Multilevel samplings for glueball calculations

Collaboration

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Based on

[arXiv: 2406.12656]	SU(3) YM
[Work in progress…]	Quenched QCD

FOR 5269

DESY.

Lattice 2024, Liverpool 28/07-03/08

Multilevel samplings for pure gauge glueballs

[G. Parisi (1983)]

[M. Lüscher, P. Weisz (2001)]

[H. Meyer (2002), (2003)]

If action and observables are local, the correlation function

$$\langle O(t_1)O(t_0)\rangle = \frac{1}{\mathcal{Z}} \int [dU]e^{-S_g[U]}\, O(U, t_1)O(U, t_0)$$

$O(t)$ = glueball operator

can be factorised into a product of integrals

$$\langle O(t_1)O(t_0)\rangle = \frac{1}{\mathcal{Z}} \int [dU_B]e^{-S_B[U_B]} \left[O^{(2)}(U_B, t_1)\right] \left[O^{(1)}(U_B, t_0)\right]$$

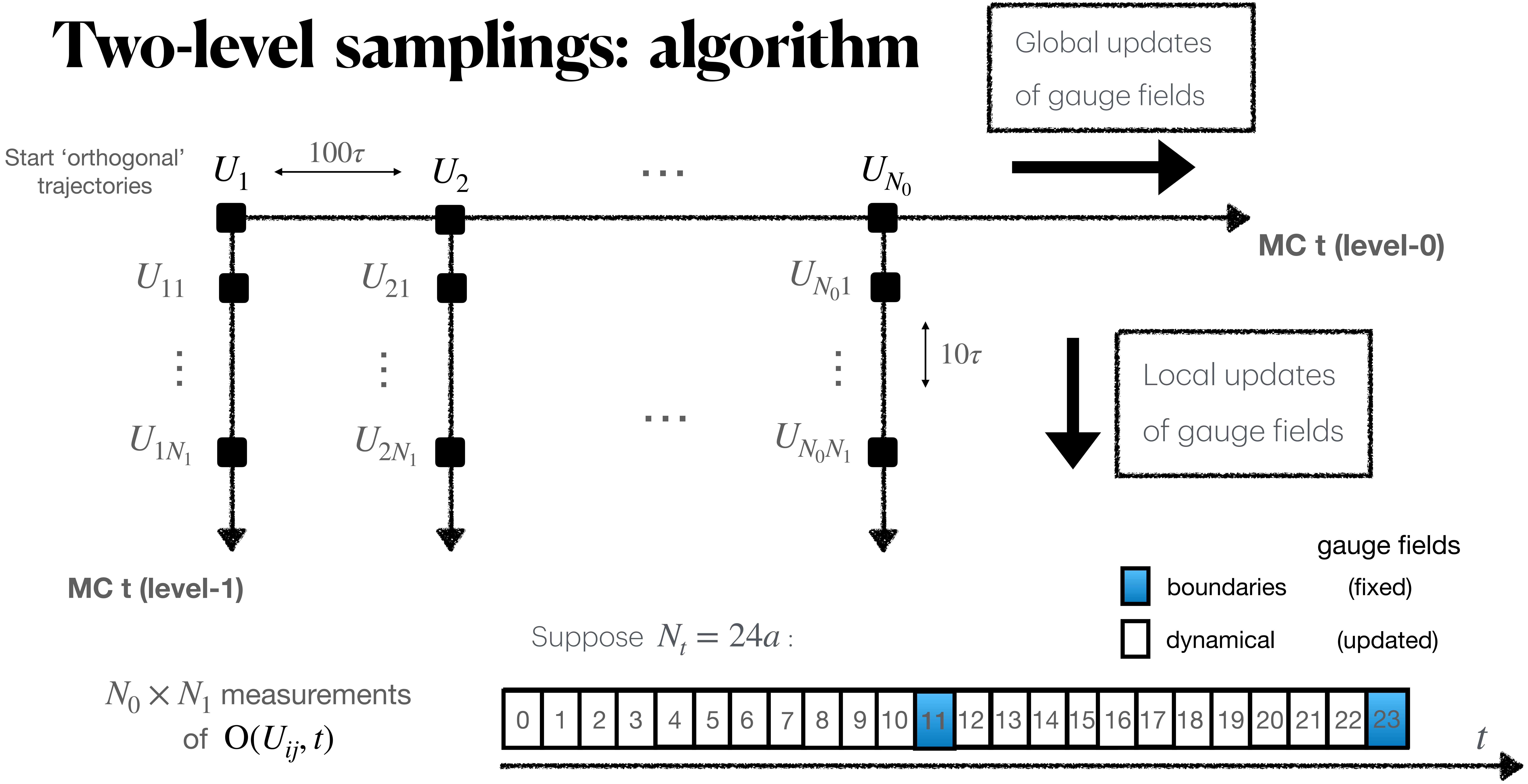
$$\left[O^{(r)}(U_B, t)\right] = \int [dU^{(r)}]\, e^{-S_r[U^{(r)}|U_B]}\, O(U^{(r)}, t)$$

Two-level samplings: algorithm

In the same way as with the standard algorithm

we generate N_0 gauge configurations (HMC)

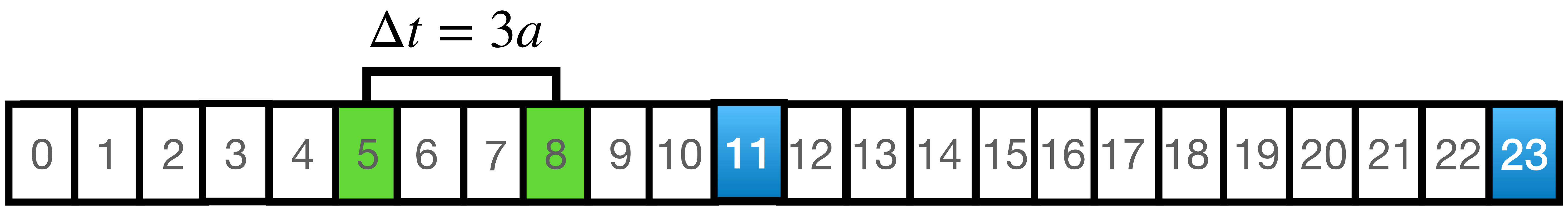
Two-level samplings: algorithm



Two-level samplings: algorithm

if t_0 and t_1 belong to the same region

e.g.



Correlate over $N_0 \times N_1$ samples

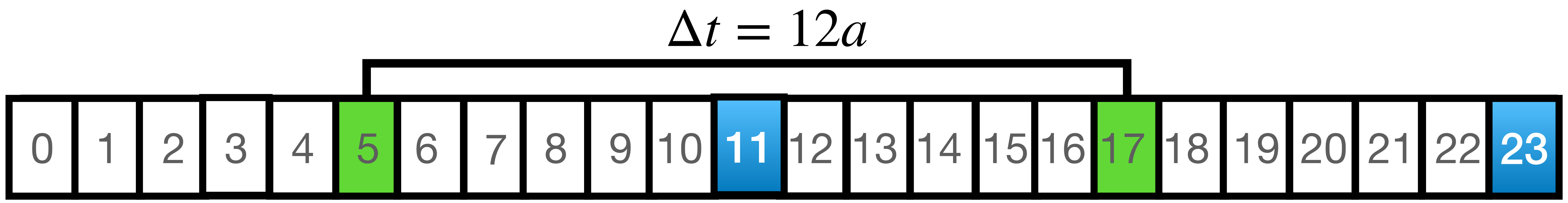
$$C(t_1 - t_0) = \langle \mathrm{O}(U_{ij}, t_1)\ \mathrm{O}(U_{ij}, t_0) \rangle = \frac{1}{N_0 N_1} \sum_{i=1}^{N_0} \sum_{j=1}^{N_1} \mathrm{O}(U_{ij}, t_1) \mathrm{O}(U_{ij}, t_0)$$

Error scaling $\quad \sigma^2(t_1 - t_0) \propto \dfrac{1}{N_0 N_1} + \ \ldots$ **Standard**

Two-level samplings: algorithm

if t_0 and t_1 belong to different regions

e.g.



Take average over level-1

$$\left[\mathrm{O}(U_i, t)\right] = \frac{1}{N_1} \sum_{j=1}^{N_1} \mathrm{O}(U_{ij}, t)$$

Correlate

$$C(t_1 - t_0) = \left\langle \left[\mathrm{O}(U_i, t_1)\right] \left[\mathrm{O}(U_i, t_0)\right] \right\rangle = \frac{1}{N_0} \sum_{i=1}^{N_0} \left[\mathrm{O}(U_i, t_1)\right] \left[\mathrm{O}(U_i, t_0)\right]$$

Error scaling

$$\sigma^2(t_1 - t_0) \propto \frac{1}{N_0 N_1^2} + \ \ldots$$

Two-level

Numerical simulations

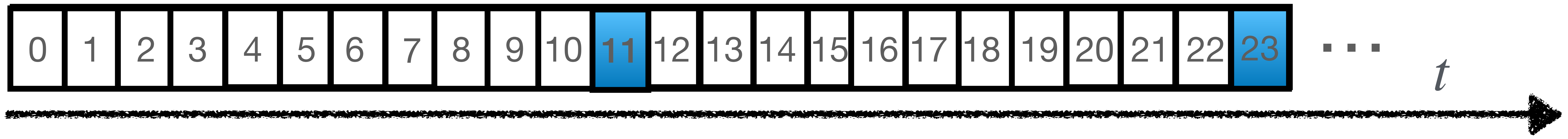
[arXiv: 2406.12656]

L.B., F. Knechtli, S. Martins, M. Peardon, S. Schaefer, J.A. Urrea-Niño

In the following:

$4D$ SU(3) theory, $\beta = 6.2$, $V = 24^3 \times 48$ $N_0 = 101$, $N_1 = 1,\ldots,1000$

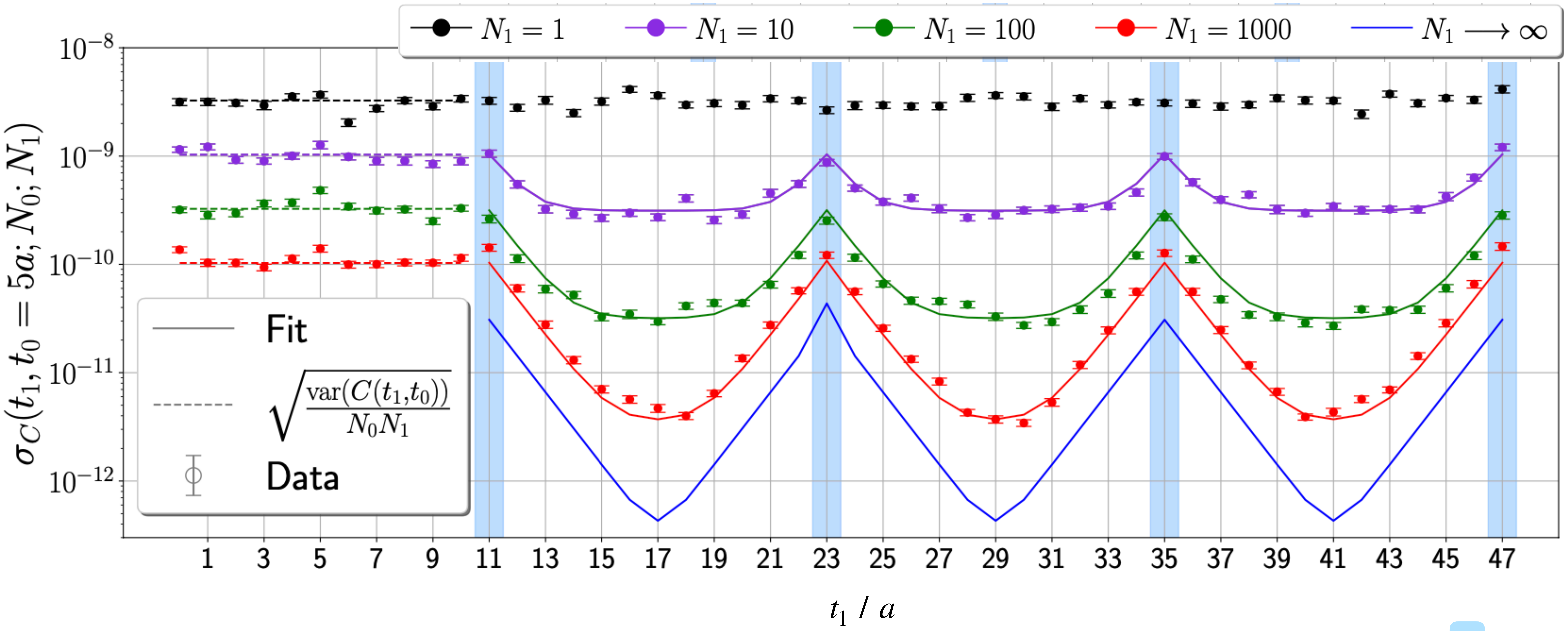
Sub-lattice decomposition on level-1:



$$C(t_1 - t_0) = \langle \mathrm{O}(U_{ij}, t_1)\ \mathrm{O}(U_{ij}, t_0) \rangle$$

Results for the statistical errors

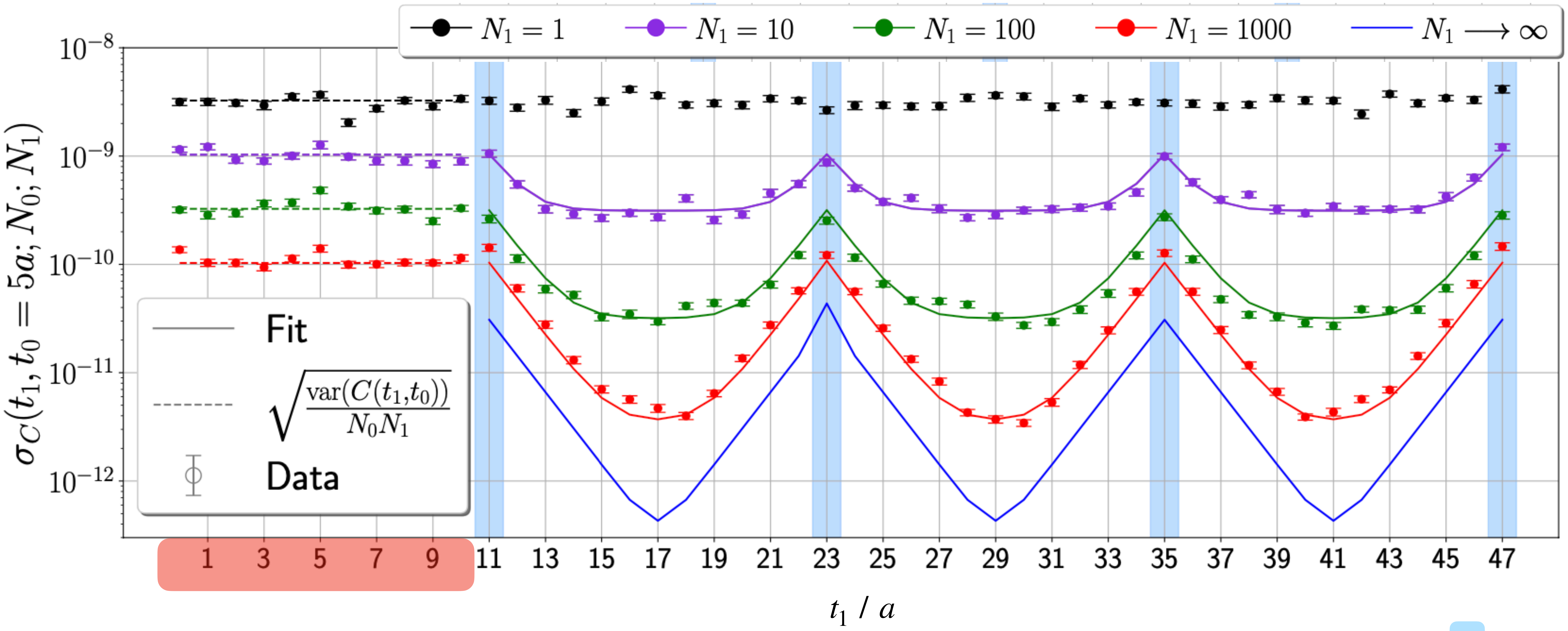
4D SU(3) theory, $\beta = 6.2$, $V/a^4 = 24^3 \times 48$



boundaries $t_B/a = 11, 23, 35, 47.$

Results for the statistical errors

4D SU(3) theory, $\beta = 6.2$, $V/a^4 = 24^3 \times 48$



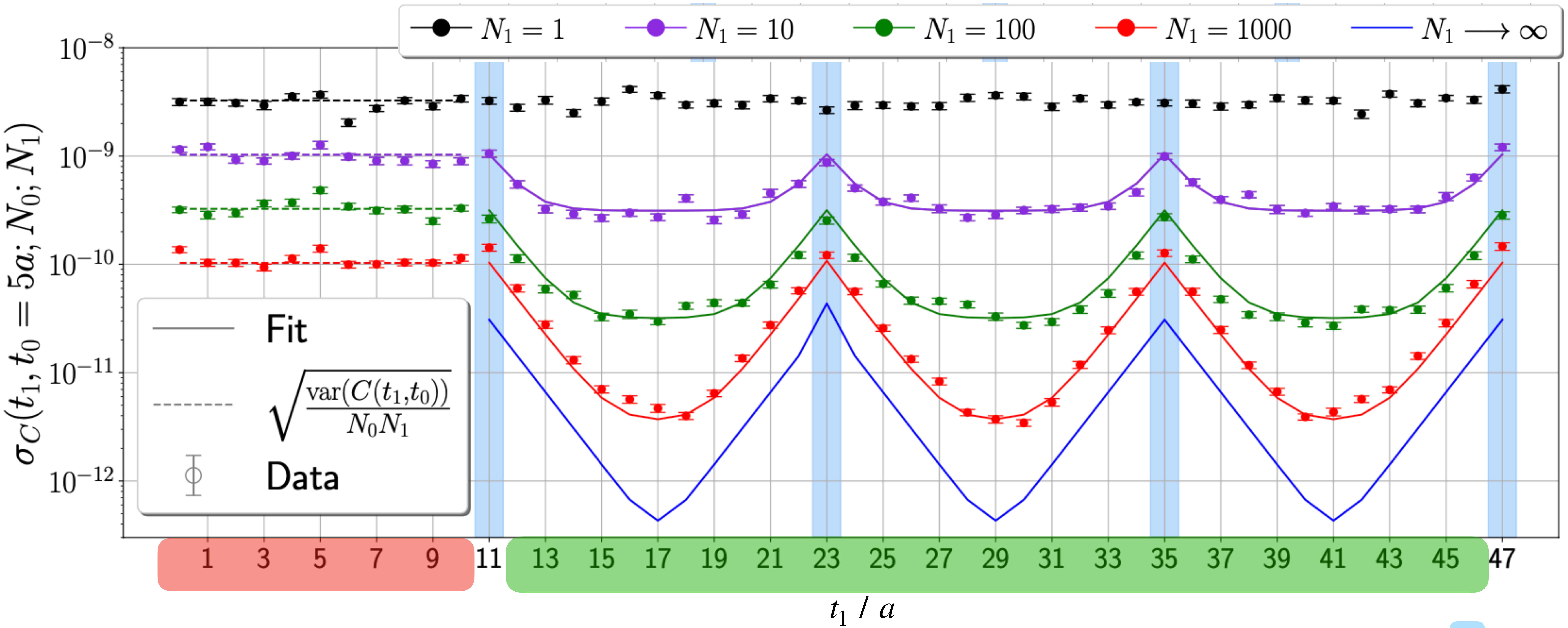
Error scales like $1/\sqrt{N_0 N_1}$

when operators in same region

boundaries $t_B/a = 11, 23, 35, 47.$

Results for the statistical errors

4D SU(3) theory, $\beta = 6.2$, $V/a^4 = 24^3 \times 48$



Error scales like $1/\sqrt{N_0 N_1}$

when operators in same region

Error scales exponentially with distance from boundaries

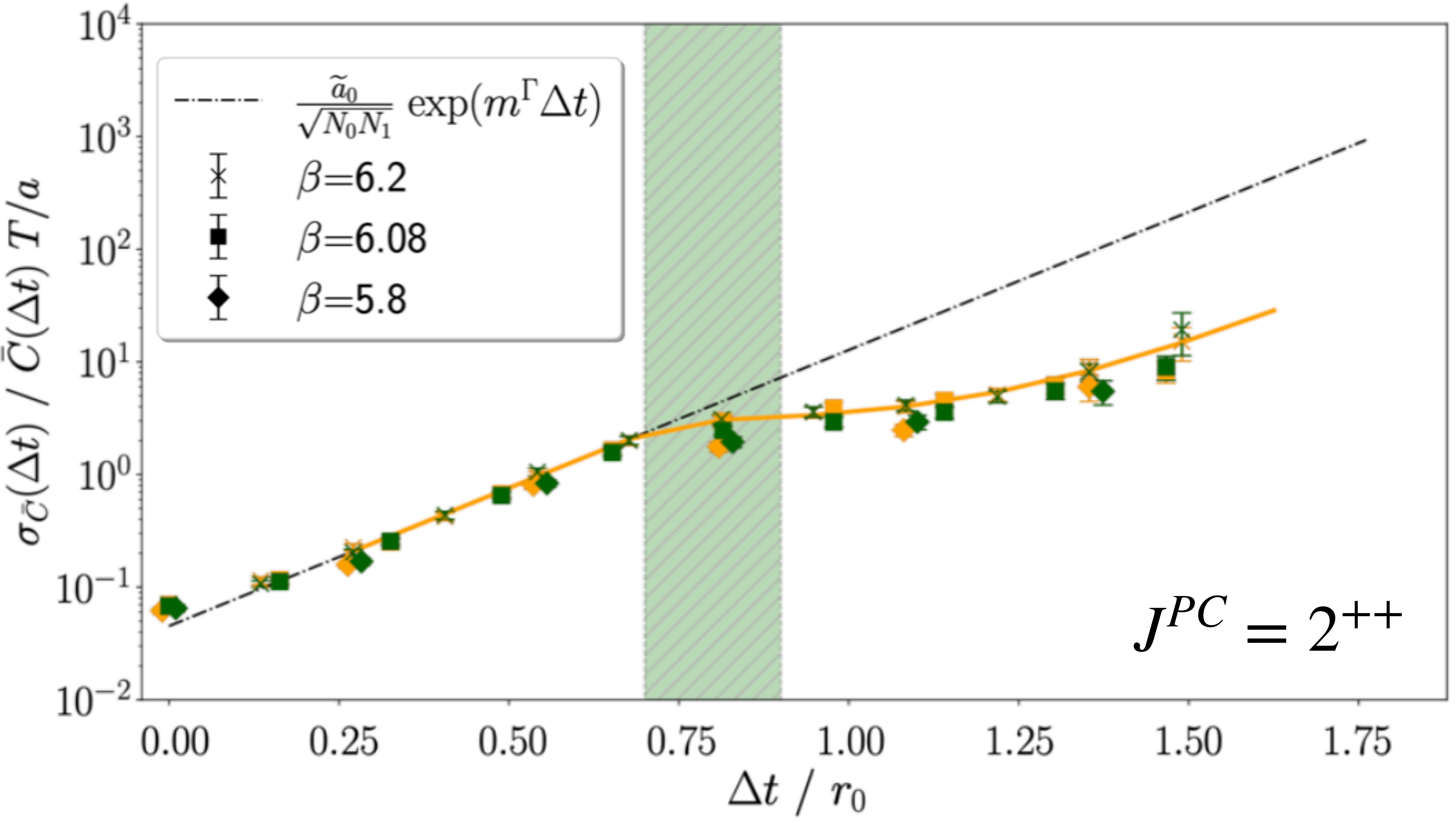
when operators in different regions

$t_B/a = 11, 23, 35, 47.$

$$\sigma_C^2(t_1, t_0) \approx \frac{c_0^2}{N_0 N_1^2} + \frac{c_1^2}{N_0 N_1}\left[e^{-m\Delta_1} + e^{-m\Delta_0}\right] + \frac{c_2^2}{N_0} e^{-m\Delta_1} e^{-m\Delta_0}$$

$$\Delta_{1,0} = |t_{1,0} - t_B|$$

Noise/Signal towards continuum limit



Weighted average

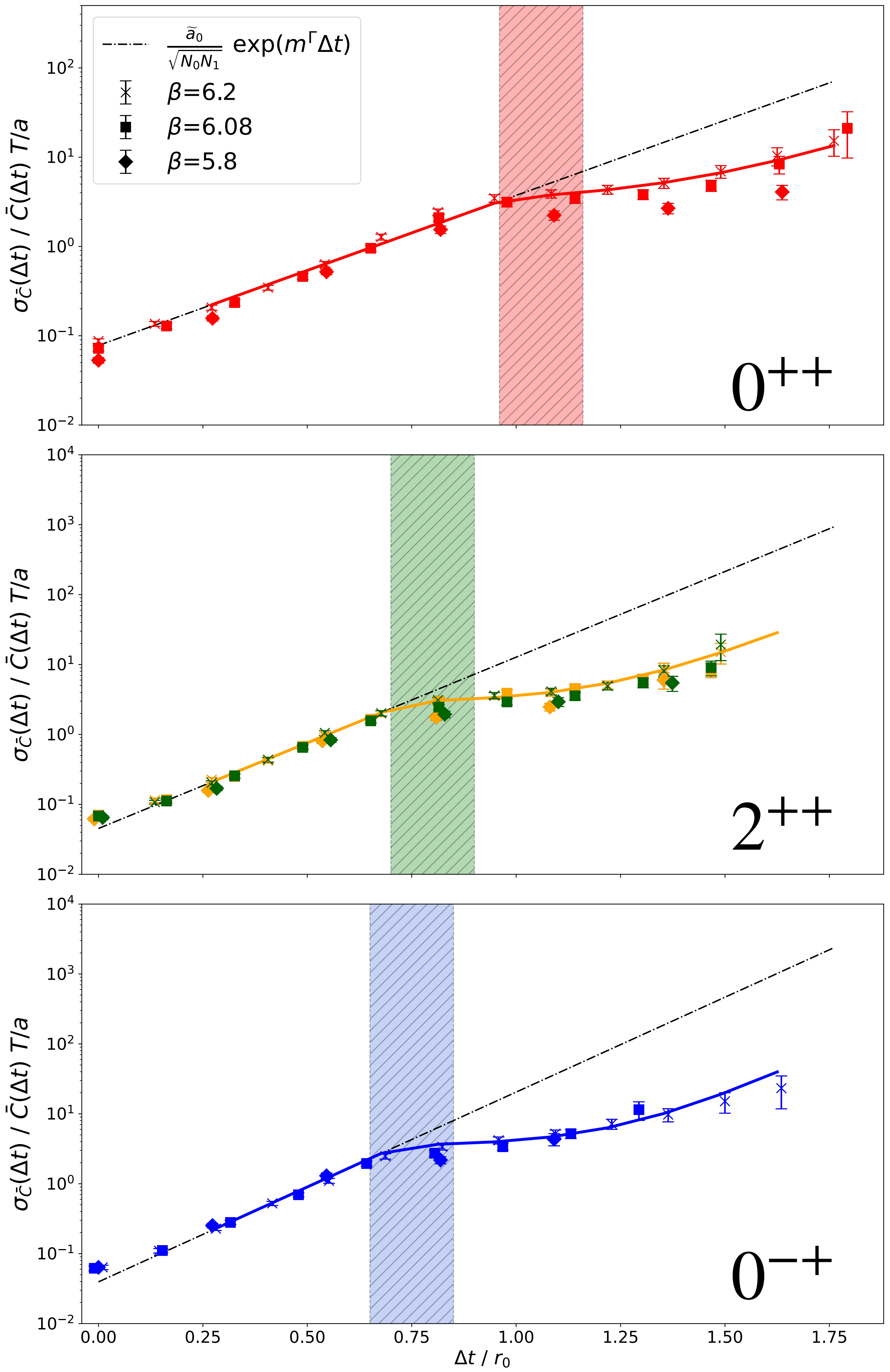
$$\overline{C}(\Delta t) = \frac{\sum_{t_0} w(t_0) C(\Delta t = t_1 - t_0)}{\sum_{t_0} w(t_0)}$$

$$w(t_0) = 1\ /\ \sigma_C^2(t_0)$$

Coloured band highlights transition between different scalings

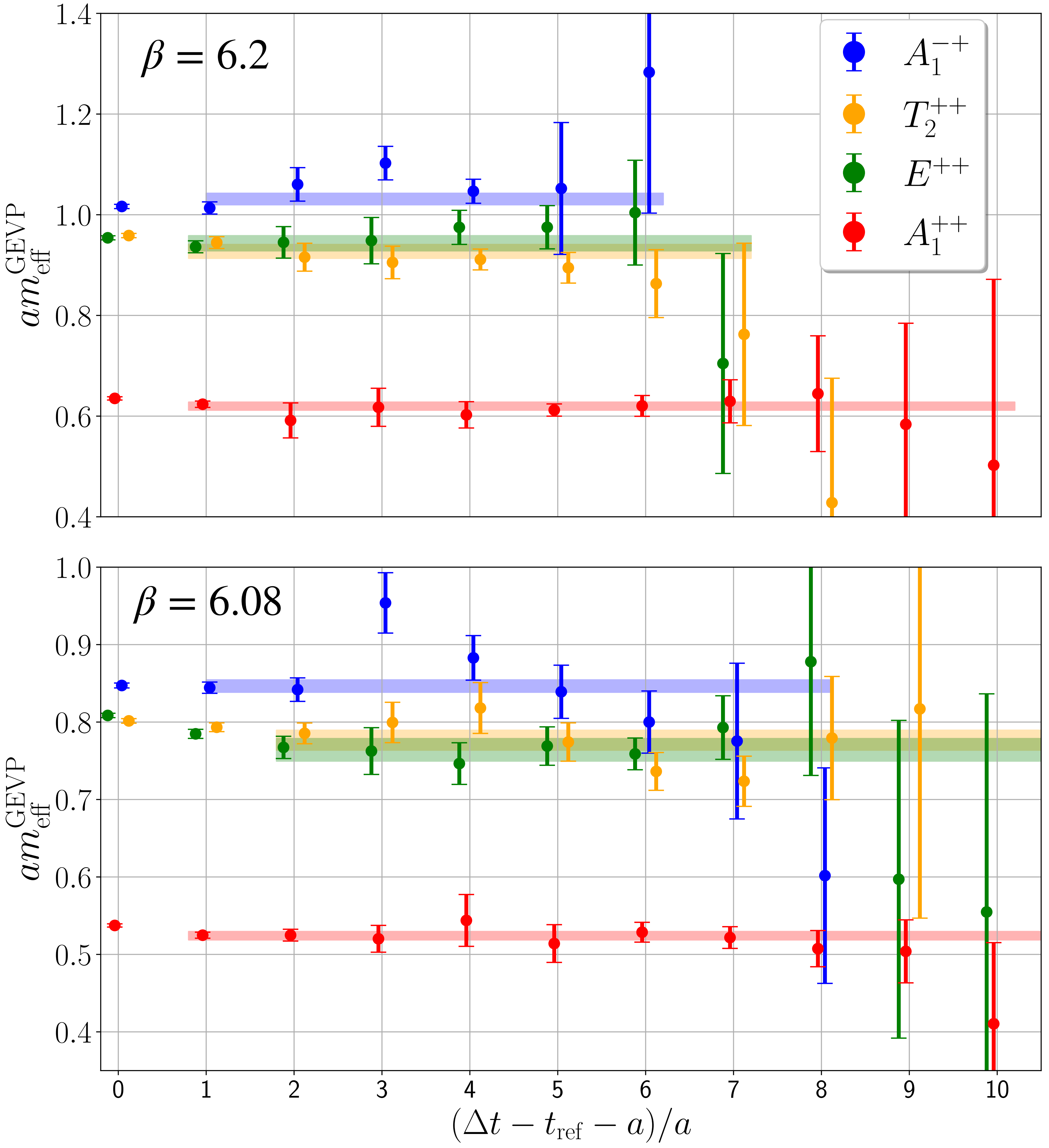
- ✦ Short distance scaling: $\dfrac{\sigma_{\overline{C}}^2(\Delta t)}{\overline{C}^2(\Delta t)} \approx \tilde{a}_0^2 \dfrac{e^{2m^\Gamma \Delta t}}{N_0 N_1}$
- ✦ Long distance scaling: $\dfrac{\sigma_{\overline{C}}^2(\Delta t)}{\overline{C}^2(\Delta t)} \approx \dfrac{\tilde{c}_0^2}{N_0 N_1^2} e^{2m^\Gamma \Delta t} + \dfrac{2\tilde{c}_1^2}{N_0 N_1} e^{2m^\Gamma \Delta t/2} + \dfrac{\tilde{c}_2^2}{N_0}$

Noise/Signal towards continuum limit



Consistency across all glueball channels

GEVP effective masses



Results agree with state-of-the-art calculations that use $\mathcal{O}(10^5)$ configs

[A. Athenodorous, M. Teper, 2020]

Glueballs in Quenched QCD

$$\langle \mathrm{O}(t_1)\mathrm{O}(t_0)\rangle = \frac{1}{\mathcal{Z}} \int [dU][d\bar{q}][dq]e^{-S_g[U]}\ \mathrm{O}(U,t_1)\mathrm{O}^\dagger(U,t_0)$$

$\mathrm{O}(U,t)$ are now purely gluonic and **fermionic** operators like $\mathrm{O}_{\pi\pi}(U,t)$ (Action is still SU(3))

Scalar glueballs candidates e.g. $f_0(1500)$, $f_0(1730)$ decay into multi-meson states like $f_0(1500) \rightarrow \pi\pi$

Need to compute

$$\langle \mathrm{O}_{\pi\pi}(t_1)\bar{\mathrm{O}}_{\pi\pi}(t_0)\rangle = \frac{1}{\mathcal{Z}} \int [dU]e^{-S[U]}\ < \mathrm{O}_{\pi\pi}(U,t_1)\mathrm{O}^\dagger_{\pi\pi}(U,t_0) >_{\mathrm{Wick}}$$

$$< \mathrm{O}_{\pi\pi}(U,t_1)\mathrm{O}^\dagger_{\pi\pi}(U,t_0) >_{\mathrm{Wick}} = \mathrm{Tr}\left[\mathrm{D}^{-1}(t_1,t_1)\mathrm{D}^{-1}(t_1,t_1)^\dagger\right] \mathrm{Tr}\left[\mathrm{D}^{-1}(t_0,t_0)\mathrm{D}^{-1}(t_0,t_0)^\dagger\right]\ +\ \ldots$$

Disconnected diagram

t_1 t_0

[J. A. Urrea-Niño talk, Tuesday]

Two-level Samplings for Glueballs in Quenched QCD

Challenge:

$$\mathrm{Tr}\left[\mathrm{D}^{-1}(t,t)\mathrm{D}^{-1}(t,t)^{\dagger}\right]$$

Propagators depend on gauge fields over all space-time

Solution:

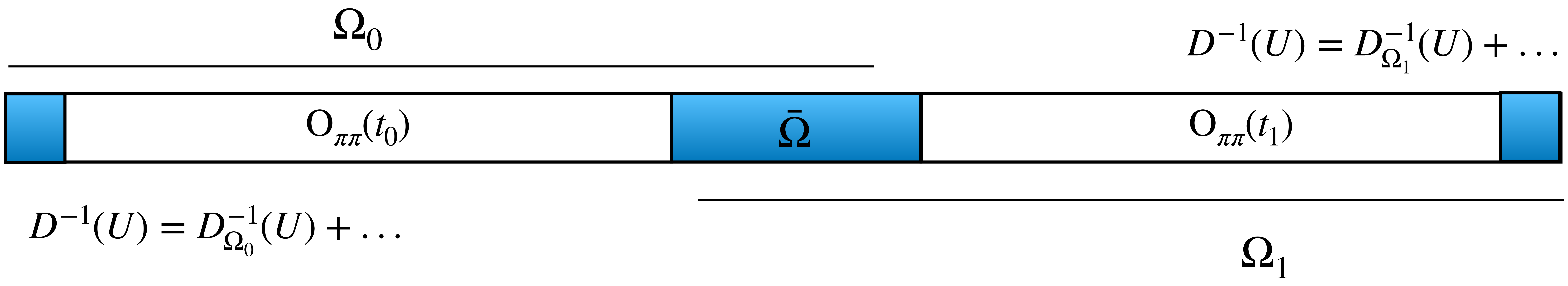
- Full factorisation of propagators in different regions allow multilevel integration

[arXiv:1601.04587]

L. Giusti, M. Cè, S. Schaefer, 2016

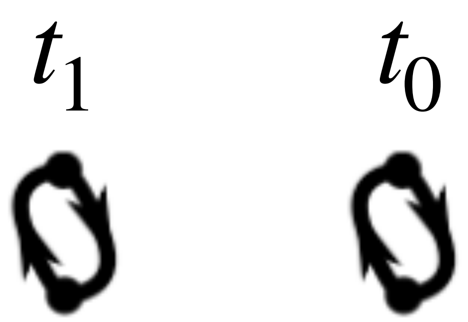
New approach:

- Propagators are numerically approximated on sub-domains of the full temporal extent



Two-level Samplings for Glueballs in Quenched QCD

This approximation allows to factorise the disconnected diagram



$$\langle O_{\pi\pi}(t_1)\bar{O}_{\pi\pi}(t_0)\rangle_{disc.} = \frac{1}{\mathcal{Z}} \int [dU] e^{-S[U]} \; \mathrm{Tr}\left[D^{-1}_{\Omega_1}(t_1, t_1) D^{-1}_{\Omega_1}(t_1, t_1)^{\dagger}\right] \; \mathrm{Tr}\left[D^{-1}_{\Omega_0}(t_0, t_0) D^{-1}_{\Omega_0}(t_0, t_0)^{\dagger}\right]$$

+ correction

Now amenable for multilevel integration:

$$= \frac{1}{\mathcal{Z}} \int [dU_B] e^{-S_B[U_B]} \; \left[O^{(1)}_{\pi\pi}(U_B, t_1)\right]_{\Omega_1} \left[O^{(0)}_{\pi\pi}(U_B, t_0)\right]_{\Omega_0} \qquad + \text{correction}$$

$$\left[O^{(r)}_{\pi\pi}(U_B, t)\right] = \int [dU^{(r)}] \; e^{-S_r[U^{(r)}, U_B]} \; \mathrm{Tr}\left[D^{-1}_{\Omega_r}(t, t) \; D^{-1}_{\Omega_r}(t, t)^{\dagger}\right]$$

(in practice we use distillation)

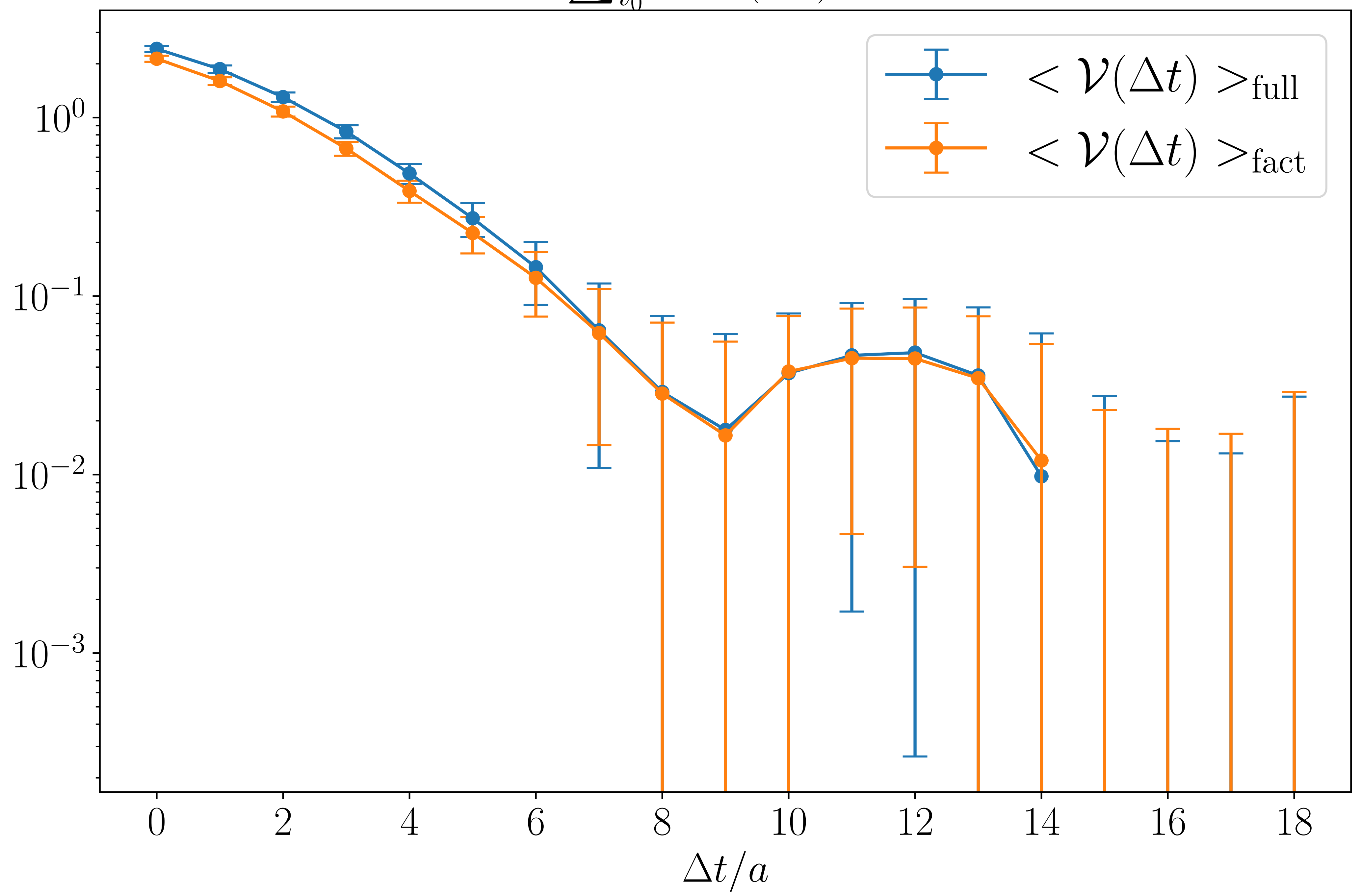
How good is the approximation?

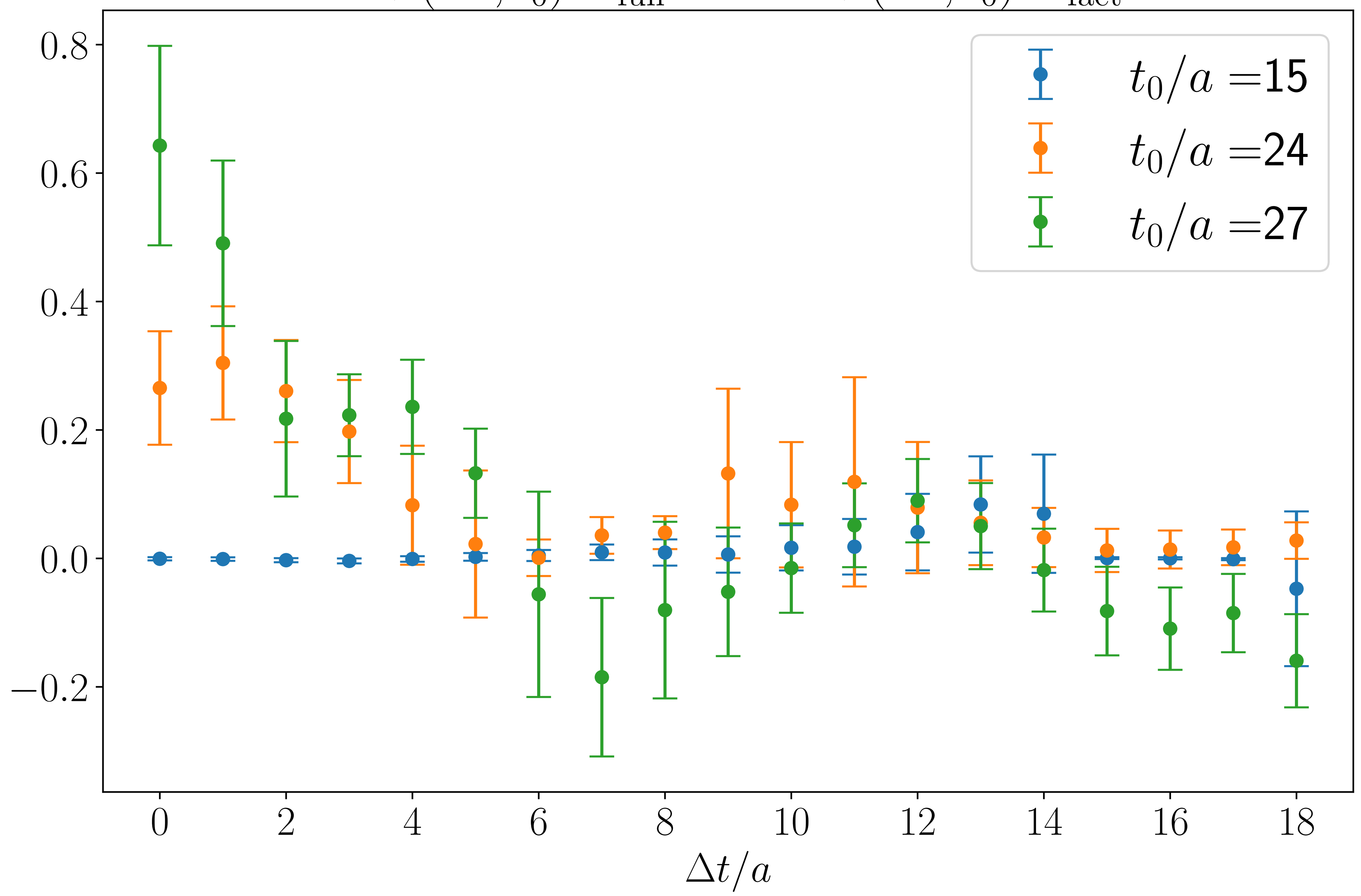
Matches well our use case

(multilevel for longer distances)

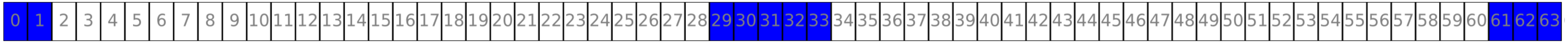
Longer the distance from the other region, better the approximation

$$< \mathcal{V}(\Delta t, t_0) > := < \mathrm{O}_{2\pi}(t_0 + \Delta t)\ \bar{\mathrm{O}}_{2\pi}(t_0) >_{disc}$$





current sub-lattice decomposition



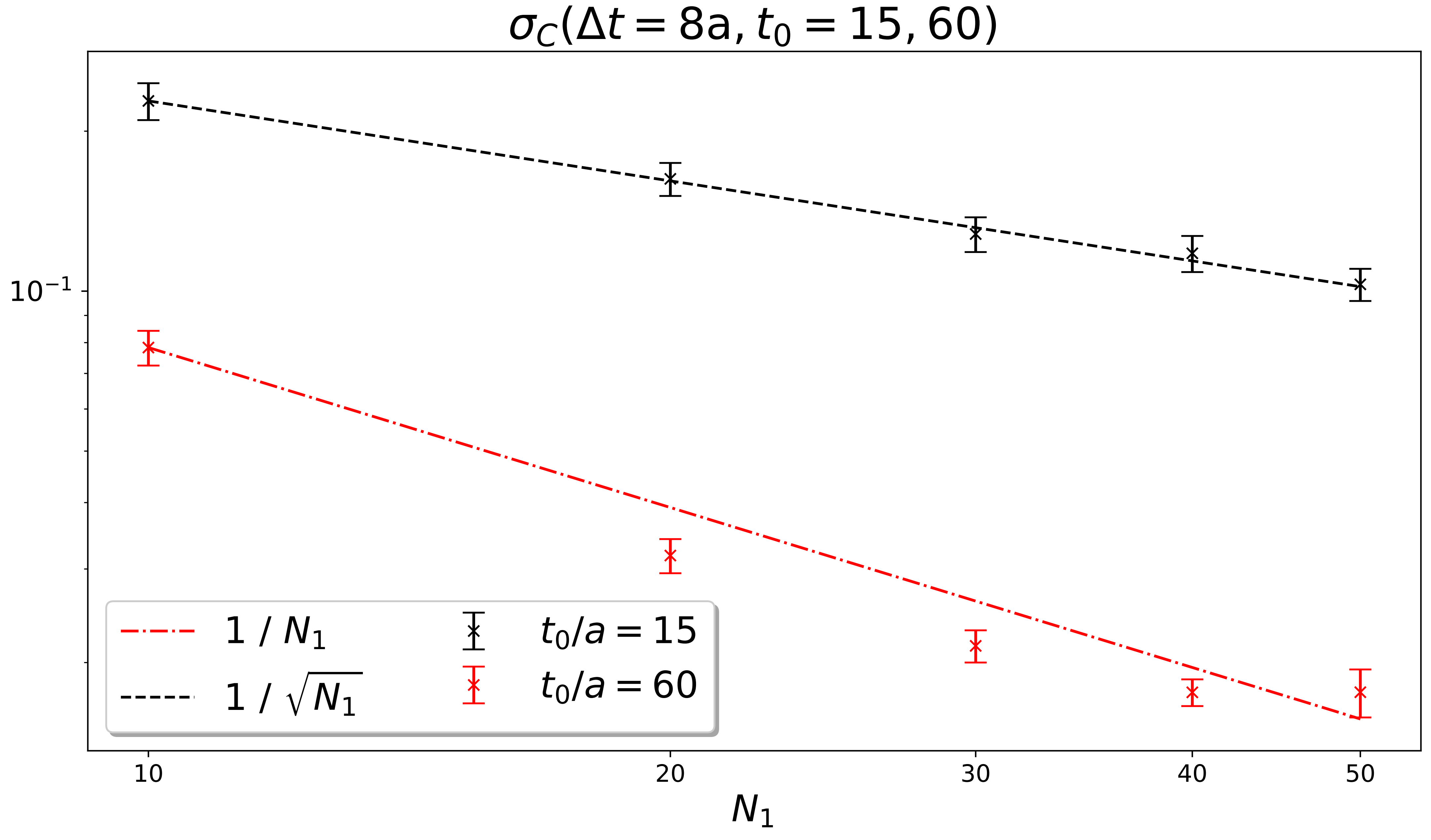
Multilevel Error Reduction

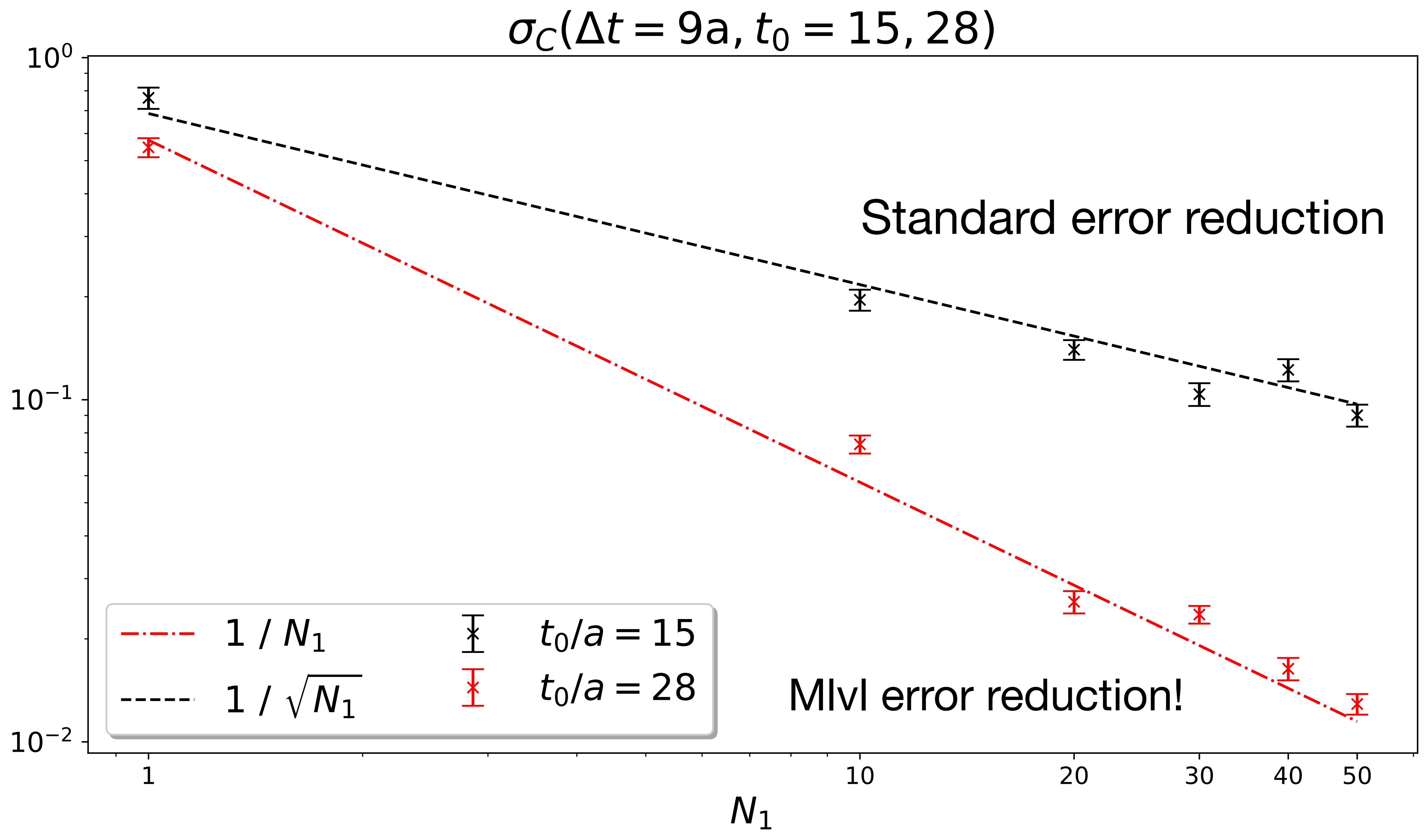
$m_\pi \approx 800$ MeV s.t. $2m_\pi \approx m_{f_0(1500)}$

$V/a^4 = 16^3 \times 64$ $N_0 = 20,\ N_1 = 50$

$$\sigma_C(\Delta t, t_0) = \text{ Error of } \langle \mathrm{O}_{\pi\pi}(t_0 + \Delta t)\ \bar{\mathrm{O}}_{\pi\pi}(t_0) \rangle$$

Preliminary





Error $\propto 1\ /\ N_1$ when the operators are located in different regions!

Conclusions

[arXiv: 2406.12656]

L.B., F. Knechtli, S. Martins, M. Peardon, S. Schaefer, J.A. Urrea-Niño

- ☑ Two-level sampling for pure gauge glueballs very effective and well understood
  - ☑ Fit ansatz describes short and long distance scalings, as well as transition point
  - ☑ GEVP effective masses agree with literature

- Two-level sampling for glueballs in quenched QCD
  - ☆ Approximating Dirac inversions to local domains enables multilevel measurements
  - ☑ Preliminary results show multilevel error reduction for $\langle O_{2\pi}(t)\bar{O}_{2\pi}(t_0)\rangle$, $\langle O_{2\pi}(t)W(t_0)\rangle$
  - ☑ Approximation improves with distance between operators

Next:

L.B., F. Knechtli, J. Finkenrath, M. Peardon, S. Schaefer, J.A. Urrea-Niño

- ☐ Study dependence of mlvl error reduction & approximation with sub-lattice decomposition, N_1, m_π
- ☐ Solve GEVP using both fermionic and gluonic operators (First quenched results ever afaik)