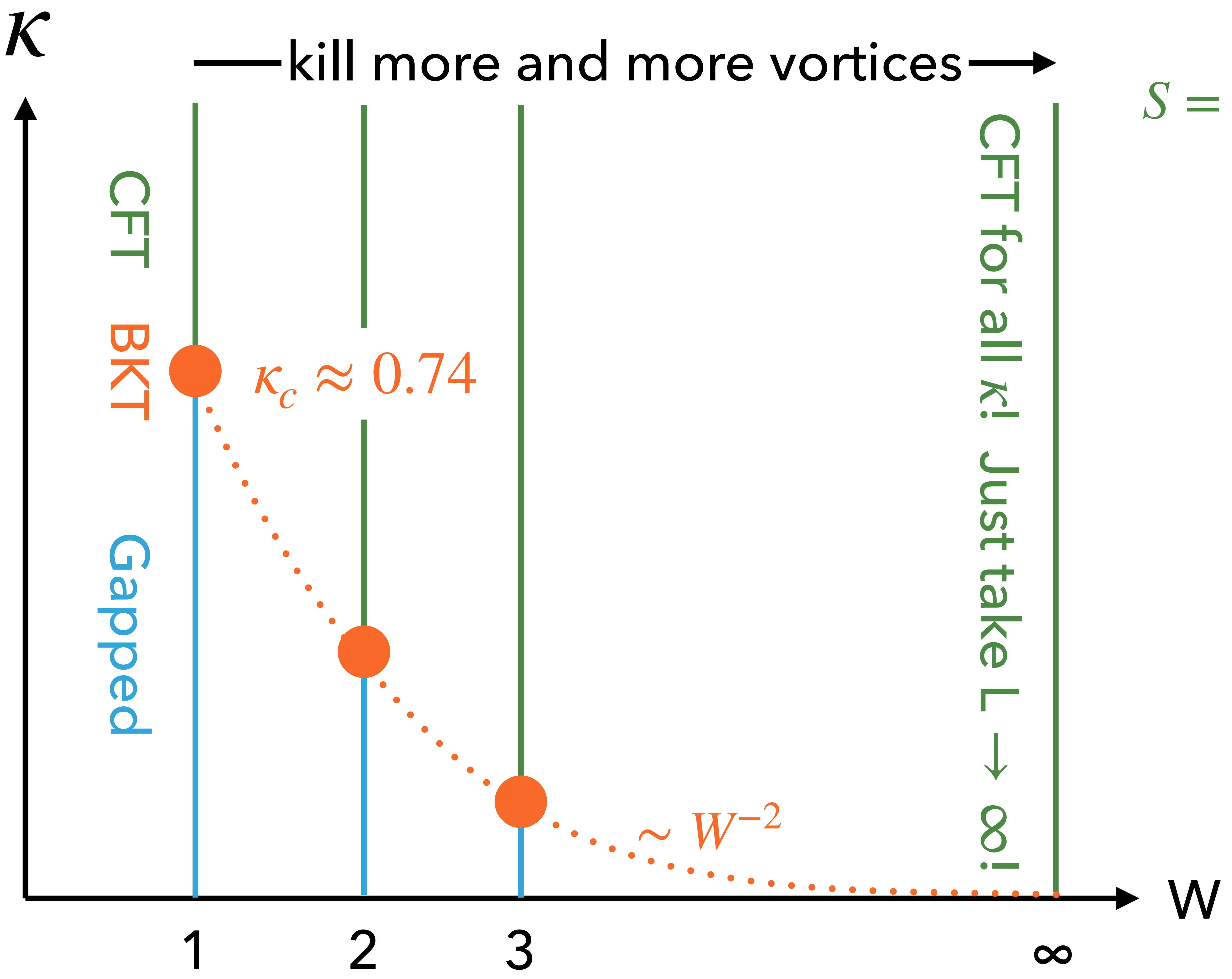
Conserved charged

$$\frac{1}{2\pi}\oint_C d\phi \to \frac{1}{2\pi}\sum_{\ell\in C}(d\varphi - 2\pi n)_\ell = -\sum_{\ell\in C} n \in \mathbb{Z}$$

Has a mixed 't Hooft anomaly between $U(1)_{\text{shift}}$ and $(\mathbb{Z}_W)_{\text{winding}}$ → No non-trivial phase!

# VORTICES TRIGGER THE TRANSITION

Kosterlitz, J. Phys. C 7, 1046 (1974), Villain, J. Phys. (Paris) 36, 581 (1975)
Janke and Nather, PRB 48, 7419 (1993)




$$S = \frac{\kappa}{2} \sum_{\ell} \left[(d\varphi)_\ell - 2\pi n_\ell\right]^2 + i \sum_p \frac{2\pi v_{\star p}}{W} (dn)_p$$

Upon path integrating $v$,

$+i \sum_p \frac{2\pi v_{\star p}}{W} (dn)_p$ yields

the constraint $dn \equiv 0 \pmod W$

Theoretical prediction: $\kappa_c \sim W^{-2}$

(to appear)

# CRITICAL MOMENTS

Generic CFT expectation

$$\left\langle \mathcal{O}_x \mathcal{O}_y \right\rangle = \frac{\#}{|x-y|^{2\Delta}}$$

← OPE coefficient

Particular CFT knowledge (without knowing $\kappa_c$)

Spin operator $e^{i\phi}$ $\quad \Delta_S(\kappa_c) = W^2/8$

Vortex operator $e^{2\pi i v/W}$ $\quad \Delta_V(\kappa_c) = 2/W^2$

(Susceptibility isn't good when W≥3)

Probes: critical moments

$$C_S = \frac{1}{L^2}\int d^2x \; r^{2\Delta_S(\kappa_c)} \left\langle e^{i(\phi_x - \phi_0)} \right\rangle$$

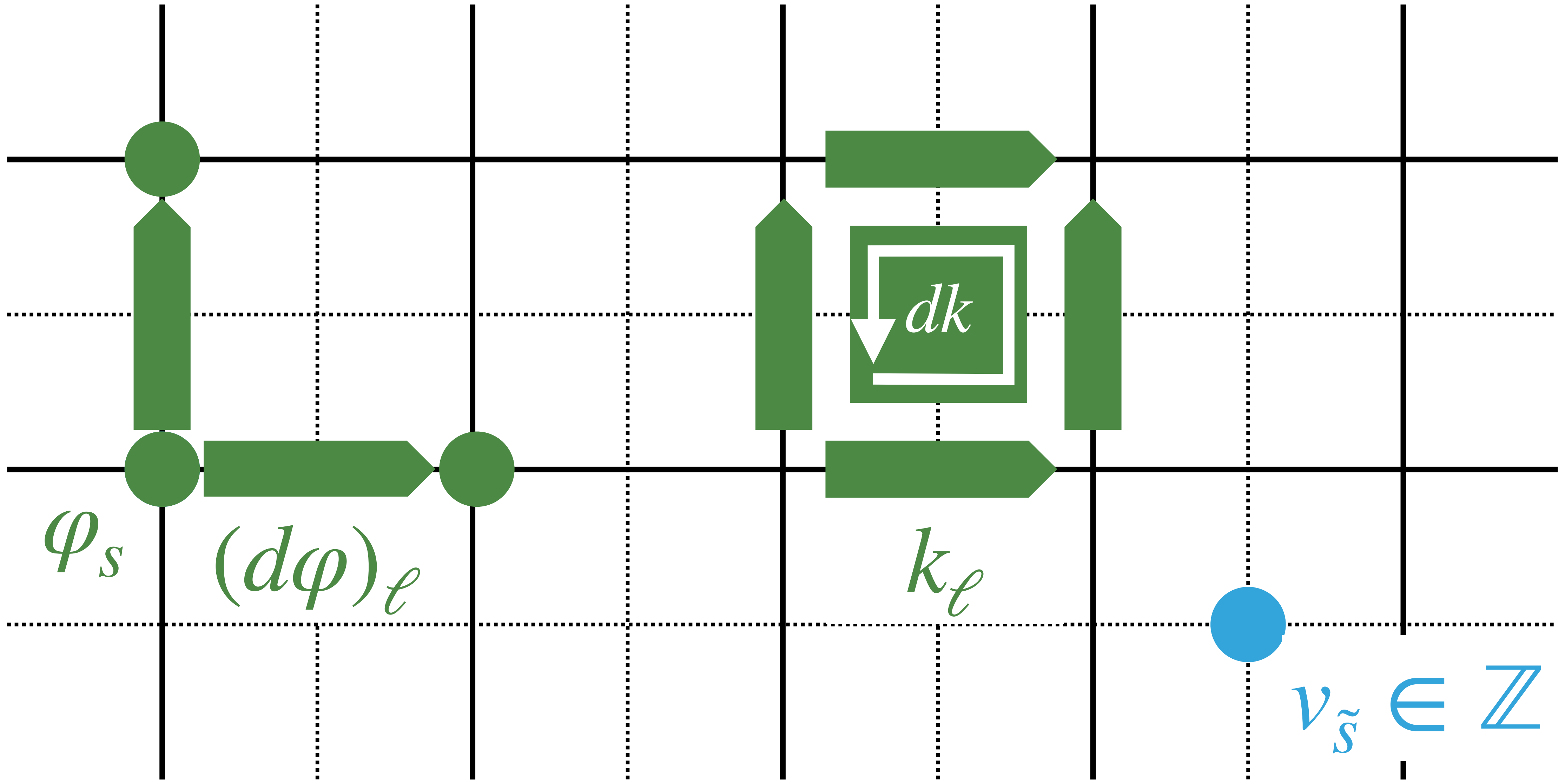
$$C_V = \frac{1}{L^2}\int d^2\tilde{x} \; \tilde{r}^{2\Delta_V(\kappa_c)} \left\langle e^{2\pi i(v_{\tilde{x}} - v_{\tilde{0}})/W} \right\rangle$$

→ constant # with L at critical $\kappa$

# SIGN PROBLEM?



$$S = \frac{\kappa}{2} \sum_{\ell} \left[ (d\varphi)_\ell - 2\pi n_\ell \right]^2 + i \sum_{p} \frac{2\pi v_p}{W} (dk)_p$$



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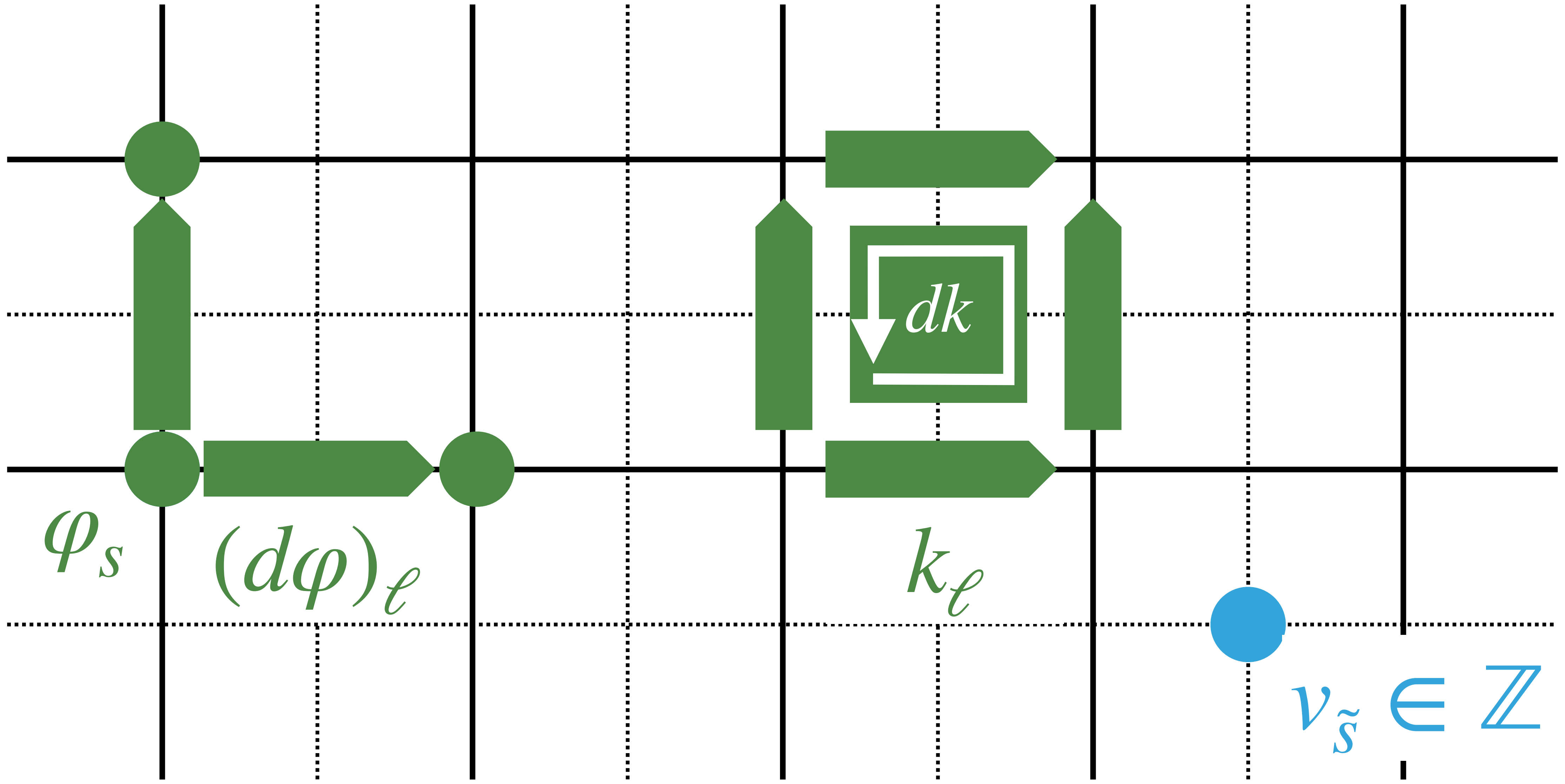
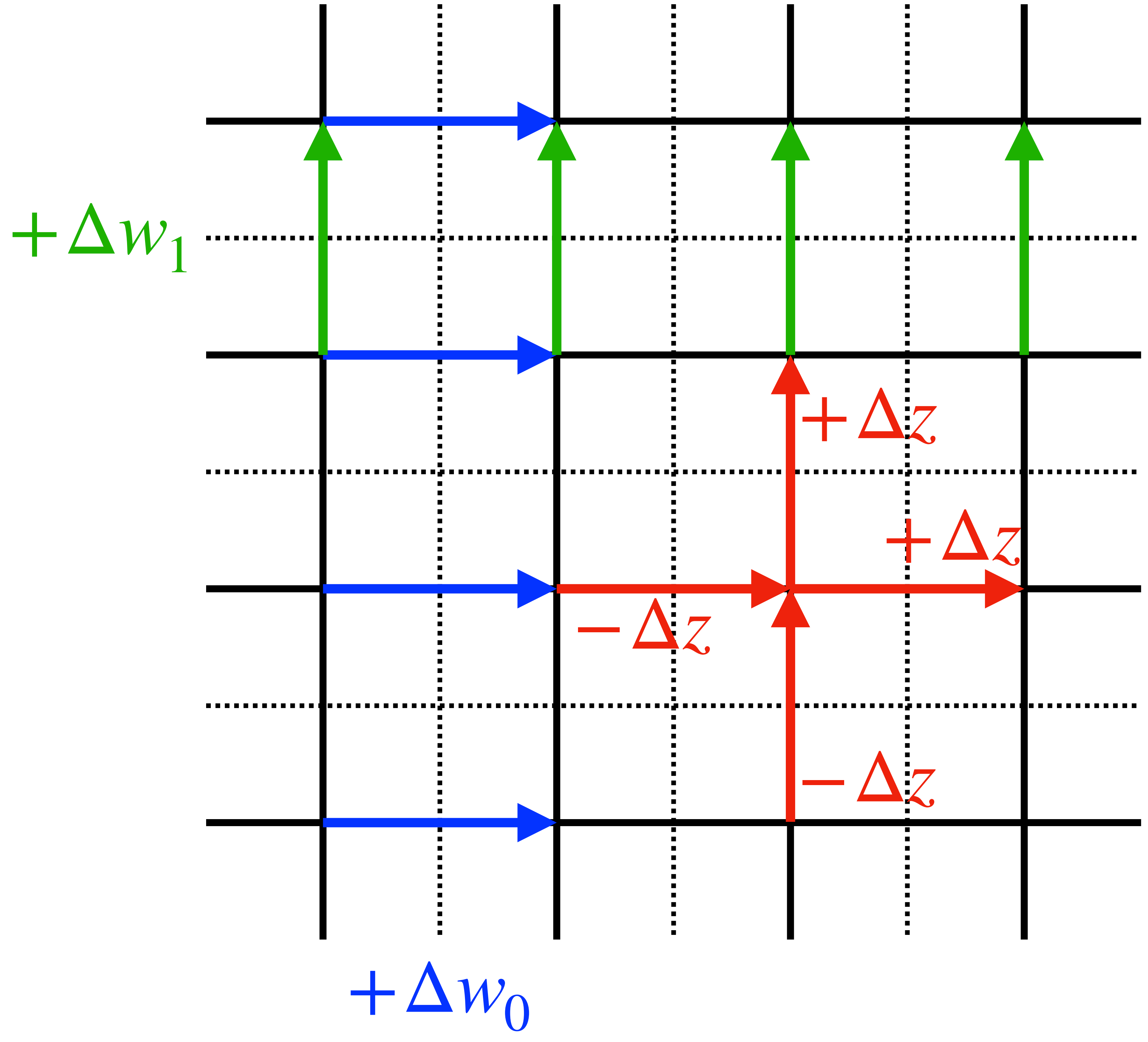



$$(dn)_p = 0$$




$$dn \equiv 0 \pmod{W}$$

Worms also work!

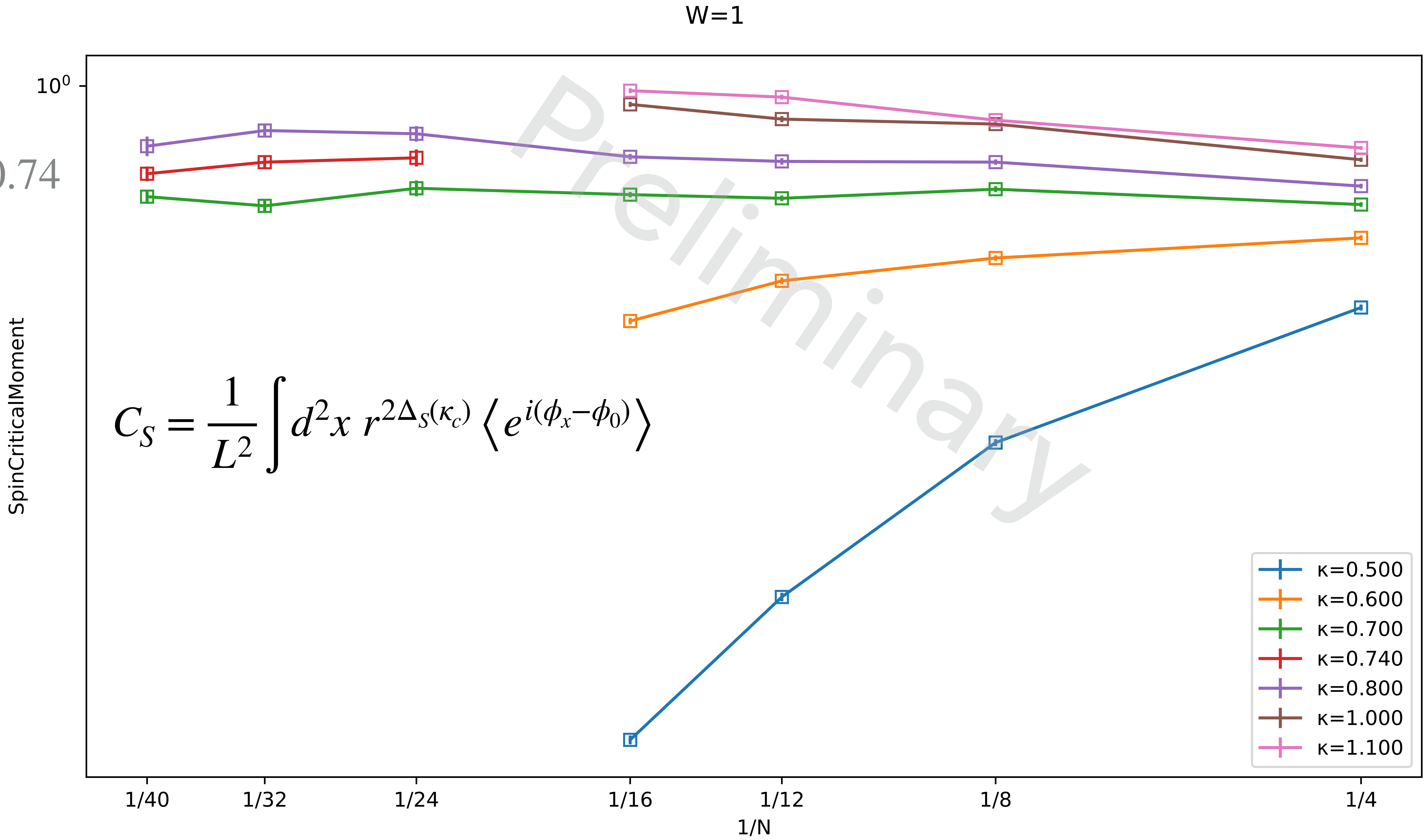
Can solve the constraint with smart update algorithms!

# RESULTS

# XY / BKT / W=1

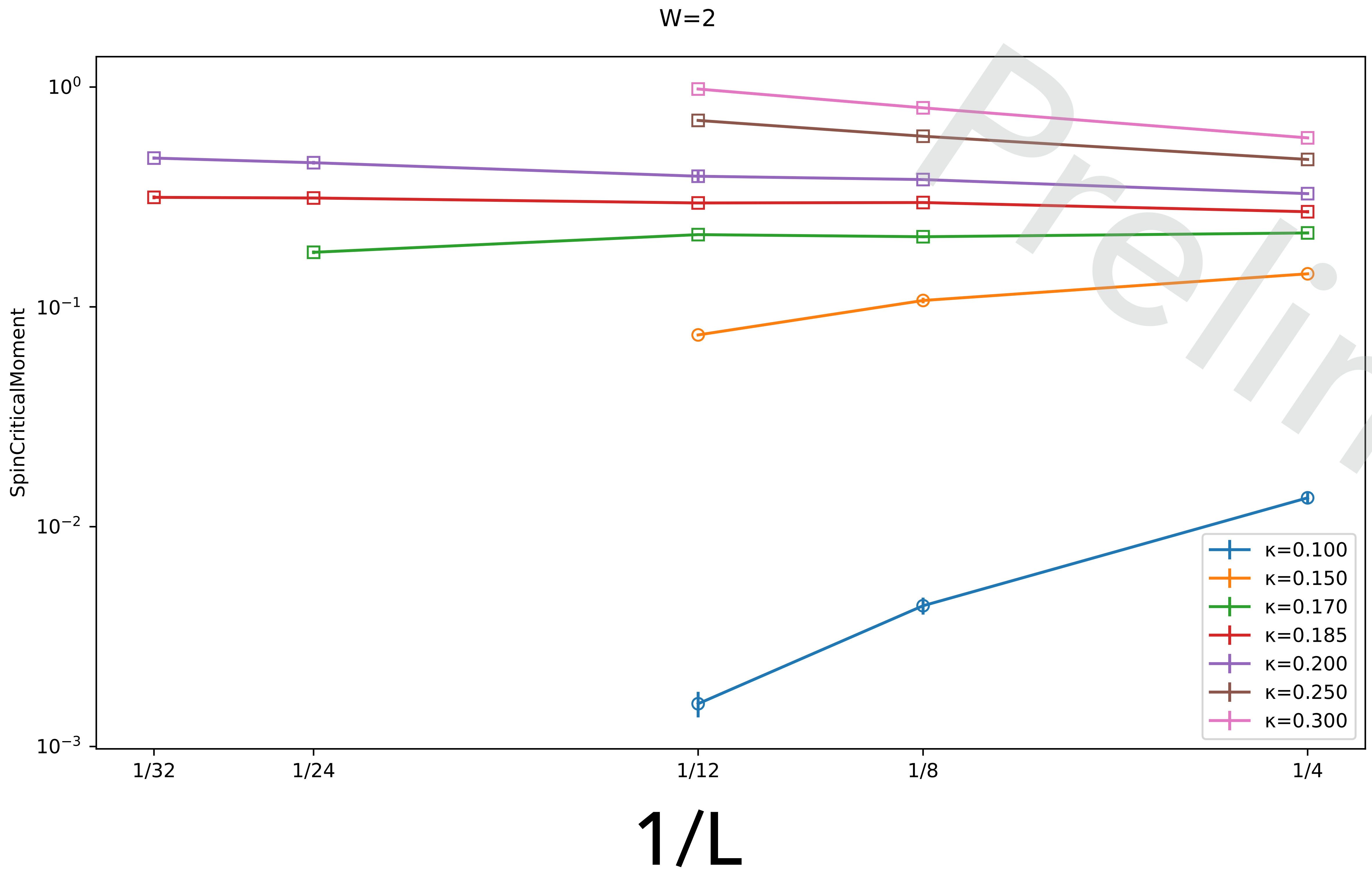
Kosterlitz, J. Phys. C 7, 1046 (1974), Villain, J. Phys. (Paris) 36, 581 (1975)
Janke and Nather, PRB 48, 7419 (1993)

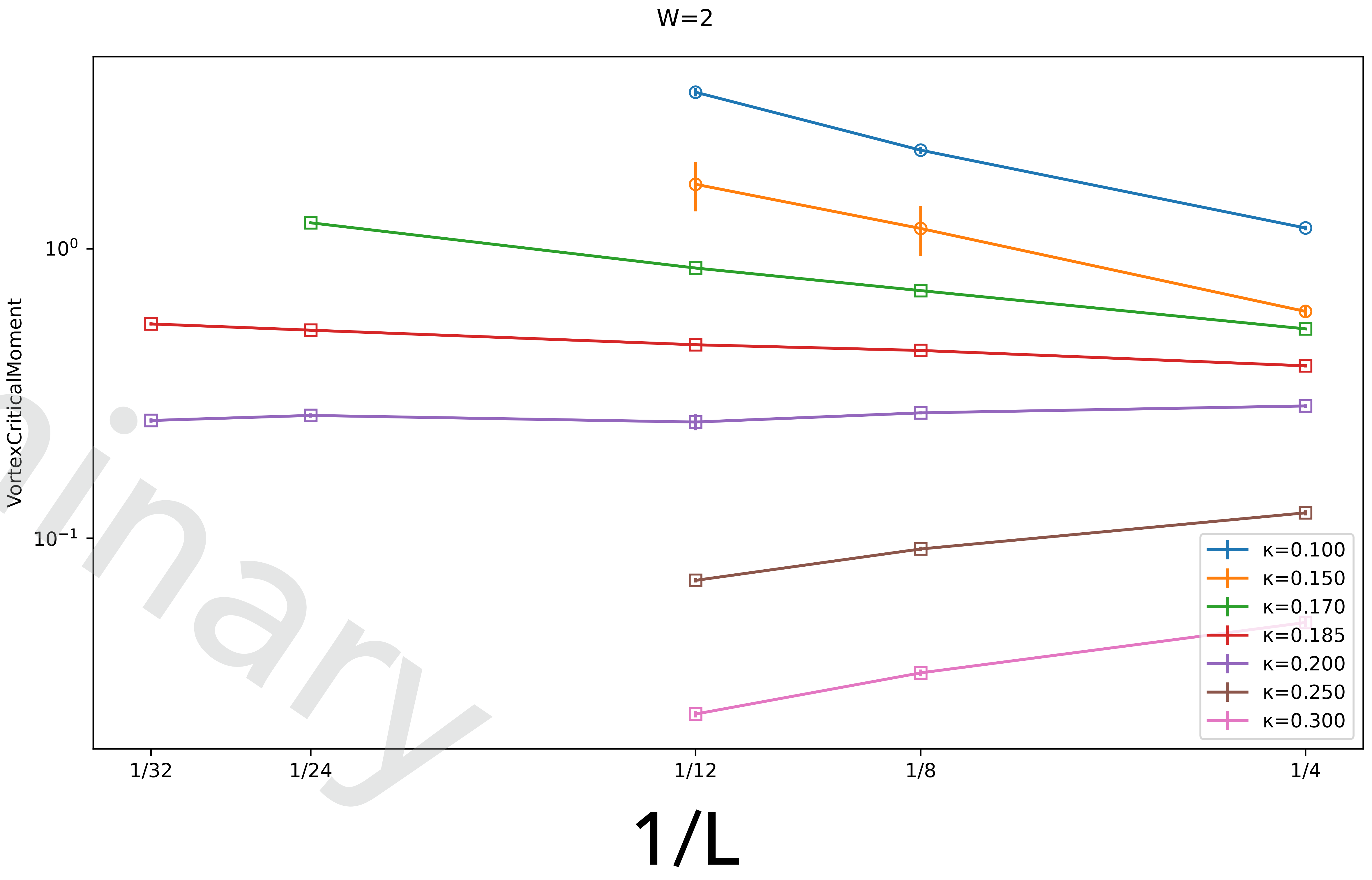
$\kappa_c \approx 0.74$




$$C_S = \frac{1}{L^2} \int d^2x \, r^{2\Delta_S(\kappa_c)} \left\langle e^{i(\phi_x - \phi_0)} \right\rangle$$

# W=2



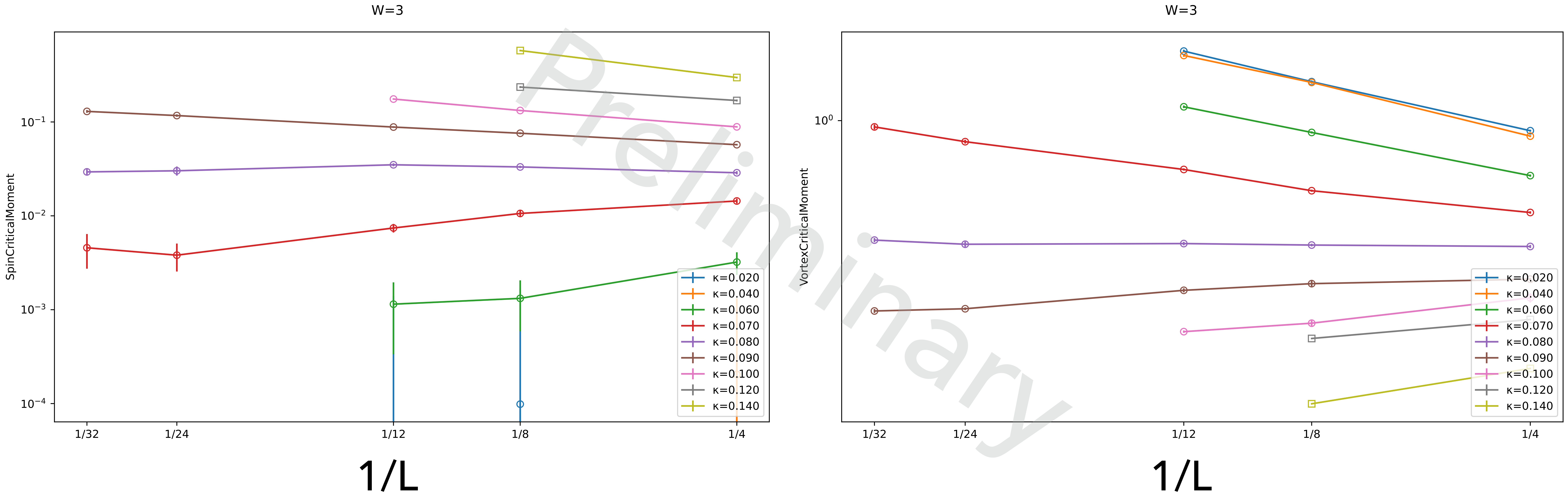






$$C_S = \frac{1}{L^2} \int d^2x \; r^{2\Delta_S(\kappa_c)} \left\langle e^{i(\phi_x - \phi_0)} \right\rangle \qquad C_V = \frac{1}{L^2} \int d^2\tilde{x} \; \tilde{r}^{2\Delta_V(\kappa_c)} \left\langle e^{2\pi i (v_{\tilde{x}} - v_{\tilde{0}})/W} \right\rangle$$

Reminder: $C_S$, $C_V$ go to a constant with large L at criticality

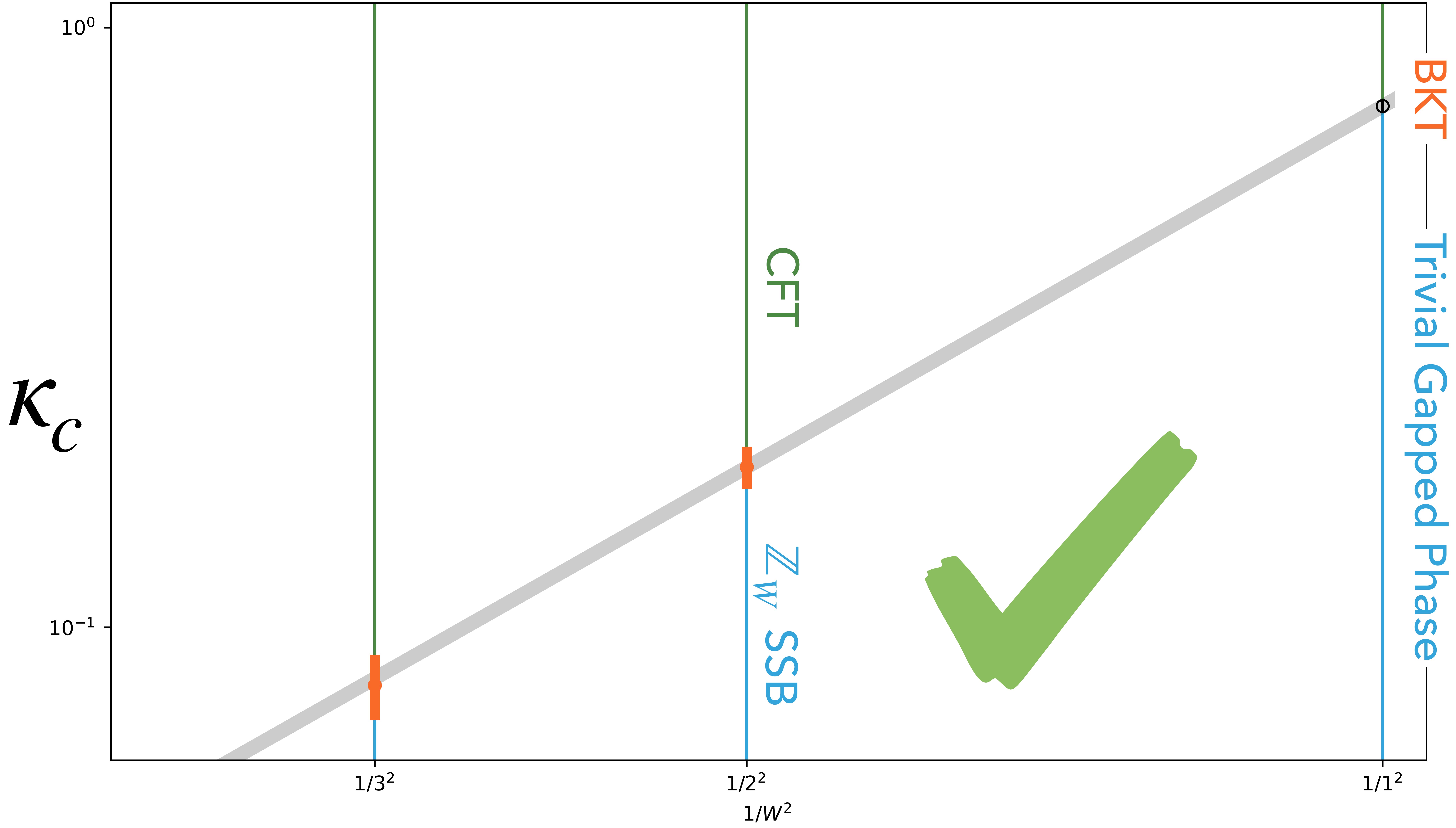
# W=3




$$C_S = \frac{1}{L^2}\int d^2x\, r^{2\Delta_S(\kappa_c)} \left\langle e^{i(\phi_x-\phi_0)} \right\rangle \qquad C_V = \frac{1}{L^2}\int d^2\tilde{x}\, \tilde{r}^{2\Delta_V(\kappa_c)} \left\langle e^{2\pi i(v_{\tilde{x}}-v_{\tilde{0}})/W} \right\rangle$$

Reminder: $C_S$, $C_V$ go to a constant with large L at criticality

# CHECK EXPECTATIONS



# VILLAINY CAN PAY

- 't Hooft anomaly on the lattice guarantees order
- one side has SSB of $\mathbb{Z}_W$ (winding)
- other side is a compact-boson CFT
- transition is ∞-order inherited from BKT
- W=∞ is easy to do and lands on a CFT with no tuning!

Code

Docs

YOU WILL NEVER FIND A MORE
WRETCHED HIVE OF SCUM AND VILLAINY.
WE MUST BE CAUTIOUS.

WITH OUR COMBINED STRENGTH,
WE CAN END THIS DESTRUCTIVE
CONFLICT AND BRING
ORDER
TO THE GALAXY.

Star Wars IV + V, Lucasfilm

# BACKUP SLIDES

# ACTION COMPARISON

| Continuum | | $N_f$=1 QED | | $N_f$=2 QED | | 3450 Chiral |
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| $+\frac{1}{8\pi}\partial_\mu\varphi\partial^\mu\varphi$ | $\rightarrow$ | $+\frac{\kappa}{2}(d\varphi - 2\pi n)_\ell^2$ | $\rightarrow$ | $+\frac{\kappa}{2}(d\varphi^j - 2\pi n^j)_{\tilde{\ell}}^2$ | $\rightarrow$ | $+\frac{\kappa}{2}(d\varphi^j - 2\pi n^j - Q_A^j a_f)_{\tilde{\ell}}^2$ |
| $+\frac{iQ}{2\pi}\varphi\epsilon^{\mu\nu}\partial_\mu a_\nu$ | $\rightarrow$ | $+\frac{iQ}{2\pi}\varphi_{\star p}(da - 2\pi r)_p$ | $\rightarrow$ | $+\frac{i}{2\pi}Q^j\varphi^j_{\star p}(da - 2\pi r)_p$ | $\rightarrow$ | $+\frac{i}{2\pi}Q_V^j\varphi^j_{\star p}(da - 2\pi r)_p$ |
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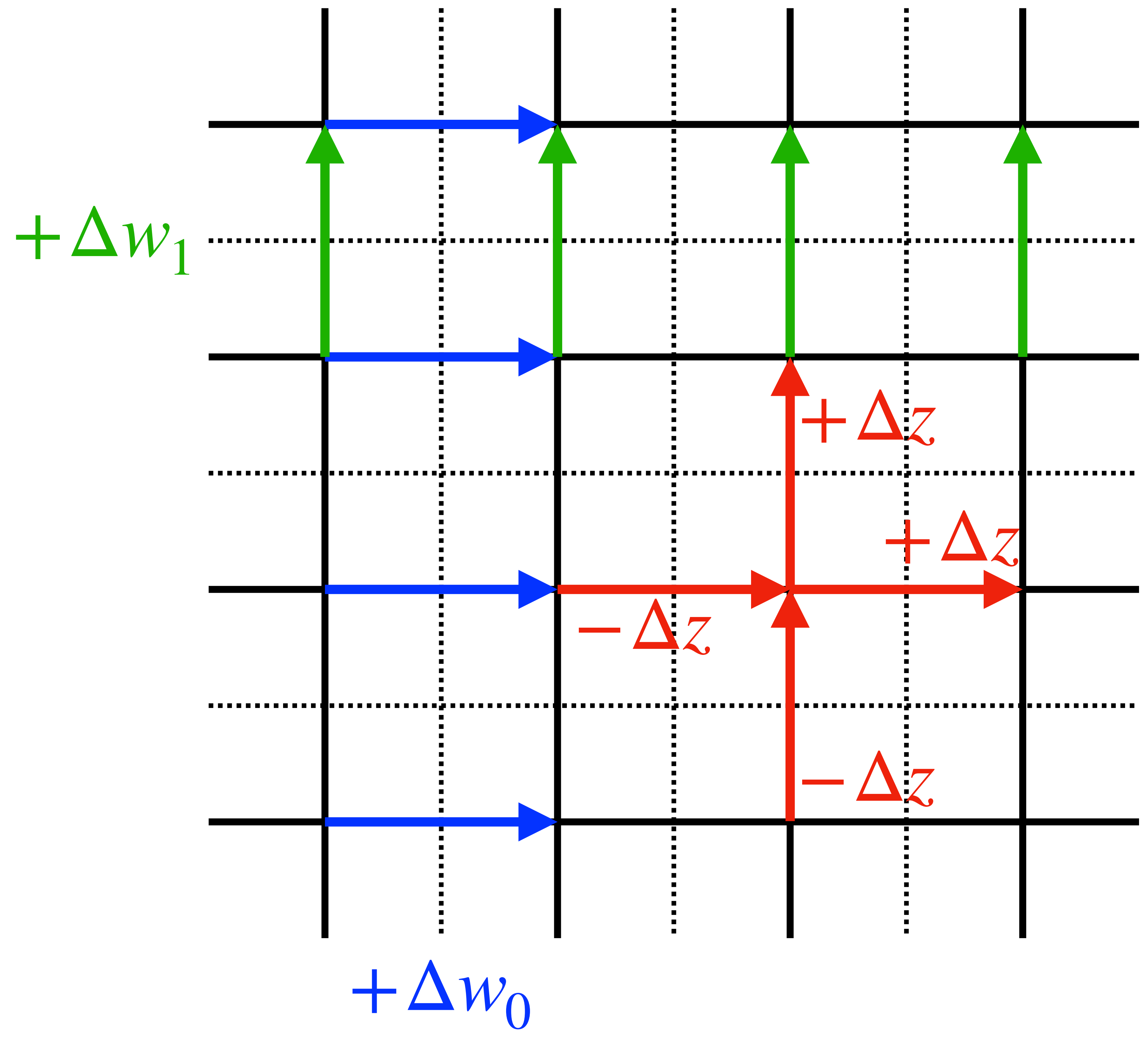
| | | | | | | $-ir_{f(\star s)}Q_A^j\chi_s^j$ |

# ALGORITHMS

Dual 3450 has terms like this

$$(dn)_p = 0$$




Worms also work!

All the dual theories have terms like this

$$dt \equiv 0 \pmod{Q}$$

$$t = x + Qy$$

global constant

local scalar

Can do standard Metropolis for both x, y

# COMPACT BOSON FROM 2D BOSONIZATION

# CHIRAL FERMIONS

QCD has nearly-massless quarks.

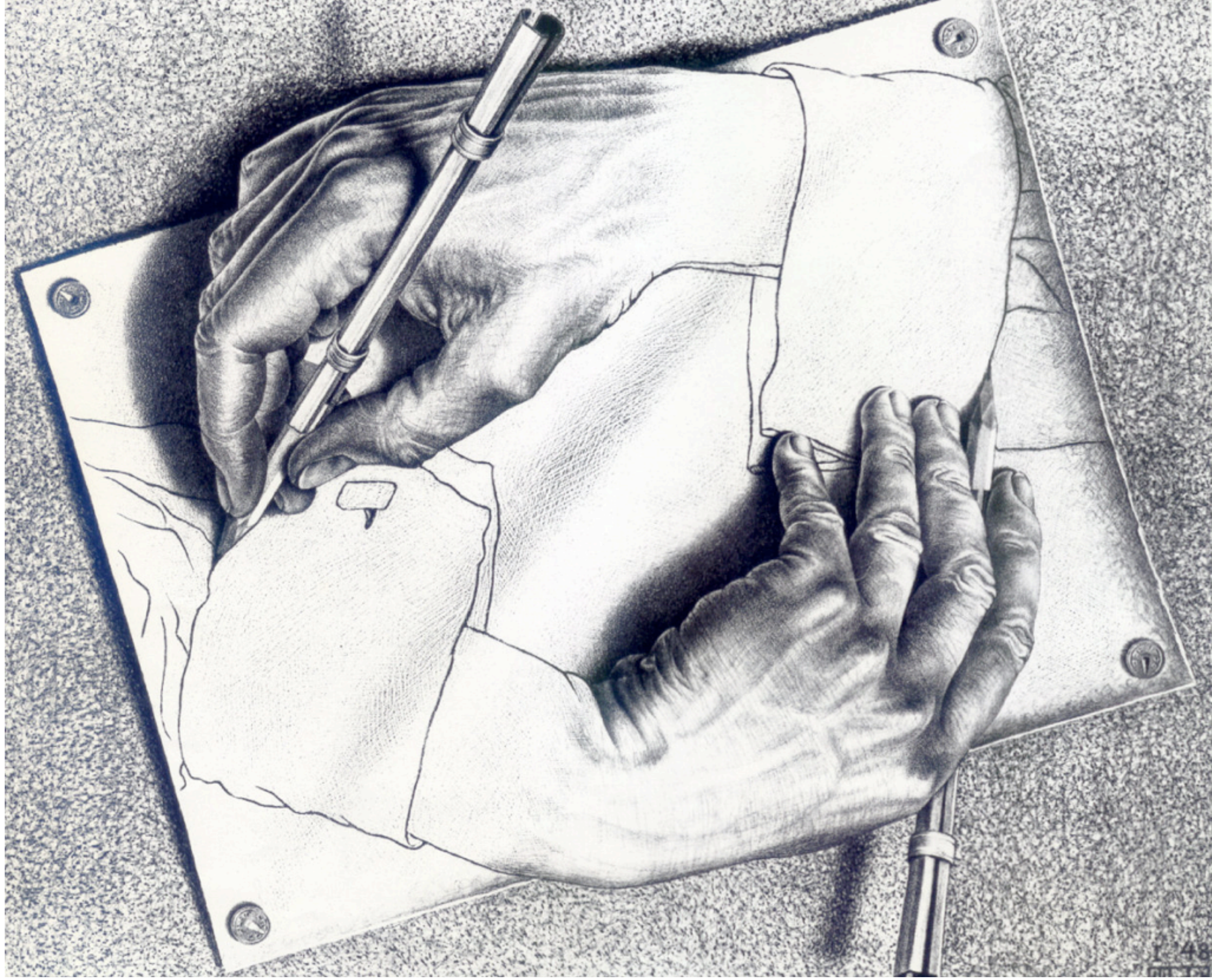
The Standard Model is a chiral gauge theory!

BSM applications…

Dream: lattice discretization which preserves as much symmetry of the continuum symmetries as possible.

Nightmare: wrong results or a lot of fine-tuning.

Reality: hard to preserve chiral symmetry.



MC Escher

# CONTINUUM WISDOM

## Symmetries

Lorentz
Chiral
Topological
Flavor …

Better preservation of symmetries help (continuum limit, renormalization …)

## ('t Hooft) Anomalies

Obstructions to gauging global symmetries

Lore:
(chiral) anomalies are absent on the lattice

## Dualities

T-duality
S-duality
particle/vortex
…

Exact maps between (lattice) theories

# OVERVIEW

Modifications of the **Villain formulation** allows you to keep

symmetries, anomalies and dualities

of continuum theories even at finite lattice spacing!

In 2D we can do a lot!

Compact boson

Schwinger model / 2D QED

3450 Chiral Gauge Theory (with exact chiral symmetry at finite spacing!)

# THE NIELSEN–NINOMIYA THEOREM

# THE NIELSEN-NINOMIYA THEOREM

$\nexists$ discretization of $D = \gamma^\mu \partial_\mu$ that has all of these desirable properties:

- **Locality** (Analyticity in $p_\mu$)
- **Correct continuum limit** ($D = \gamma^\mu p_\mu$ for $a|p| \ll 1$)
- **No doublers** (D invertible except at $|p| \longrightarrow 0$)
- **Chiral Symmetry** ($\{\Gamma, D\} = 0$)

Lore:
chiral anomalies come from UV divergences and are therefore absent on the lattice

"A lattice theory will not correctly reproduce anomalous symmetry currents in the continuum limit, unless that symmetry is broken explicitly by the lattice regulator. **This means we would be foolish to expect to construct a lattice theory with exact chiral symmetry.**"

D. B. Kaplan, "Chiral symmetry and lattice fermions", arXiv:0912.2560

# WHAT TO CONCEDE?

| | Locality | No Doublers | Chiral Symmetry |
|---|---|---|---|
| Wilson | ✓ | ✓ | ✗ |
| Staggered | ✓ | ✗ | ✓ |
| Domain Wall and Overlap | ✗ | ✓ | Ginsparg-Wilson<br>$\{\Gamma, D\} = aD\Gamma D$<br>exact in the continuum limit |

# RECENT DEVELOPMENTS

Pessimistic conclusion is based on the textbook view that

- Only fermions have anomalies.
- Anomalies have to do with regulator dependence.

But, recent developments (since 2019) show

- Purely bosonic systems can have anomalies!
- Anomalies can occur in systems with a finite number of DOFs.

hep-th:

Sulejmanpasic, Shao, Seiberg, Lam, Fazzi, Gorantla, Gattringer, Cheng…

cond-mat:

Lieb+Shutz+Mattis, Kitaev, Kapustin+Thorngren, …

# BOSONIZATION

Coleman, Jackiw, Suskind (1975) Annals Phys. 93 267
Coleman (1975) PRD11 2088; Annals Phys. 101 239

It's been known since the 1970s that in 2D

$$\det D(a) = \int_{\varphi \sim \varphi + 2\pi} \mathscr{D}\varphi \exp\left[-\int d^2x \left(\frac{1}{8\pi}\partial_\mu\varphi\partial^\mu\varphi + \frac{iQ}{2\pi}\epsilon^{\mu\nu}a_\mu\partial_\nu\varphi\right)\right]$$

"compact boson"

$$S_f = \int d^2x\, \bar{\psi}\gamma^\mu(\partial + iQa)_\mu\psi \qquad S_b = \int d^2x\, \frac{1}{8\pi}(d\varphi)^2 + iQa_\mu\frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\varphi$$

Discretize D
(Wilsonian)
Nielsen-Ninomiya
Sadness + pain

$$J^V_\mu = \bar{\psi}\gamma_\mu\psi$$

$$J^A_\mu = \bar{\psi}\gamma_5\gamma_\mu\psi$$

$$J^w_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu}\partial^\nu\varphi$$

$$J^s_\mu = \frac{i}{4\pi}\partial_\mu\varphi$$

Discretize det D
(modified Villain)
Symmetries, Anomalies, Dualities
A garden of delights

## [Discretizing spacetime, bosonizing] ≠0

Bosonize first for fun and profit!

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$$J^V_\mu = \bar\psi\gamma_\mu\psi$$

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$$J^w_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu}\partial^\nu\varphi = \frac{1}{2\pi}\star d\varphi$$

Not conserved if there are vortices

$$J^s_\mu = \frac{i}{4\pi}\partial_\mu\varphi = \frac{i}{4\pi}d\varphi$$

Conserved by EOMs

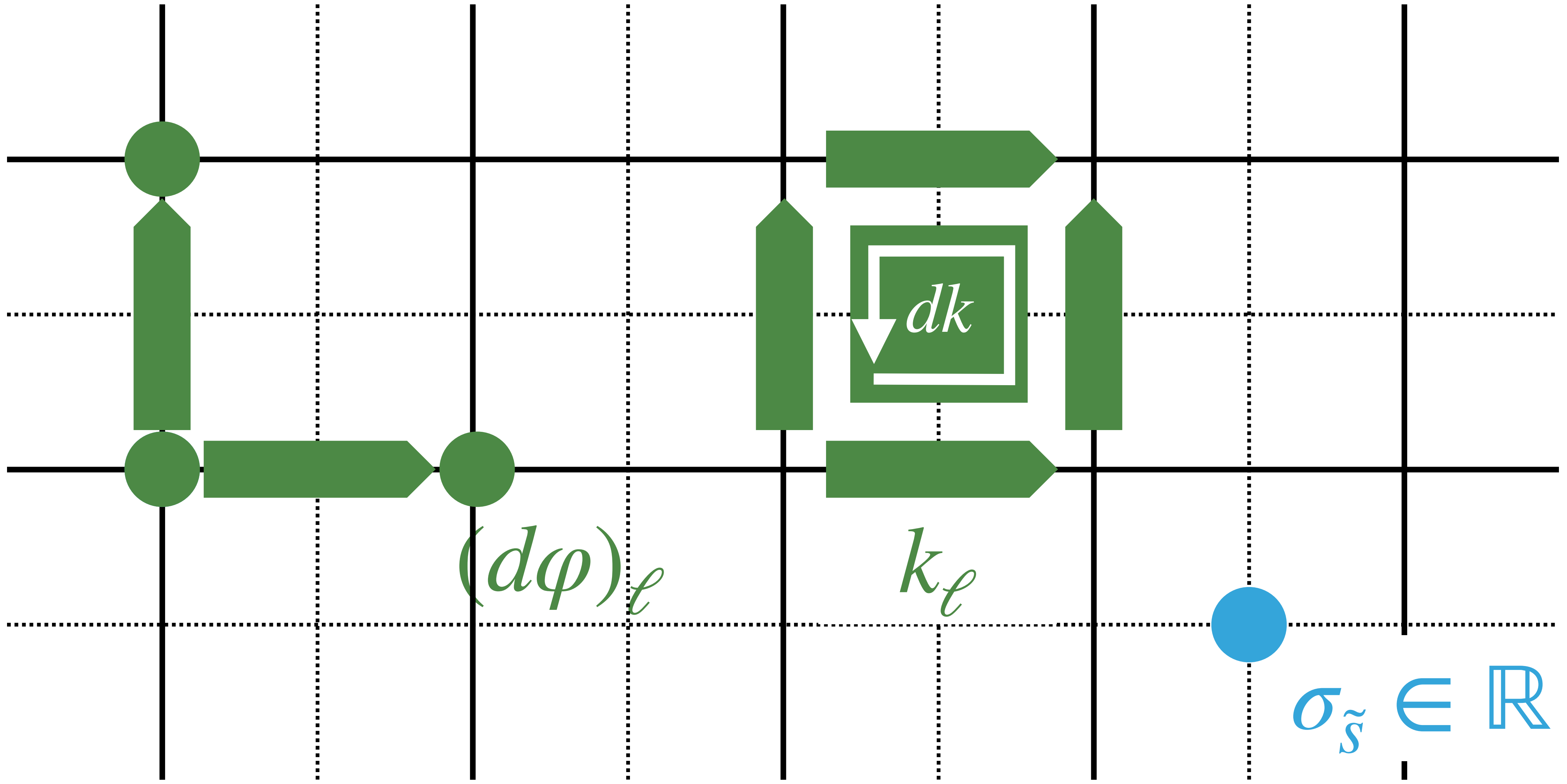
[Discretizing spacetime, bosonizing] ≠0

Bosonize first for fun and profit!

# MIXED 't HOOFT ANOMALY + VACUUM STRUCTURE

Gorantla, Lam, Seiberg, Shao 2021
PRB 103 205116, J Math Phys 62 102301

$$S = \frac{\kappa}{2}\sum_{\ell}\left[(d\varphi)_\ell - 2\pi k_\ell\right]^2 + i\sum_{p}\sigma_{\star p}\,(dk)_p$$




$$j_s = \kappa(d\varphi - 2\pi k)$$

$$j_w = \star\frac{1}{2\pi}(d\varphi - 2\pi k)$$

$$d\star j_s = \kappa\, d\star(d\varphi - 2\pi n) = 0 \quad (\varphi \text{ EOM})$$

$$d\star j_w = \frac{1}{2\pi}(d^2\varphi - 2\pi dk) = -dk = 0 \quad (\sigma \text{ EOM})$$

$$S = \frac{\kappa}{2}\sum_{\ell}\left[(d\varphi)_\ell - 2\pi k_\ell - a_\ell\right]^2 + i\sum_{p}\sigma_{\star p}(dk + r)_p$$

$$a \to a + d\alpha + 2\pi m$$
$$\varphi \to \varphi + \alpha + 2\pi n$$
$$k \to k + dn - m$$
$$r \to r + dm$$

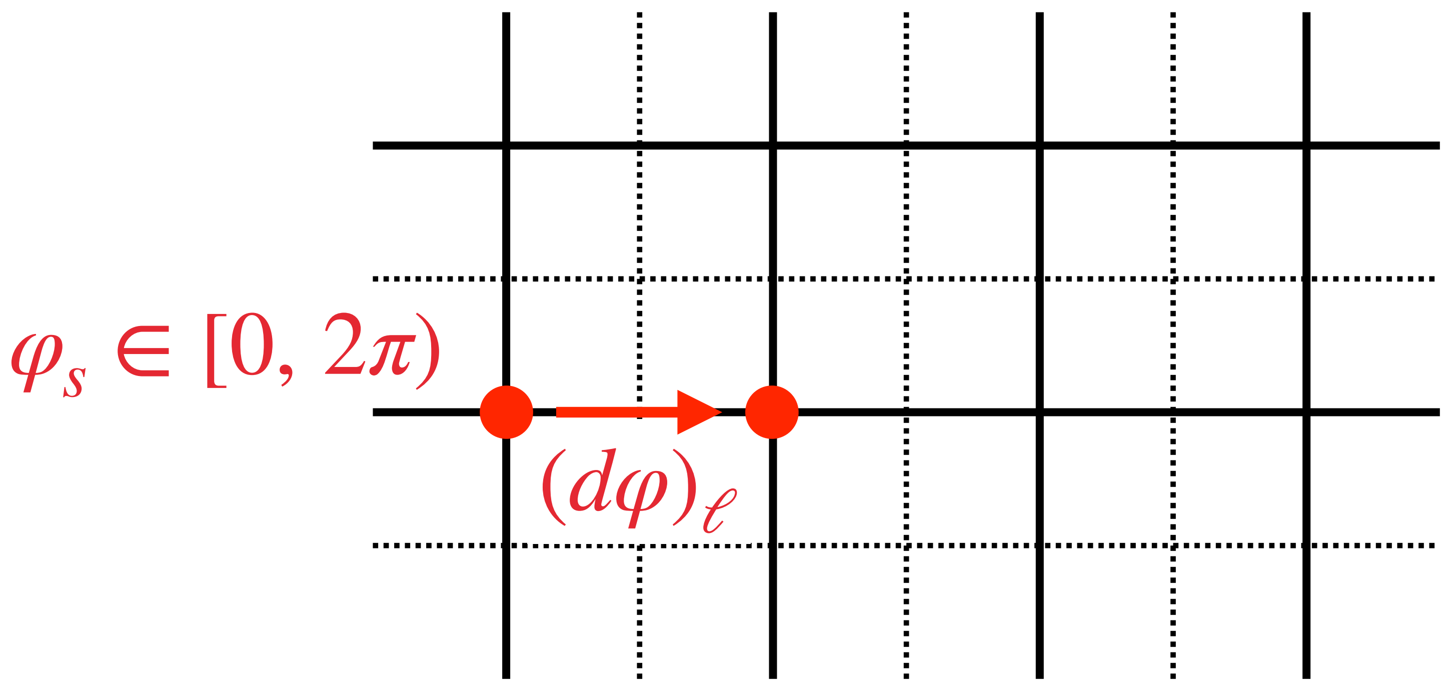
$$j_w = \star\frac{1}{2\pi}(d\varphi - 2\pi k - a)$$

(NOT 0 by σ EOM)

# XY A LA VILLAIN

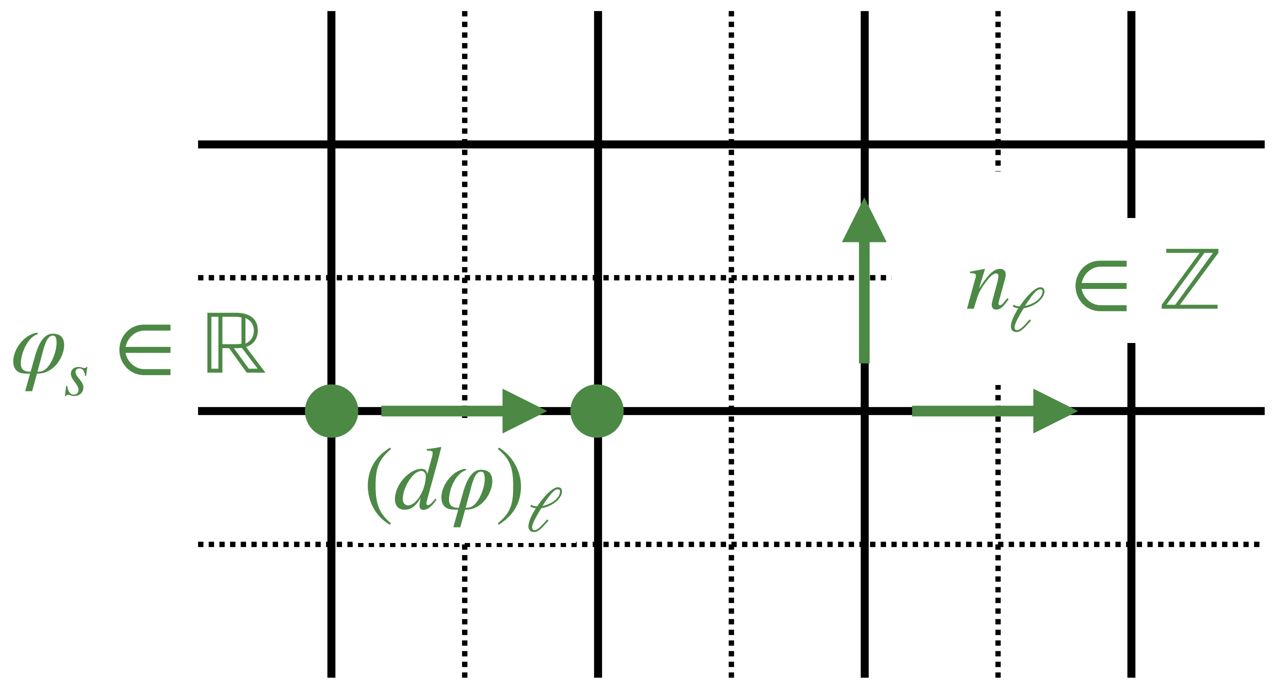
J. Villain (1975) J. Phys France 36 6 581-590

$$S = \frac{\kappa}{2} \sum_{\ell} \left[1 - \cos(d\varphi)_\ell\right]$$




$$S = \frac{\kappa}{2} \sum_{\ell} \left[(d\varphi)_\ell - 2\pi n_\ell\right]^2$$

(quadratic!)




plaquette vorticity $\in \{-1, 0, +1\}$

plaquette vorticity $\in \mathbb{Z}$

Very easy to keep only vorticity $0$ (mod $W$)

# DUALITY

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$$S = \frac{\kappa}{2}\sum_\ell \left[(d\varphi)_\ell - 2\pi n_\ell\right]^2 + i\sum_p \frac{2\pi v_{\star p}}{W}(dn)_p$$

$$\sum_{c^r}(dA)_{c^r}B_{\star c^r} = \sum_{c^{r-1}}(-1)^r A_{c^{r-1}}(dB)_{\star c^{r-1}}$$

Lattice summation by parts

$$S = \frac{\kappa}{2}\sum_\ell \left[(d\varphi)_\ell - 2\pi n_\ell\right]^2 + i\sum_\ell 2\pi(\delta v)_\ell\, n_\ell / W$$

Poisson resum link-by-link

$$\sum_n \exp\left\{-\frac{\kappa}{2}(\theta - 2\pi n)^2 + in\tilde\theta\right\} = \frac{1}{\sqrt{2\pi\kappa}}\sum_m \exp\left\{-\frac{1}{2\kappa}\left(m - \frac{\tilde\theta}{2\pi}\right)^2 - i\left(m - \frac{\tilde\theta}{2\pi}\right)\theta\right\}$$

$$S = \frac{\kappa}{2}\sum_\ell \left[(d\varphi)_\ell - 2\pi k_\ell\right]^2 + i\sum_p \frac{2\pi v_{\star p}}{W}(dk)_p$$

trade path integration variables (and strong/weak)

$$\theta \to d\varphi, \tilde\theta \to d\sigma$$

$$\tilde S = \frac{\tilde\kappa}{2}\sum_{\tilde\ell}\left[(d\sigma)_{\tilde\ell} - 2\pi\tilde k_{\tilde\ell}\right]^2 + i\sum_{\tilde p}\varphi_{\star\tilde p}(d\tilde k)_{\tilde p}$$

Exact T-duality on the lattice!

$$\tilde\kappa = \frac{1}{(2\pi)^2\kappa}$$

# DUALITY: A WORLDLINE FORMULATION

Gorantla, Lam, Seiberg, Shao 2021

PRB 103 205116, J Math Phys 62 102301

$$S = \frac{\kappa}{2}\sum_{\ell}\left[(d\varphi)_\ell - 2\pi n_\ell\right]^2 + i\sum_{p}\frac{2\pi v_{\star p}}{W}(dn)_p$$

Lattice summation by parts

$$S = \frac{\kappa}{2}\sum_{\ell}\left[(d\varphi)_\ell - 2\pi n_\ell\right]^2 + i\sum_{\ell} 2\pi(\delta v)_\ell\, n_\ell/W$$

Poisson resum n link-by-link (exact lattice duality)

trade path integration variables (and strong/weak)

$$S = \frac{1}{2\kappa}\sum_{\ell}\left[m - \delta v/W\right]_\ell^2 - i\sum_{s}(\delta m)_s\,\varphi_s$$

$(m_\ell, v_p \in \mathbb{Z})$ Worldline: m counts bosons

# DUALITY: A WORLDLINE FORMULATION

Gorantla, Lam, Seiberg, Shao 2021
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$$S = \frac{\kappa}{2}\sum_{\ell}\left[(d\varphi)_\ell - 2\pi n_\ell\right]^2 + i\sum_p \frac{2\pi v_{\star p}}{W}(dn)_p$$

Path integrate v to get **constraint** $dn \equiv 0 \pmod{W}$

Lattice summation by parts

$$S = \frac{\kappa}{2}\sum_{\ell}\left[(d\varphi)_\ell - 2\pi n_\ell\right]^2 + i\sum_{\ell} 2\pi(\delta v)_\ell\, n_\ell / W$$

Poisson resum n link-by-link (exact lattice duality)

trade path integration variables (and strong/weak)

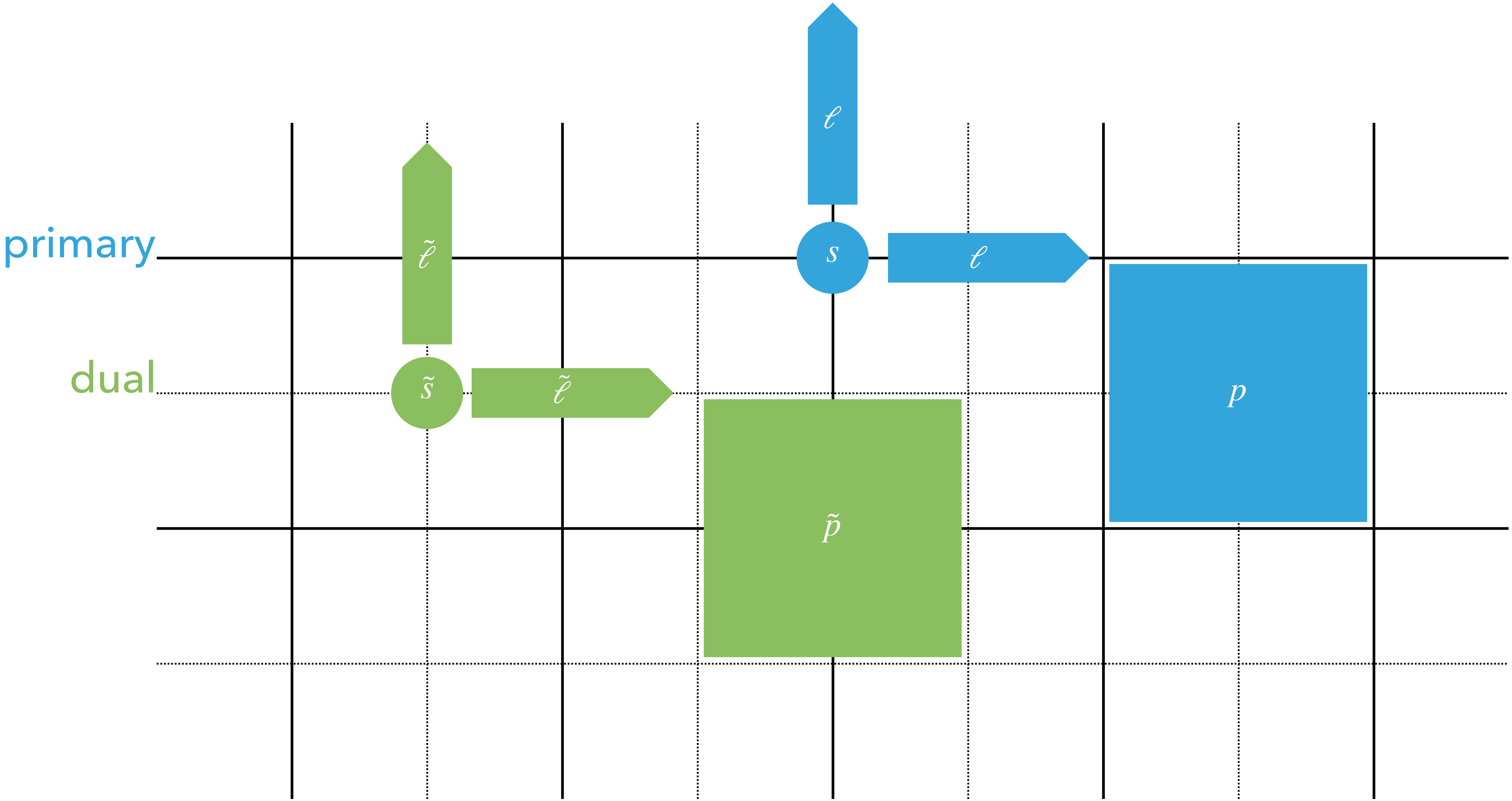
$$S = \frac{1}{2\kappa}\sum_{\ell}\left[m - \delta v/W\right]_\ell^2 - i\sum_s (\delta m)_s \varphi_s$$

$(m_\ell, v_p \in \mathbb{Z})$ Worldline: m counts bosons

Path integrate $\varphi$ to get **constraint** $\delta m=0$

# LATTICE EXTERIOR CALCULUS

# LATTICE

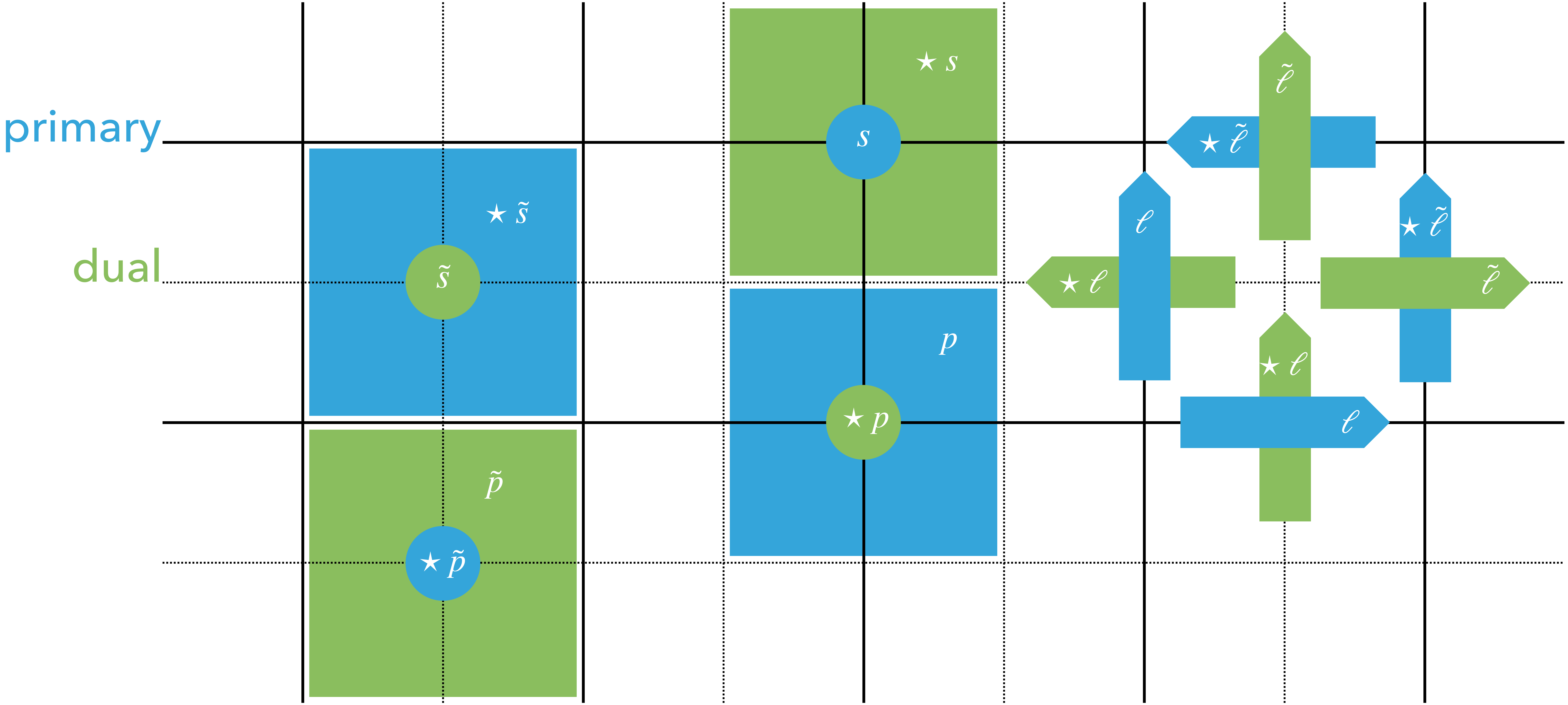


# LATTICE EXTERIOR CALCULUS: HODGE STAR

$\star^2 = (-1)^{r(d-r)}$

Review: Sulejmanpasic + Gattringer (2019)
Nucl. Phys. B 943 114616 1901.02637

★ maps between primary and dual lattices

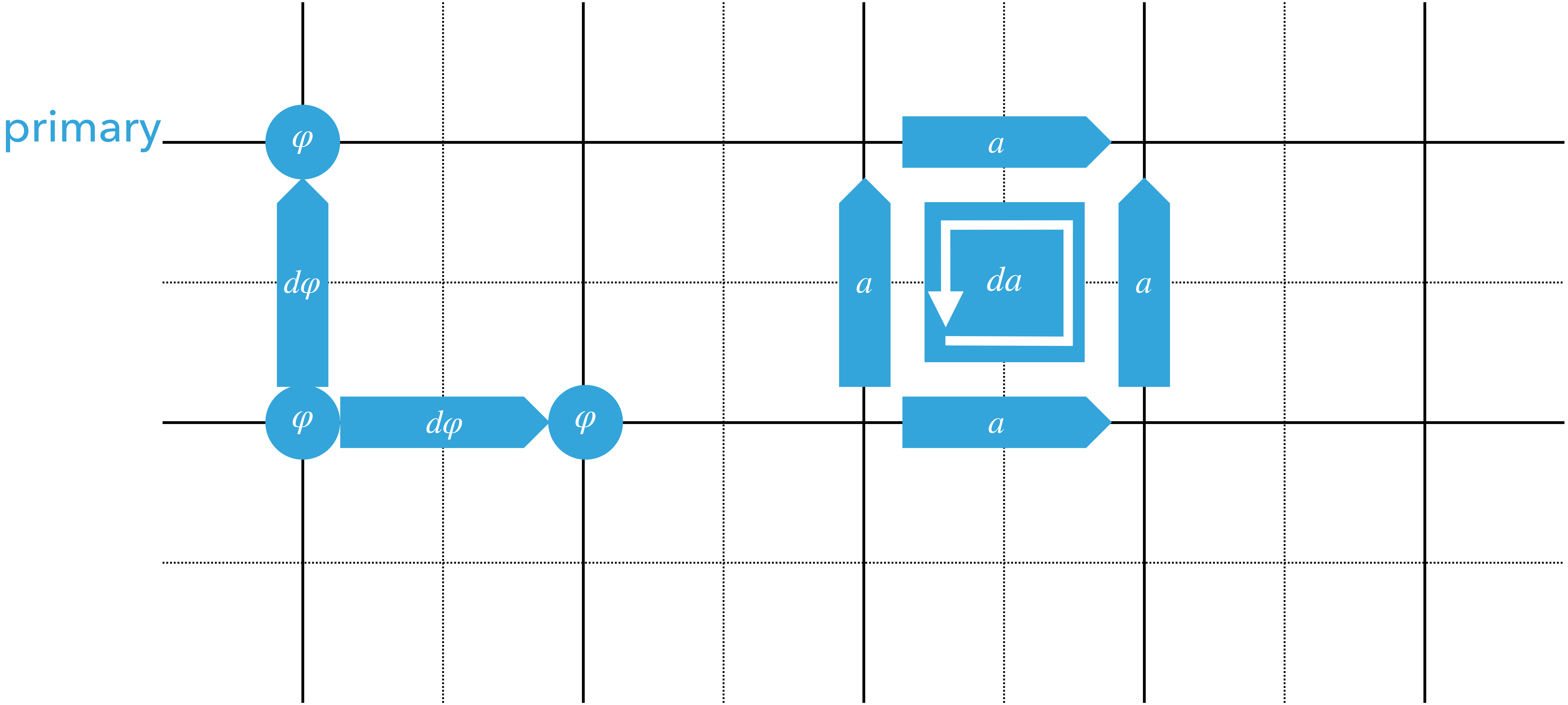


# LATTICE EXTERIOR CALCULUS: d

$$d^2 = 0$$

Review: Sulejmanpasic + Gattringer (2019)
Nucl. Phys. B 943 114616 1901.02637

d increases the degree of the form by 1



# CLASSIC LORE

# CHIRAL SYMMETRY + ANOMALIES

Dirac fermion in 2D

$$S = \int d^2x \, \bar{\psi}\gamma^\mu \partial_\mu \psi$$

Classically

$$U(1)_V: \qquad \psi \to e^{i\alpha}\psi$$

$$U(1)_A: \qquad \psi \to e^{i\Gamma\beta}\psi \qquad \{\Gamma, \gamma^\mu\} = 0$$

A **mixed 't Hooft anomaly** is revealed by turning on a background $U(1)_V$ field A

$$\Delta\mathscr{L} = QA_\mu j_V^\mu \qquad Q \in \mathbb{Z}$$

't Hooft anomaly becomes ABJ anomaly if A is made dynamical.

# CHARGE-Q 2D QED

$$\mathscr{L} = \frac{1}{4e^2} f_{\mu\nu}^2 + \bar{\psi}\gamma^\mu(\partial_\mu - iQa_\mu)\psi$$

Axial current is not conserved

$$\frac{1}{Q}\partial_\mu j_A^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu} f_{\mu\nu} \equiv q$$

Topological charge density

Topological charge is integer

$$\mathcal{Q} = \int d^2x \, q \in \mathbb{Z}$$

Axial charge transforms

$$\Delta S = 2Q\mathcal{Q} \in 2Q\mathbb{Z}$$

# CHARGE-Q 2D QED

$$\mathscr{L} = \frac{1}{4e^2} f_{\mu\nu}^2 + \bar{\psi}\gamma^\mu(\partial_\mu - iQa_\mu)\psi$$

$$\mathcal{Q} = \int d^2x \, q \in \mathbb{Z} \qquad q \equiv \frac{1}{2\pi}\epsilon^{\mu\nu} f_{\mu\nu}$$

Axial charge changes by $$2Q\mathcal{Q} \in 2Q\mathbb{Z}$$

ABJ anomaly breaks $U(1)_A$ $$U(1)_A \to \mathbb{Z}_{2Q}$$

The global chiral symmetry is $$G_A = \frac{\mathbb{Z}_{2Q}}{\mathbb{Z}_2} \simeq \mathbb{Z}_Q$$

$\psi \longrightarrow -\psi$ is a gauge transformation

# SUBTLETIES ON THE LATTICE

Typically $\mathcal{Q} = \int d^2x\, q \notin \mathbb{Z}$ so the ABJ anomaly can't correctly reduce $U(1)_A$

Everything on the lattice is already finite, naively there is no room for the subtleties allegedly required in the continuum.

The obvious approach $\bar{\psi}\gamma^\mu\partial_\mu\psi \rightarrow \bar{\psi}_x\gamma^\mu(\psi_{x+\hat{\mu}} - \psi_{x-\hat{\mu}})$ leads to fermion doubling.

**The charges of the doublers cancel the ABJ anomaly!**

# RECENT DEVELOPMENTS

see Aleksey Cherman, 14:35 Theoretical Developments

Pessimistic conclusion is based on the textbook view that

- Only fermions have anomalies.
- Anomalies have to do with regulator dependence.

hep-th:

Sulejmanpasic, Shao, Seiberg, Lam, Fazzi, Gorantla, Gattringer, Cheng…

cond-mat:

Lieb+Shutz+Mattis, Kitaev, Kapustin+Thorngren, …

But, recent developments (since 2019) show

- Purely bosonic systems can have anomalies!
- Anomalies can occur in systems with a finite number of DOFs.

# MIXED 't HOOFT ANOMALY + VACUUM STRUCTURE

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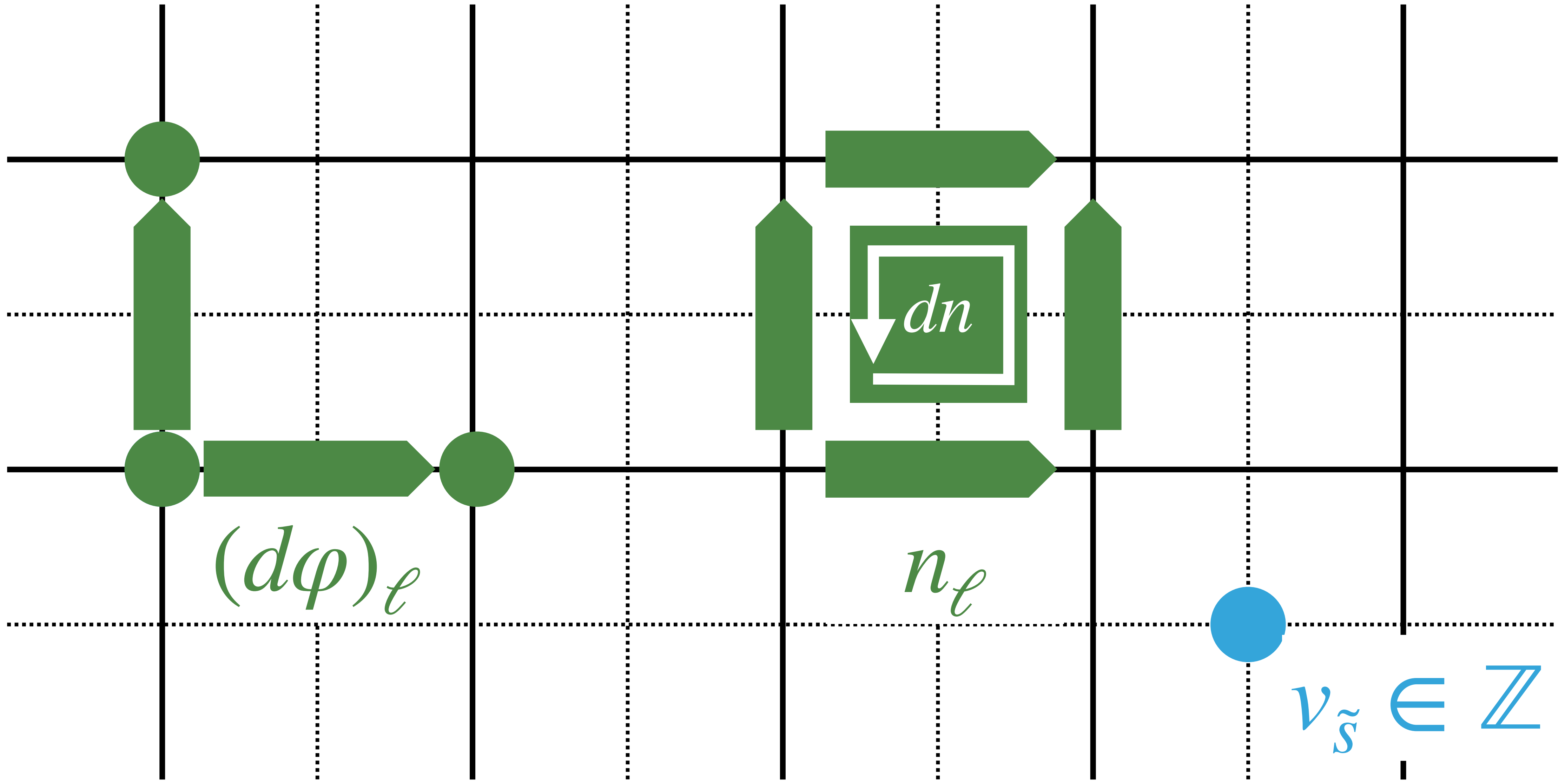



Can't simultaneously gauge both $U(1)_{\text{shift}}$ and $(\mathbb{Z}_W)_{\text{winding}}$

$\mathscr{S}_\ell$ $\quad \mathscr{W}_{\tilde{\ell}}$ Background fields

$$Z[\mathscr{S} + 2\pi\mathscr{K}, \mathscr{W}] = Z[\mathscr{S}, \mathscr{W}]\, e^{i\sum_\ell \mathscr{K}_\ell \mathscr{W}_{\star\ell}} \neq Z[\mathscr{S}, \mathscr{W}]$$

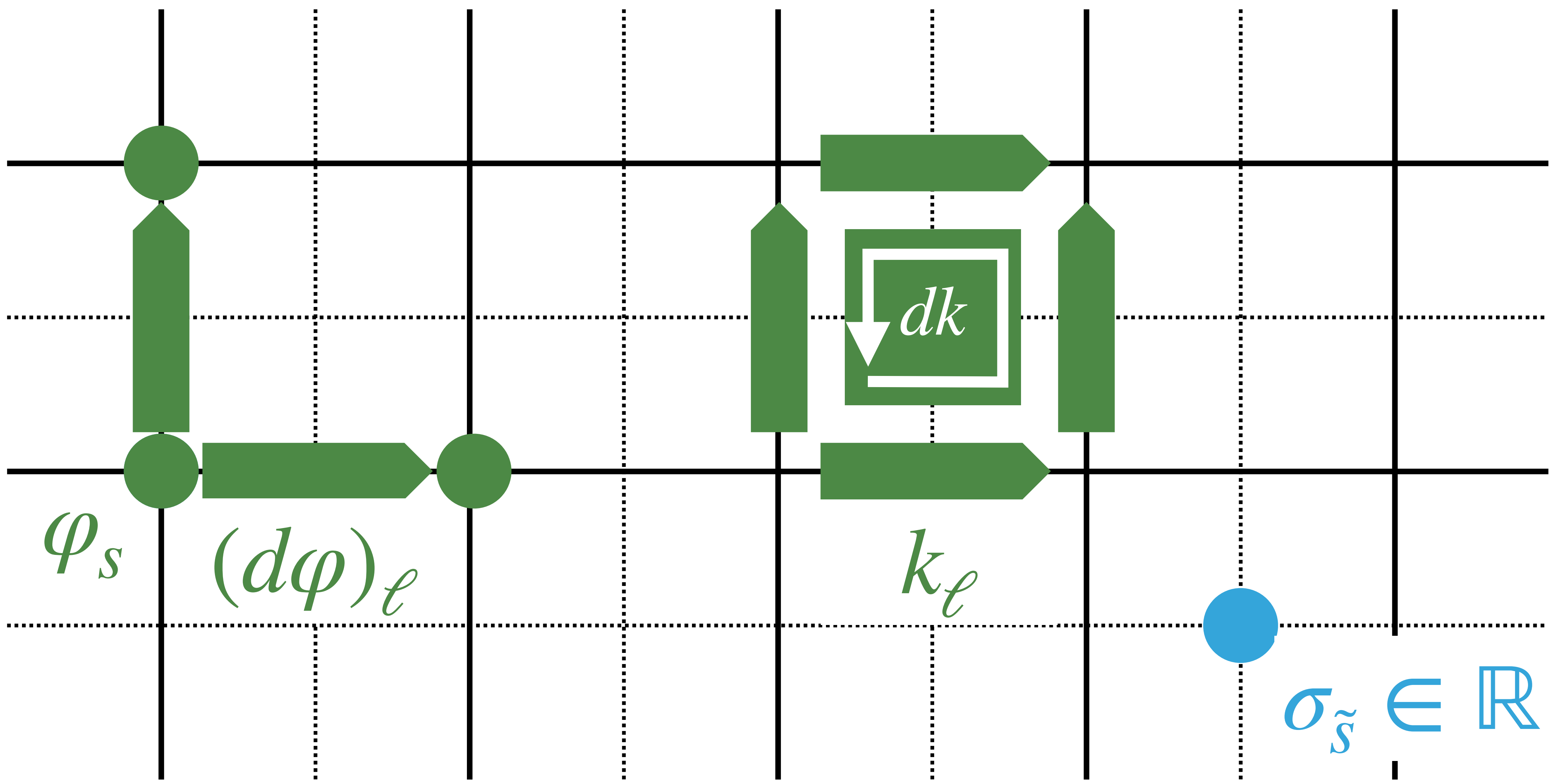
and therefore the lattice model doesn't have a gapped phase!


# SIGN PROBLEM?

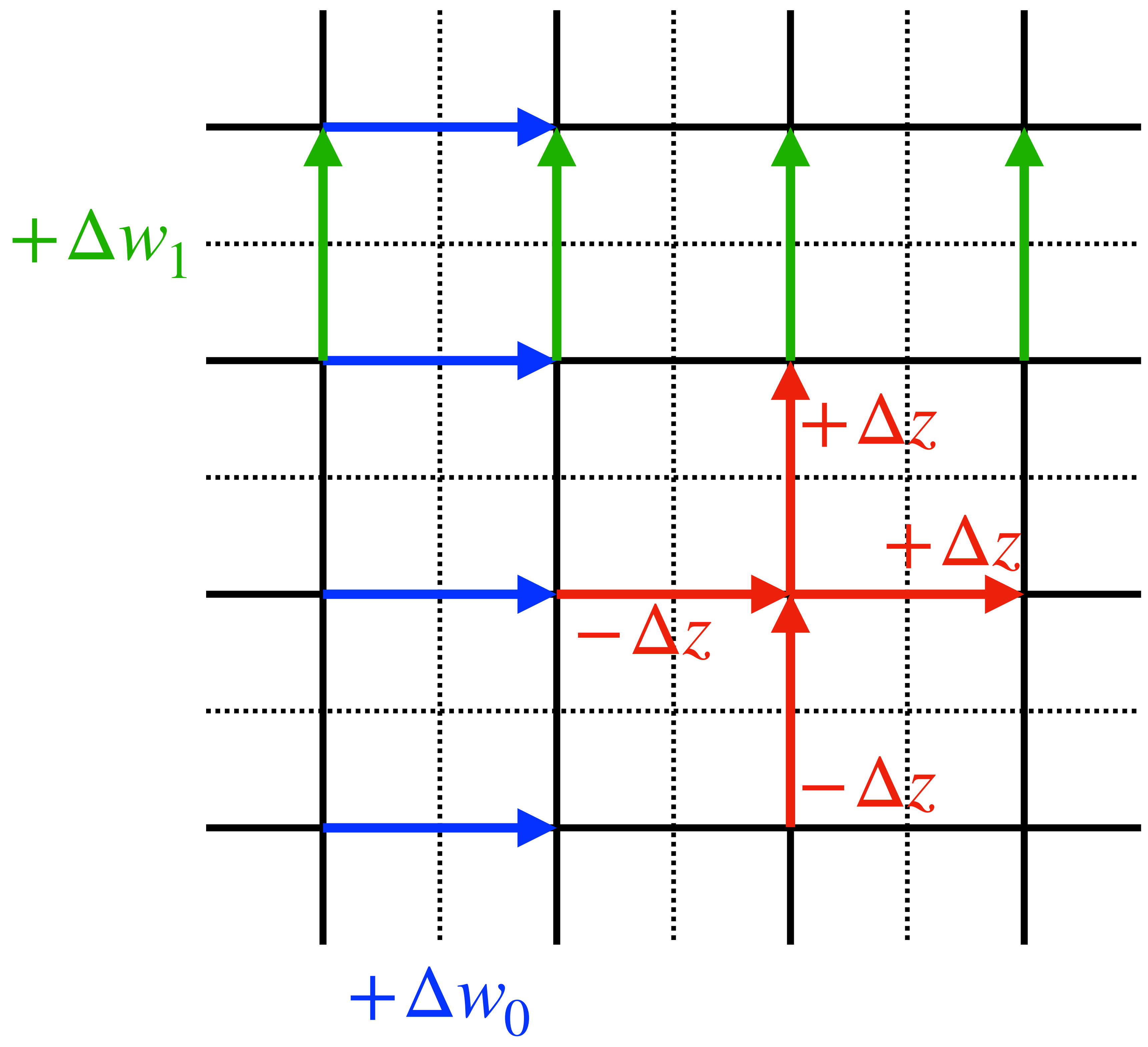
# SIGN PROBLEM?

EB, Cherman, Jacobson
2310.17539

$$S = \frac{\kappa}{2}\sum_{\ell}\left[(d\varphi)_\ell - 2\pi k_\ell\right]^2 + i\sum_{p}\sigma_{\star p}\,(dk)_p$$




$$(dk)_p = 0$$




Worms also work! (forthcoming)

Can solve the constraint with smart update algorithms!

# THE VILLAIN WORM (W=∞, BUT SIMILAR FOR FINITE W)

$$S = \frac{\kappa}{2}\sum_{\ell}\left[(d\varphi)_\ell - 2\pi k_\ell\right]^2 + i\sum_{p}\sigma_{\star p}\,(dk)_p \qquad (dk)_p = 0$$

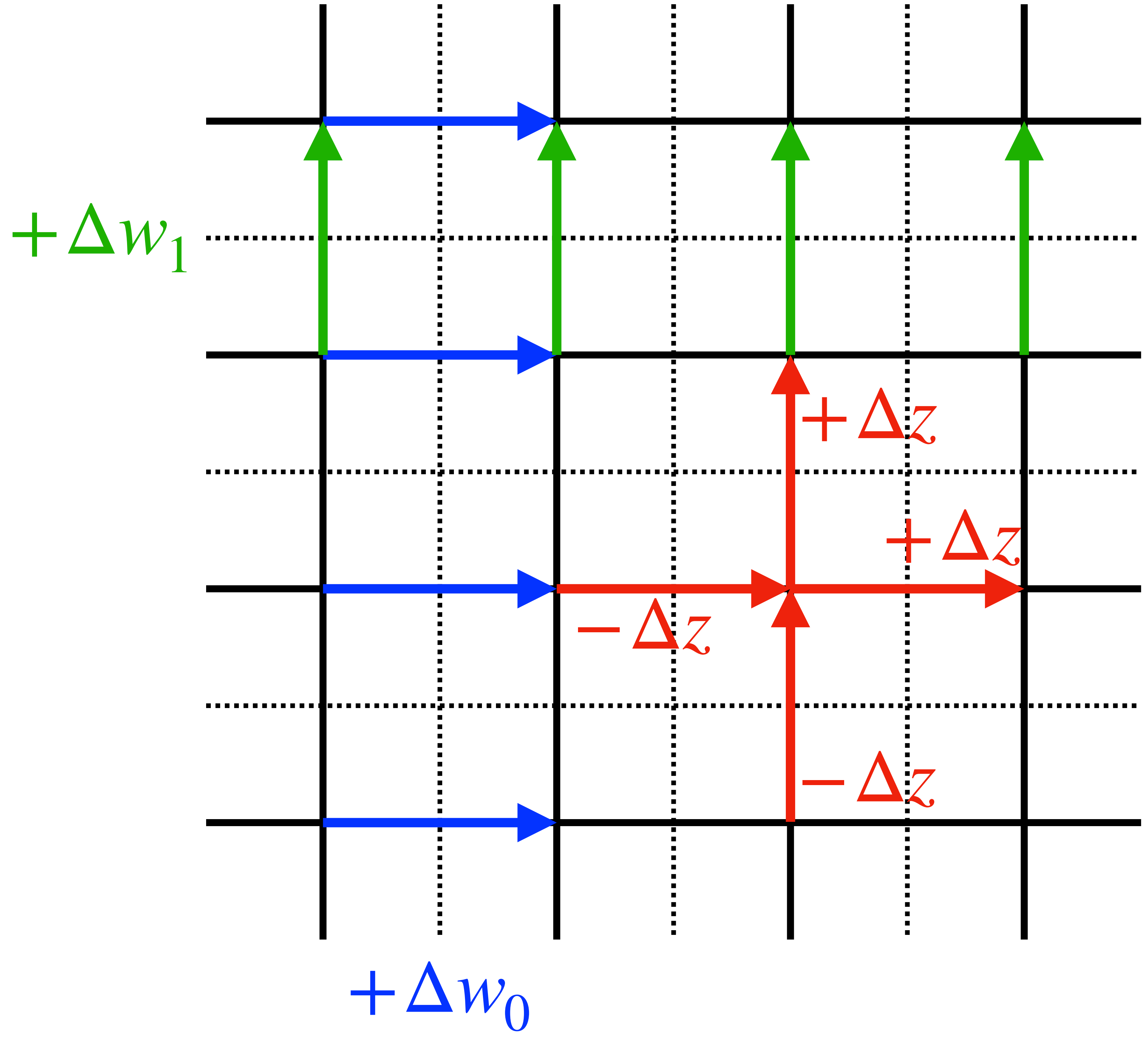
$$V_{\tilde{x}\tilde{y}} = \left\langle e^{i(\sigma_{\tilde{x}}-\sigma_{\tilde{y}})}\right\rangle$$

$$= \frac{1}{\mathcal{Z}}\int D\phi\, Dn\, D\sigma\, e^{-S}\, e^{i(\sigma_{\tilde{x}}-\sigma_{\tilde{y}})}$$

$$= \frac{1}{\mathcal{Z}}\int D\phi\, Dn\, e^{-S}\left[dn_p = \delta_{py} - \delta_{px}\right]$$

$$\mathcal{G} = \int D\phi\, Dn\, d\tilde{h}\, d\tilde{t}\, e^{-S}\left[dn_p = \delta_{p\tilde{t}} - \delta_{p\tilde{h}}\right]$$

$$V_{\tilde{x}\tilde{y}} = \frac{\left\langle \delta_{\tilde{x}\tilde{h}}\delta_{\tilde{y}\tilde{t}}\right\rangle_{\mathcal{G}}}{\left\langle \delta_{\tilde{h}\tilde{t}}\right\rangle_{\mathcal{G}}}$$




Can measure otherwise-difficult correlators during evolution

# THE VILLAIN WORM (W=∞, BUT SIMILAR FOR FINITE W)

$$S = \frac{\kappa}{2} \sum_{\ell} \left[ (d\varphi)_\ell - 2\pi k_\ell \right]^2 + i \sum_p \sigma_{\star p} \, (dk)_p \qquad (dk)_p = 0$$

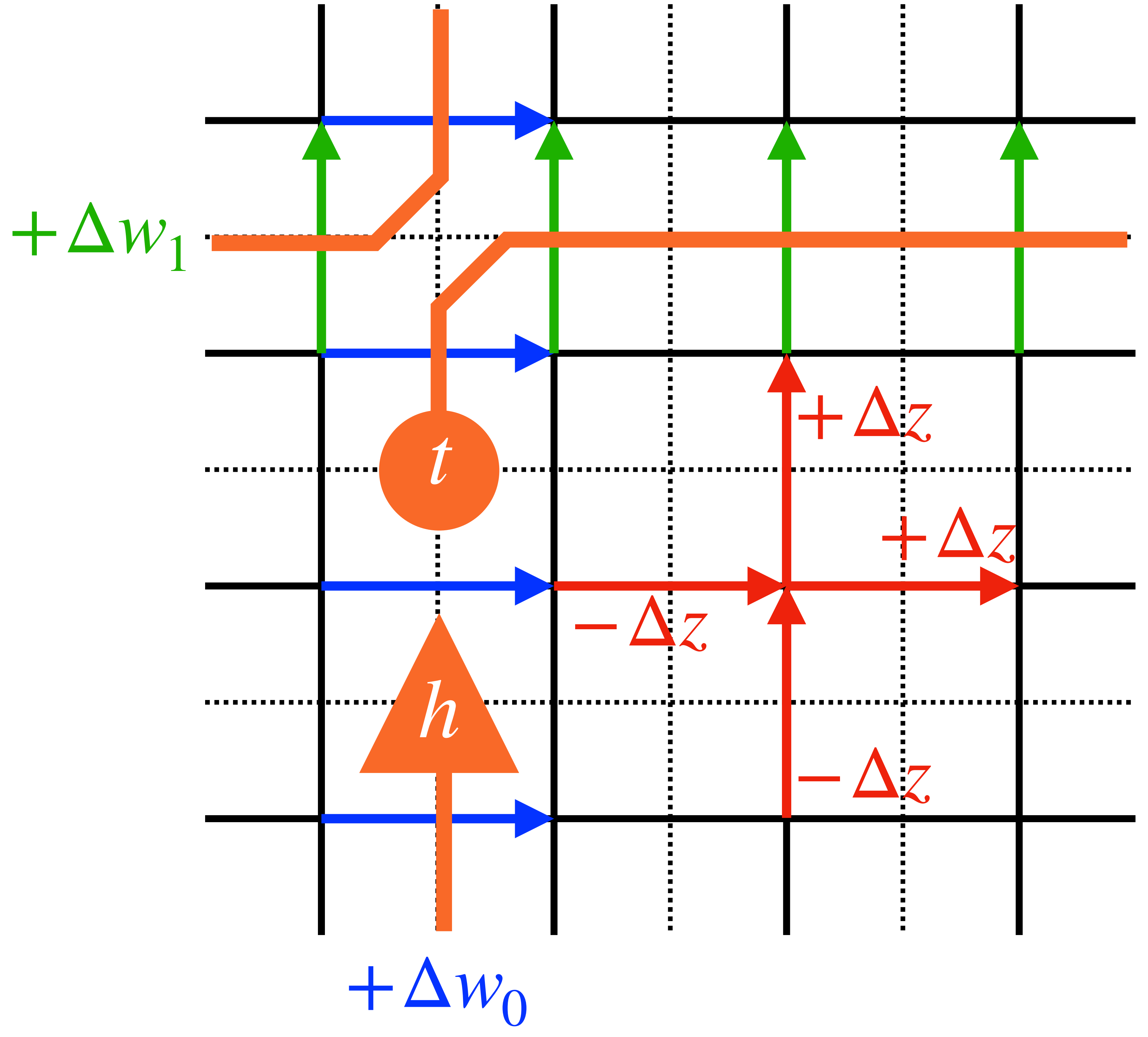
$$V_{\tilde{x}\tilde{y}} = \left\langle e^{i(\sigma_{\tilde{x}} - \sigma_{\tilde{y}})} \right\rangle$$

$$= \frac{1}{\mathcal{Z}} \int D\phi \; Dn \; D\sigma \; e^{-S} \; e^{i(\sigma_{\tilde{x}} - \sigma_{\tilde{y}})}$$

$$= \frac{1}{\mathcal{Z}} \int D\phi \; Dn \; e^{-S} \left[ dn_p = \delta_{py} - \delta_{px} \right]$$

$$\mathcal{G} = \int D\phi \; Dn \; d\tilde{h} \; d\tilde{t} \; e^{-S} \left[ dn_p = \delta_{p\tilde{t}} - \delta_{p\tilde{h}} \right]$$

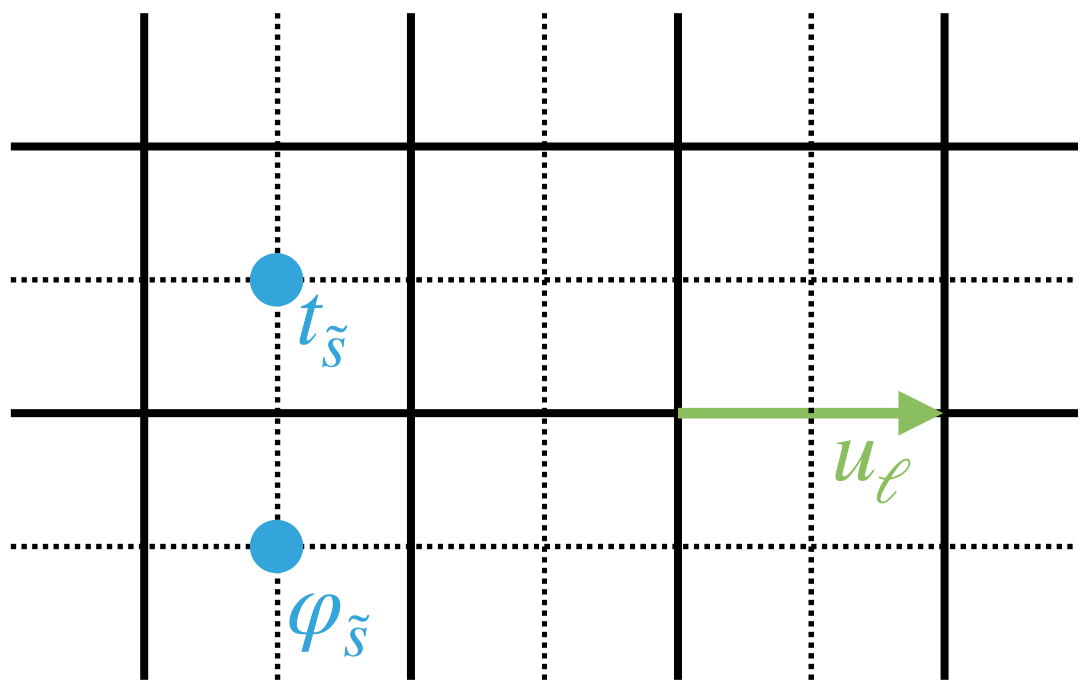
$$V_{\tilde{x}\tilde{y}} = \frac{\left\langle \delta_{\tilde{x}\tilde{h}} \delta_{\tilde{y}\tilde{t}} \right\rangle_{\mathcal{G}}}{\left\langle \delta_{\tilde{h}\tilde{t}} \right\rangle_{\mathcal{G}}}$$




Can measure otherwise-difficult correlators during evolution

# $N_F$=2 QED

# DUAL 2D CHARGE-Q QED




$$\varphi \in \mathbb{R}$$
$$t \in \mathbb{Z}$$
$$u \in \mathbb{Z}$$

Exact manipulations:

- Hubbard-Stratonovich
- Integrate out $a_\ell$ and $r_p$
- Shifting integrated variables

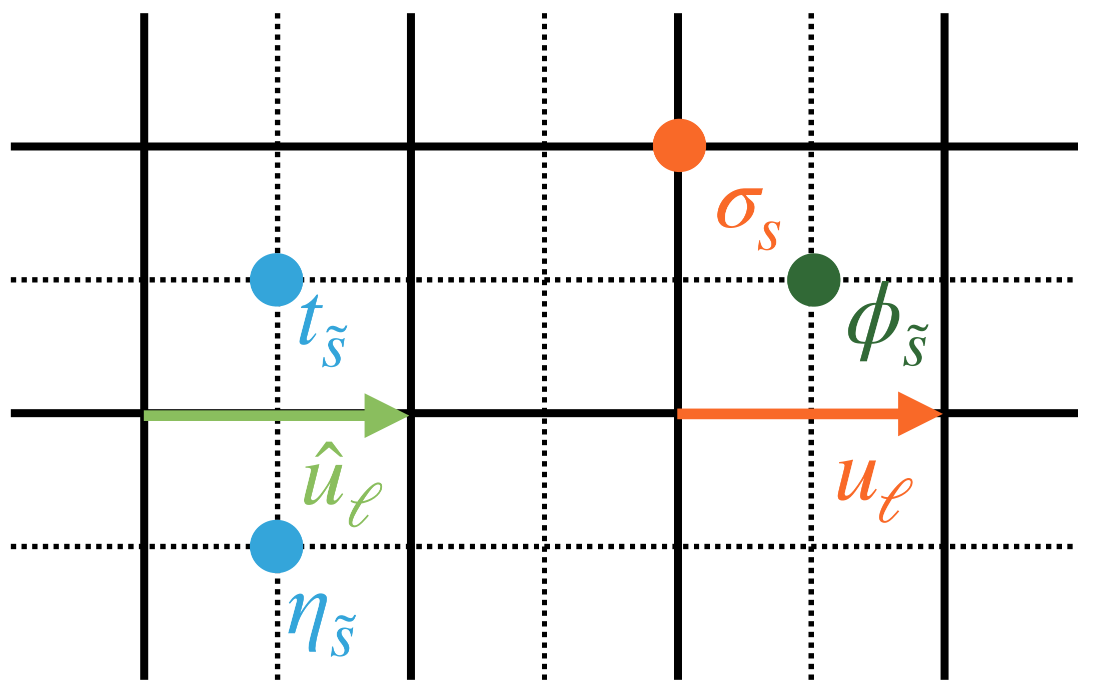
$$S = \frac{\kappa}{2}\left[d\varphi - \frac{2\pi}{Q}dt\right]^2_{\tilde{\ell}} + \frac{1}{2\beta}\left(\frac{Q}{2\pi}\right)^2\left(\varphi - \frac{2\pi}{Q}t\right)^2_{\tilde{s}} - \frac{2\pi i}{Q}(dt)_{\star\ell}u_\ell$$

Massive (free!) Schwinger boson

Constraint:
$dt \equiv 0 \pmod{Q}$

Can solve the constraint with smart update algorithms!

# DUAL 2D CHARGE-Q QED $N_F=2$




$\eta \in \mathbb{R}$ $\quad \sigma \in \mathbb{R}$

$t \in \mathbb{Z}$ $\quad u \in \mathbb{Z}$

$\hat{u} \in \mathbb{Z}$ $\quad \phi \in \mathbb{R}$

$$S = \frac{1}{4\kappa(2\pi)^2}[d\sigma - 2\pi u]_\ell^2 + i\phi_{\star p}(du)_p$$

Villain compact boson σ + constraint

$$+\frac{\kappa}{4}\left[d\eta - \frac{2\pi}{Q}dt\right]_{\tilde{\ell}}^2 + \frac{1}{2\beta}\left(\frac{Q}{2\pi}\right)^2\left[\eta - \frac{2\pi}{Q}t\right]_{\tilde{s}}^2 - \frac{2\pi i}{Q}\hat{u}_\ell(dt)_{\star\ell}$$

Schwinger boson

Constraint:

$dt \equiv 0 \pmod{Q}$

# DUAL 2D CHARGE-Q QED $N_F=2$: CONTINUUM LIMIT

$$S = \frac{1}{4\kappa(2\pi)^2}[d\sigma - 2\pi u]^2_\ell + i\phi_{\star p}(du)_p$$

Villain compact boson σ + constraint

$$+\frac{\kappa}{4}\left[d\eta - \frac{2\pi}{Q}dt\right]^2_{\tilde{\ell}} + \frac{1}{2\beta}\left(\frac{Q}{2\pi}\right)^2\left[\eta - \frac{2\pi}{Q}t\right]^2_{\tilde{s}} - \frac{2\pi i}{Q}\hat{u}_\ell(dt)_{\star\ell}$$

Schwinger boson

Constraint: $dt \equiv 0 \pmod{Q}$

Continuum c=1 compact boson CFT + Schwinger boson limit is simple:

$$\kappa = \frac{1}{4\pi} \qquad \frac{\beta}{N^2} = \frac{1}{2e^2L^2} \qquad N \to \infty$$

(self-dual)

**NO FINE TUNING!**

# EXTRA SYMMETRY OF DUAL 2D CHARGE-Q QED $N_F=2$

$$S \ni \frac{1}{4\kappa(2\pi)^2}[d\sigma - 2\pi u]_\ell^2 + i\phi_{\star p}(du)_p$$

Self-dual under Poisson resummation (Exact!)

$$\sum_n \exp\left\{-\frac{k}{2}(\theta - 2\pi n)^2 + in\phi\right\} = \frac{1}{\sqrt{2\pi k}}\sum_m \exp\left\{-\frac{1}{2k}\left(m - \frac{\phi}{2\pi}\right)^2 - i\left(m - \frac{\phi}{2\pi}\right)\theta\right\}$$

when $\kappa = 1/4\pi$, which we can do link-by-link.

This is exact T-duality at finite lattice spacing and protects $\kappa$ from renormalization!

Weird surprise: **NO FINE TUNING!**

# CHIRAL SYMMETRY

# VILLAIN COMPACT SCALAR

Villain (1975) J. Phys. France 36 581-589

$$\det D(a) = \int \mathscr{D}\varphi \exp\left[-\int d^2x \left(\frac{1}{8\pi}\partial_\mu\varphi\partial^\mu\varphi + \frac{iQ}{2\pi}\epsilon^{\mu\nu}a_\mu\partial_\nu\varphi\right)\right] \qquad \varphi \sim \varphi + 2\pi$$

Villain: Let's use the fact that $\quad U(1) \cong \mathbb{R}/2\pi\mathbb{Z}$

and represent $\varphi$ by two dynamical lattice fields: $\varphi_s \in \mathbb{R}$ and $n_\ell \in \mathbb{Z}$.

$$\begin{aligned} \varphi_s &\to \varphi_s + 2\pi k_s \\ n_\ell &\to n_\ell + 2\pi(dk)_\ell \end{aligned} \qquad (k_s \in \mathbb{Z})$$

so that on the lattice $\partial\varphi \to (d\varphi - 2\pi n)_\ell$ is invariant.

Winding of $\varphi$ is integer $\quad -\frac{1}{2\pi}\sum_{\ell\in C}(d\varphi - 2\pi n)_\ell = \sum_{\ell\in C} n_\ell \quad$ for closed curves $C$.

# VILLAIN GAUGE FIELD

Villain (1975) J. Phys. France 36 581-589

$$\mathscr{L} = \frac{1}{4e^2} f_{\mu\nu}^2 + \text{bosonized matter}$$

Again, split $a_\mu$ into two dynamical fields, $a_\ell \in \mathbb{R}$ and $r_p \in \mathbb{Z}$

$$a_\ell \to a_\ell + (dh)_\ell + 2\pi m_\ell \qquad (h_s \in \mathbb{R})$$ small gauge transformations

$$r_p \to r_p + (dm)_p \qquad (m_\ell \in \mathbb{Z})$$ large gauge transformations

and $f_{\mu\nu} \to (da - 2\pi r)_p$

Instanton number is integer $-\frac{1}{2\pi} \sum_p (da - 2\pi r)_p = \sum_p r_p$

# TOPOLOGICAL QUANTITIES ARE INTEGER AT FINITE SPACING!

# AXIAL SYMMETRY CORRECTLY REDUCED

Under the global axial U(1)$_A$ transformation $\varphi_p \to \varphi_p + \delta$

$$S \ni \frac{iQ}{2\pi}\varphi_{\star p}(da - 2\pi r)_p \to S + \frac{iQ}{2\pi}\delta_{\star p}(da - 2\pi r)_p = \frac{iQ}{2\pi}\delta \sum_p (da - 2\pi r)_p = iQ\delta$$

so that the only invariant choices are $\delta = 2\pi k/Q$ with integer k,

matching the global chiral symmetry

$$G_A = \frac{\mathbb{Z}_{2Q}}{\mathbb{Z}_2} \simeq \mathbb{Z}_Q$$

# 3450 DUAL

$$S_{3450,\text{ dual}} = \frac{\kappa}{2}\frac{1}{5}\left((d\phi) - 2\pi v\right)^2_{\star\ell} + \frac{1}{2\kappa}\frac{1}{20(2\pi)^2}\left(2(d\psi) - 2\pi((dy) - 4v)\right)^2_{\tilde{\ell}}$$

$$+\frac{1}{2\beta}\frac{1}{(2\pi)^2}(4\phi + 2\psi - 2\pi y)^2_{\tilde{s}} + i\sigma_{\star\tilde{p}}(dv)_{\tilde{p}} - i\pi\hat{n}_{\star\tilde{\ell}}(dy)_{\tilde{\ell}} - 2\pi i\frac{4}{5}v_{\star\ell}v_{f(\ell)}$$

Sets dv=0    $dy \equiv 0 \pmod 2$    Total derivative when dv=0

Constraints that don't give sign problems with clever algorithms

$v, y, \hat{n} \in \mathbb{Z}$

$\phi, \chi, \sigma \in \mathbb{R}$

Warning: getting here requires horrible algebra.

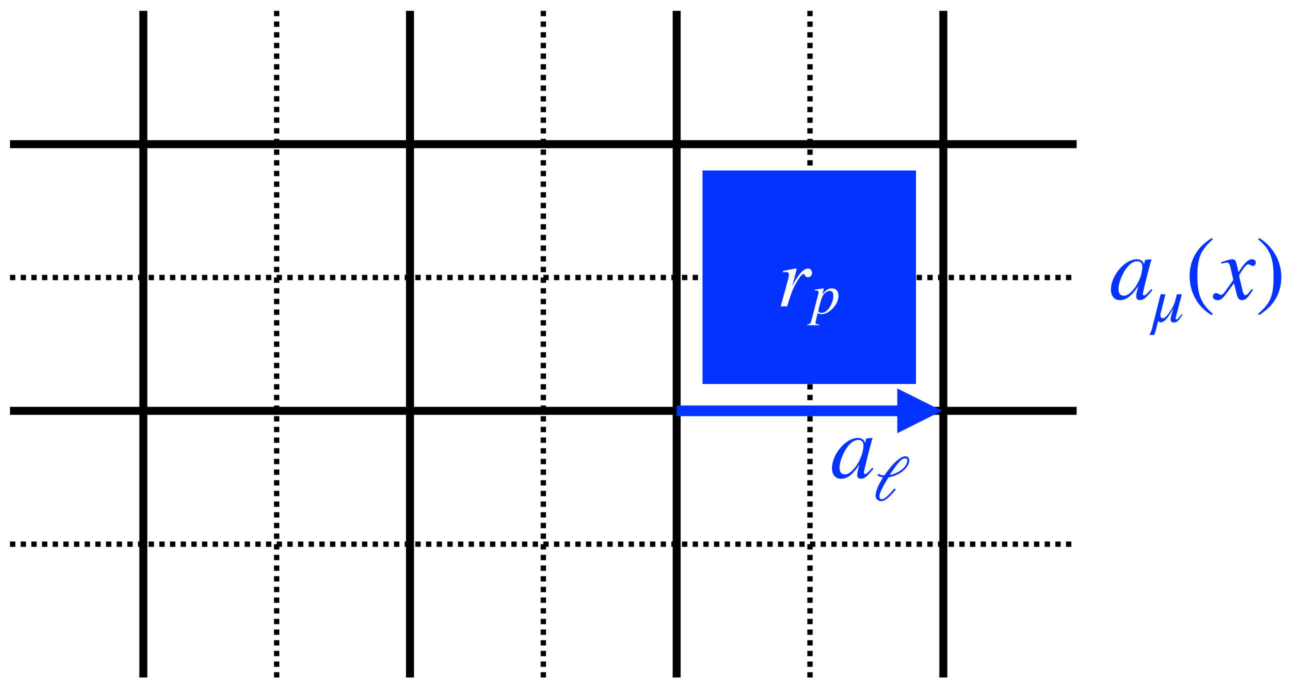
In fact we got tired of trying to do it for generic charge assignments, and just specialized to 3450.

# GAUGE THEORY

# MODIFIED VILLAIN U(1) GAUGE THEORY

$$S_f = \int d^2x\ \bar{\psi}\gamma^\mu(\partial + iQa)_\mu\psi \qquad S_b = \int d^2x\ \frac{1}{8\pi}(d\varphi)^2 + iQa_\mu\frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\varphi$$

"Just integrate a" ? $S_g = \int d^2x\ \frac{1}{4e^2}f_{\mu\nu}^2$ $\quad U(1) = \mathbb{R}/2\pi\mathbb{Z}$ (same trick as the compact boson)




$$S_g = \frac{\beta}{2}\sum_p [da - 2\pi r]_p^2$$

$$\left(\beta = \frac{1}{2e^2}\right)$$

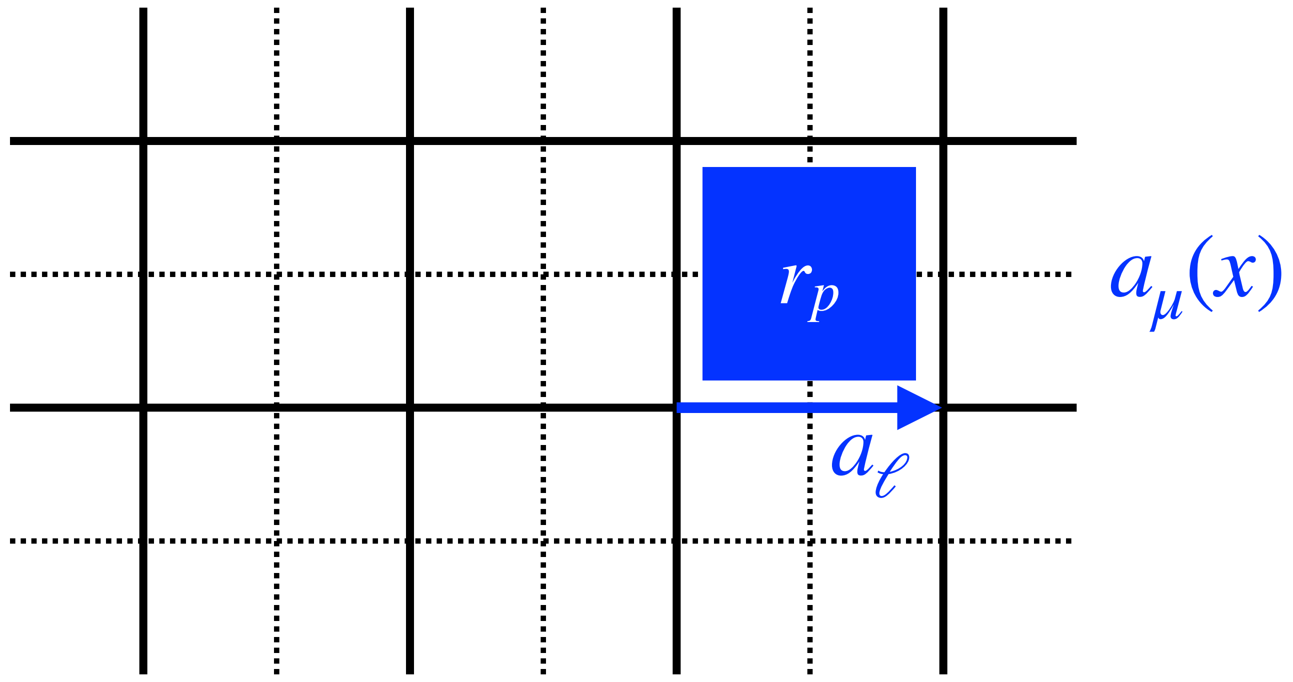
Gauge transformation

$$a \to a + d\lambda + 2\pi m$$

$$r \to r + dm$$

$(\lambda \in \mathbb{R},\ m \in \mathbb{Z})$

# LATTICE-EXACT TOPOLOGY




$$S_g = \frac{\beta}{2} \sum_p [da - 2\pi r]_p^2$$

Gauge transformation

$$a \to a + d\lambda + 2\pi m$$

$$r \to r + dm$$

$(\lambda \in \mathbb{R},\ m \in \mathbb{Z})$

Integer-valued magnetic flux

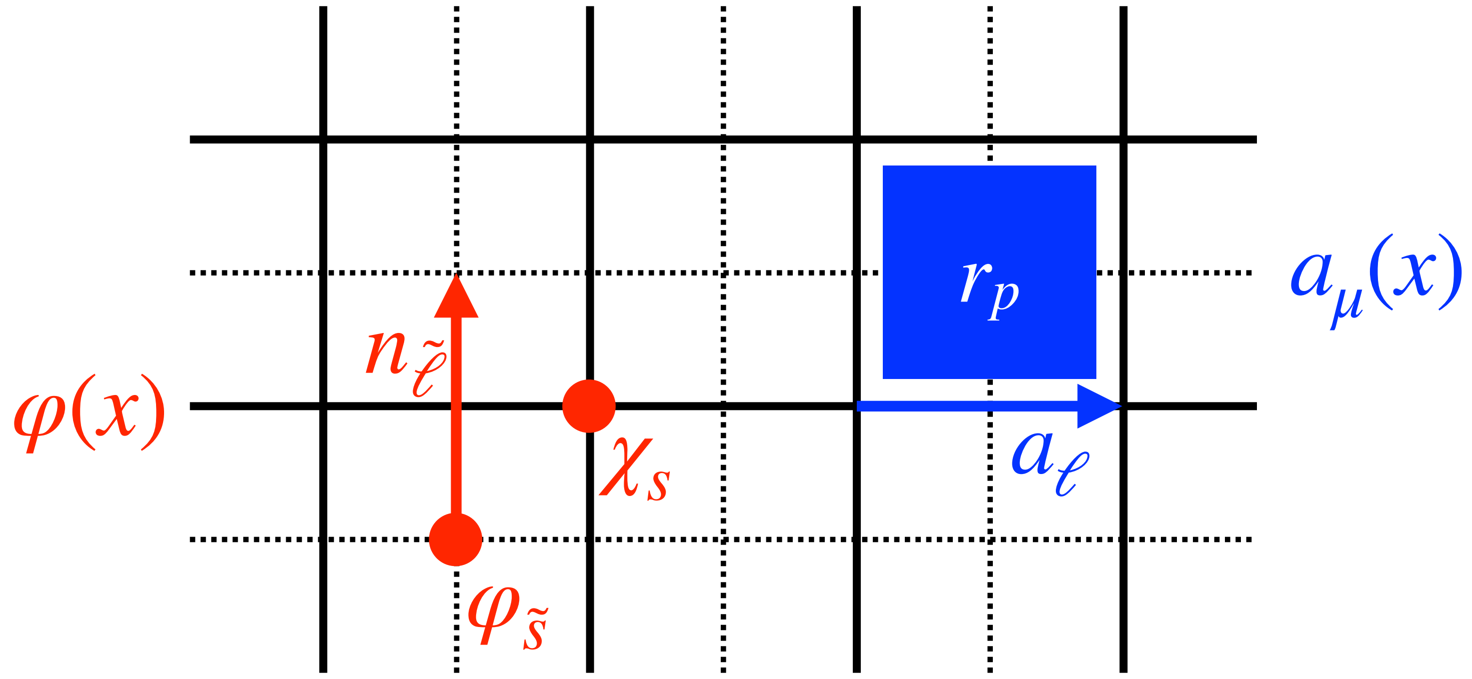
$$\Phi_B = \frac{1}{2\pi} \int_\Sigma da \to \frac{1}{2\pi} \sum_p [da - 2\pi n]_p = -\sum_p n_p \in \mathbb{Z}$$

Always-integer flux means we have a chance of correctly breaking

$$U(1)_A \to (\mathbb{Z}_Q)_A$$

and getting the chiral anomaly right

# 2D CHARGE-Q QED




Not a Wilsonian formulation!

Links are in the algebra, not the group!

$$\mathscr{L} = \frac{1}{4e^2} f_{\mu\nu}^2 \quad \rightarrow \quad \frac{\beta}{2}(da - 2\pi r)_p^2 \qquad -iQ a_\ell n_{\star\ell}$$

required for gauge invariance

$$+\frac{1}{8\pi}\partial_\mu \varphi \partial^\mu \varphi \quad \rightarrow \quad +\frac{\kappa}{2}(d\varphi - 2\pi n)_{\tilde{\ell}}^2 \qquad -i\chi_s (dn)_{\star s}$$

path integrating $\chi$ ensures no dynamical $\varphi$ vortices, preserving U(1)$_V$

$$+\frac{iQ}{2\pi}\varphi \epsilon^{\mu\nu}\partial_\mu a_\nu \quad \rightarrow \quad +\frac{iQ}{2\pi}\varphi_{\star p}(da - 2\pi r)_p$$

2019 onwards: Cheng, Gattringer, Gorantla, Fazzi, Lam, Seiberg, Shao,Sulejmanpasic,…

# ELECTRIC SYMMETRY AND A MIXED 't HOOFT ANOMALY

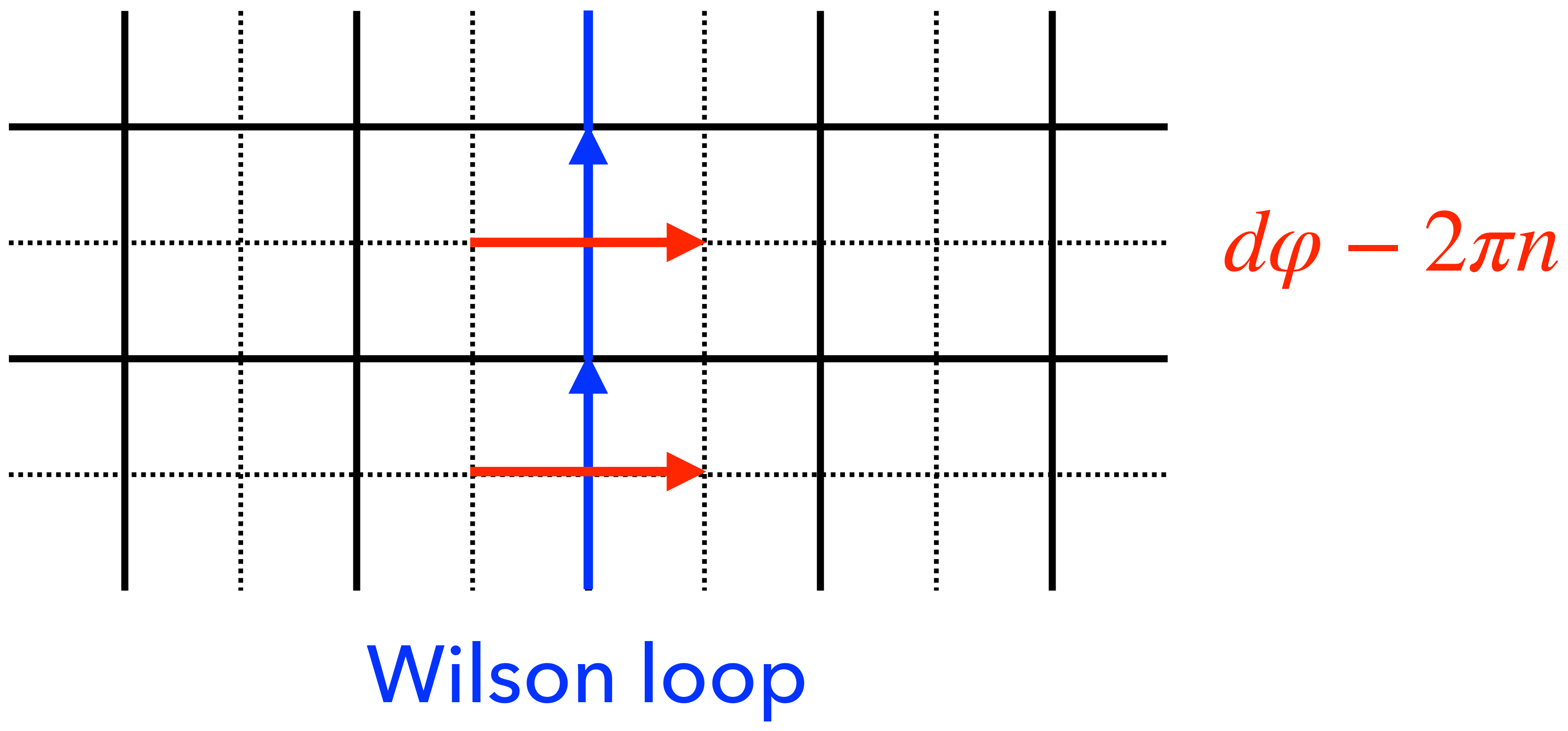
Electric $\mathbb{Z}_Q$ 1-form symmetry transforms Wilson loops

$$\exp\left\{i\oint dx^\mu a_\mu\right\} \to \exp\left\{2\pi i k/Q\right\} \exp\left\{i\oint dx^\mu a_\mu\right\}$$

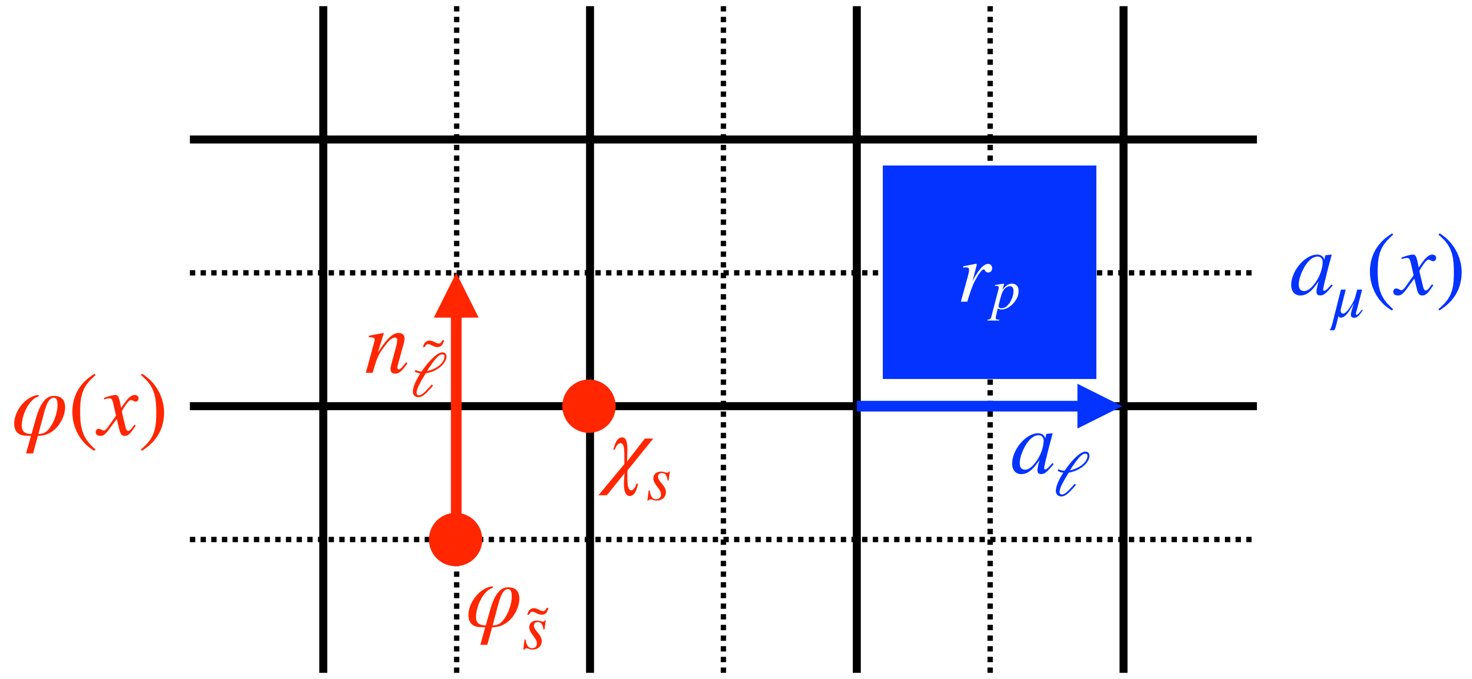
The axial and electric generators are charged under the other symmetry.

Background fields sourcing one explicitly breaks the other.

Also maintained exactly on the lattice!



# 2D CHARGE-Q QED




Symmetries act locally

Captures the anomalies exactly

**even at finite lattice spacing**!

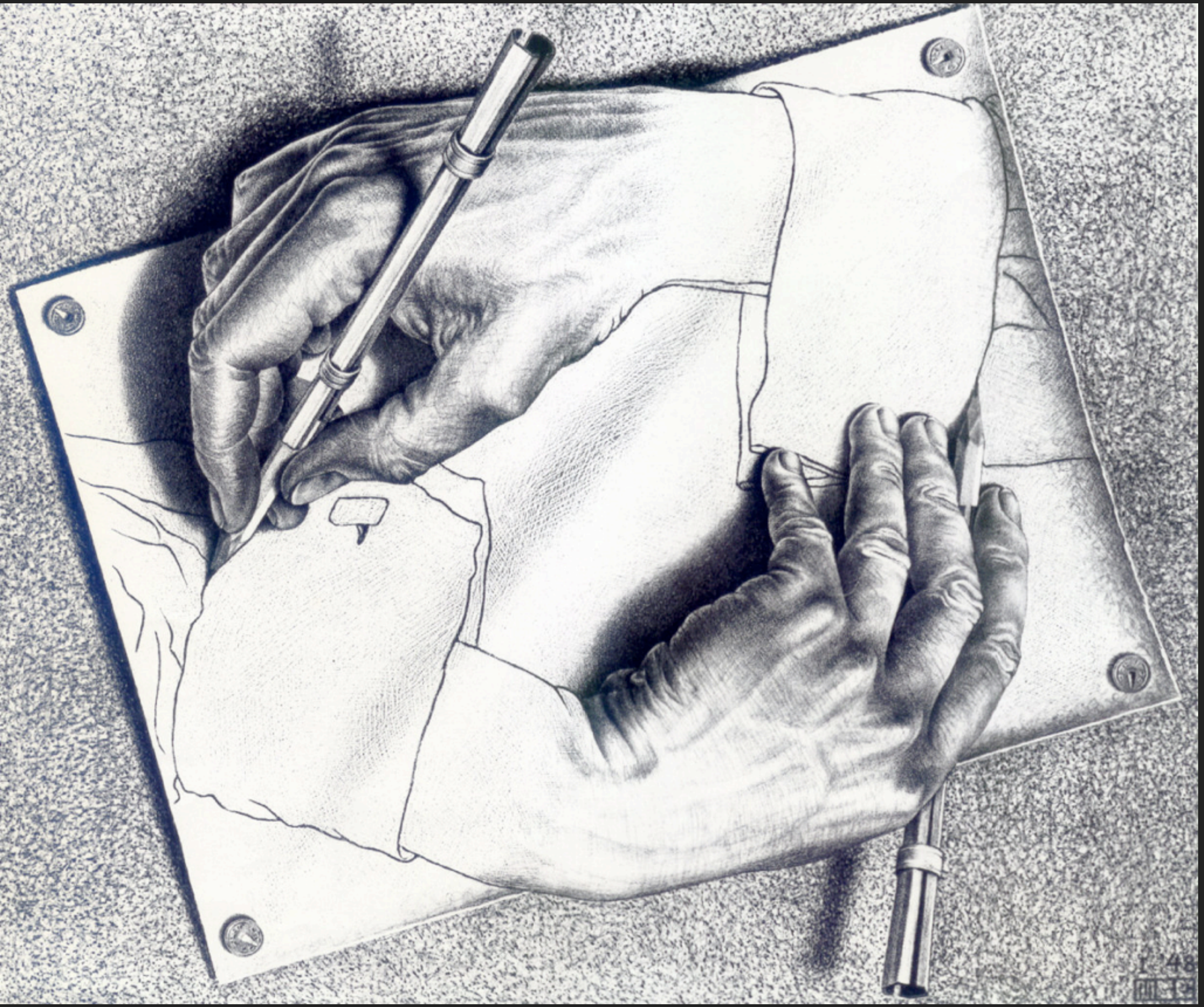
$$\mathscr{L} = \frac{1}{4e^2} f_{\mu\nu}^2 \quad \to \quad \frac{\beta}{2}(da - 2\pi r)_p^2 \quad -iQa_\ell n_{\star\ell}$$

required for gauge invariance

$$+\frac{1}{8\pi}\partial_\mu\varphi\partial^\mu\varphi \quad \to \quad +\frac{\kappa}{2}(d\varphi - 2\pi n)_{\tilde\ell}^2 \quad -i\chi_s(dn)_{\star s}$$

path integrating $\chi$ ensures no dynamical $\varphi$ vortices, preserving $U(1)_V$

$$+\frac{iQ}{2\pi}\varphi\epsilon^{\mu\nu}\partial_\mu a_\nu \quad \to \quad +\frac{iQ}{2\pi}\varphi_{\star p}(da - 2\pi r)_p$$


MC Escher

# CHIRAL GAUGE THEORY

# THE 3450 MODEL

**4 Weyl Fermions ⟷ 2 Dirac Fermions**

| L | R | V | A | |
|---|---|---|---|---|
| 3 | 5 | 8 | -2 | $\psi$ |
| 4 | 0 | 4 | 4 | $\hat{\psi}$ |

Anomaly cancellation:

$$(Q_L^2 - Q_R^2) + (\hat{Q}_L^2 - \hat{Q}_R^2) = Q_V Q_A + \hat{Q}_V \hat{Q}_A = 0$$

# BOSONIZED 3450 ACTION

| Continuum | | $N_f=1$ QED | | $N_f=2$ QED | | 3450 Chiral |
|---|---|---|---|---|---|---|
| $\mathscr{L} = \frac{1}{4e^2} f_{\mu\nu}^2$ | $\rightarrow$ | $\frac{\beta}{2}(da - 2\pi r)_p^2$ | $\rightarrow$ | $\frac{\beta}{2}(da - 2\pi r)_p^2$ | $\rightarrow$ | $\frac{\beta}{2}(da - 2\pi r)_p^2$ |
| $+\frac{1}{8\pi}\partial_\mu\varphi\partial^\mu\varphi$ | $\rightarrow$ | $+\frac{\kappa}{2}(d\varphi - 2\pi n)_\ell^2$ | $\rightarrow$ | $+\frac{\kappa}{2}(d\varphi^j - 2\pi n^j)_{\tilde{\ell}}^2$ | $\rightarrow$ | $+\frac{\kappa}{2}(d\varphi^j - 2\pi n^j - Q_A^j a_f)_{\tilde{\ell}}^2$ |
| $+\frac{iQ}{2\pi}\varphi\epsilon^{\mu\nu}\partial_\mu a_\nu$ | $\rightarrow$ | $+\frac{iQ}{2\pi}\varphi_{\star p}(da - 2\pi r)_p$ | $\rightarrow$ | $+\frac{i}{2\pi}Q^j\varphi^j_{\star p}(da - 2\pi r)_p$ | $\rightarrow$ | $+\frac{i}{2\pi}Q_V^j\varphi^j_{\star p}(da - 2\pi r)_p$ |
| | | $-iQa_\ell n_{\star\ell}$ | $\rightarrow$ | $-iQ^j n^j_{\star\ell}a_\ell$ | $\rightarrow$ | $-iQ_V^j n^j_{\star\ell}a_\ell$ |
| | | $-i\chi_s(dn)_{\star s}$ | $\rightarrow$ | $+in^j_{\star\ell}(d\chi^j)_\ell$ | $\rightarrow$ | $+in^j_{\star\ell}(d\chi^j)_\ell$ |
| | | | | | | $-ir_{f(\star s)}Q_A^j\chi_s^j$ |

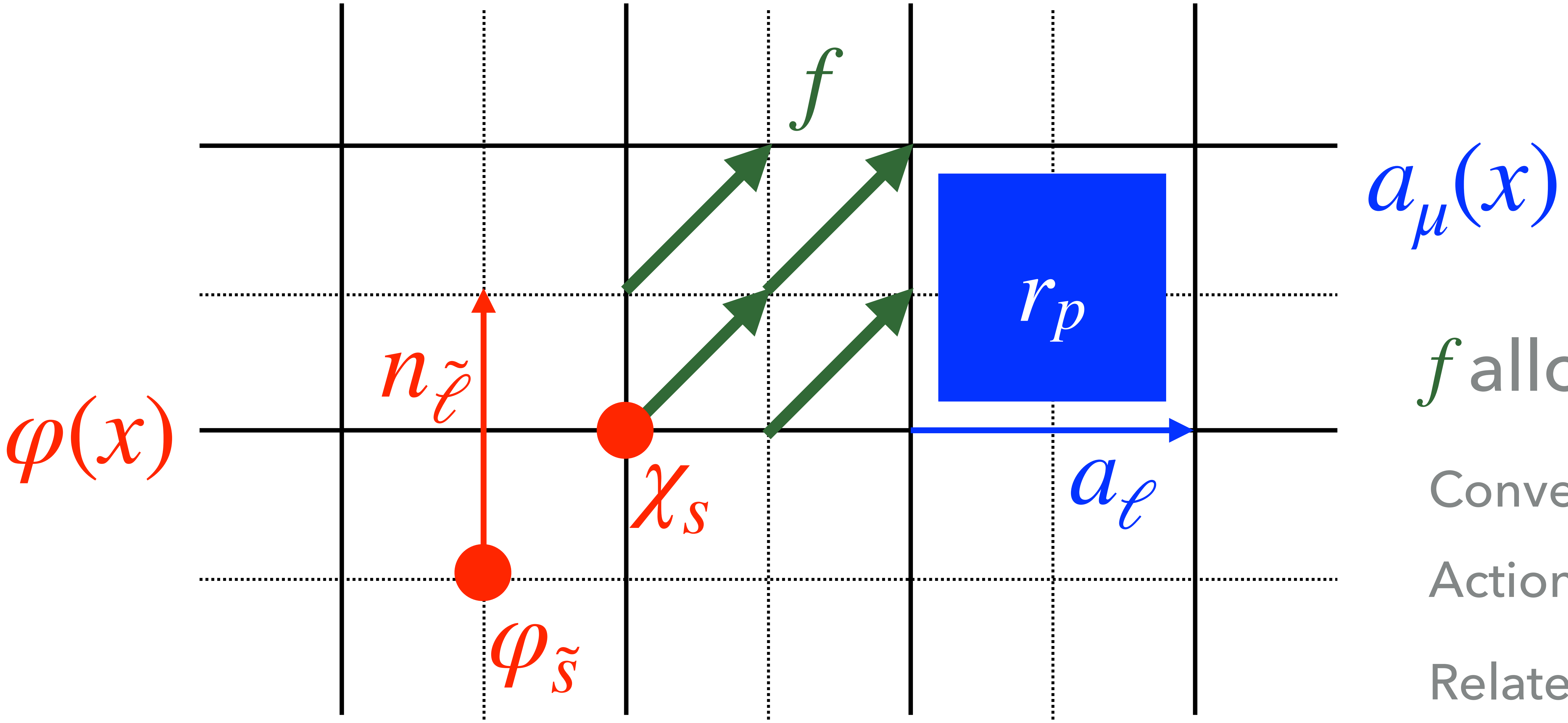
Have a similar sign-problem-free dual formulation.

# BOSONIZED 3450 ACTION

$$S_{3450} = \frac{\beta}{2}\left[da - 2\pi r\right]_p^2 + \frac{\kappa}{2}\left[d\varphi^j - 2\pi n^j - Q_A^j a_f\right]_{\tilde{\ell}}^2 + \frac{i}{2\pi} Q_V^j \varphi^j_{\star p}\left[da - 2\pi r\right]_p$$

$$-iQ_V^j n^j_{\star \ell} a_\ell + i n^j_{\star \ell}(d\chi)^j_\ell - i r_{f(\star s)} Q_A^j \chi_s^j$$

$\Delta S_{3450} \propto Q_V^j Q_A^j$ and vanishes due to the anomaly cancellation condition




$$f(\vec{x}) = \vec{x} + \left(\frac{1}{2}, \frac{1}{2}\right)$$

$f$ allows coupling between nearby cells

Conventional choice but you have to pick something

Action is $\mathbb{Z}_4$ invariant but it's not trivial!

Related to the cup product

Jacobson and Sulejmanpasic PRD 107 125017 (2023) 2303.06160

# QUESTIONS / OBJECTIONS

- Are we stuck in 2D?
  - For now, but there has been a lot of progress on continuum bosonization in D > 2

    Since 2015: Aharony, Gomis, Karch, Kapustin, Komargodski, Son, Seiberg, Senthil, Thorngren, Tong, Witten, …
- In eg. D=3 this will require lattice Chern-Simons…
  - Recent breakthrough construction from Jacobson + Sulejmanpasic, 2023

    Jacobson and Sulejmanpasic PRD 107 125017 (2023) 2303.06160
- Does the Villain trick work for nonabelian groups?
  - I don't know; we haven't found an example with hope but I don't know a theorem excluding the possibility. That may be my own ignorance.
- The Arf invariant…
  - Our constructions actually have gauged $(-1)^F$

# SUMMARY

- A new route around the Nielsen-Ninomiya Theorem
- Exact, locally-acting chiral symmetry
- Non-Wilsonian construction with algebra-valued DOFs, leveraging latticized differential geometry.
- Works for chiral gauge theories!

# DREAMS

- Numerically cheap compared to eg. overlap
- Maybe we can 'back out' D itself from det(D) by measuring correlators?
- What do you want to know about the 3450 model?