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Rethinking Quantum Field Theory

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Energy-momentum tensor in the 2d O(3) non-linear sigma model on the lattice

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Sigma models

- Two-dimensional non-linear sigma models play a role in a variety of models in statistical mechanics, as well as in string theory and the AdS/CFT correspondence.
- Consider scalar fields $\phi(x)$ in a $d = 2$ euclidean spacetime Σ mapped into S^N. The action is
$$S[\phi] = \frac{1}{2\lambda^2}\int_\Sigma d^2x\, \partial_\mu\phi^i\partial_\mu\phi^i$$
where the fields are subject to the constraint $\phi^i\phi^i = 1$. This is the $O(N)$ non-linear sigma model.
- For now, we consider $N = 3$. To generate a perturbative expansion we have to solve the constraint. We do this by parametrizing the field ϕ via
$$\phi = \{\sigma, \pi\} \quad \text{where} \quad \pi = (\pi^1, \pi^2),\ \sigma = (1 - \pi^2(x))^{1/2}.$$

Perturbation theory

- Using this parametrization we can formulate a perturbation expansion around $\pi = 0$. The action takes the standard form of a non-linear sigma model
$$S(\pi) = \frac{1}{2}\int d^2x\, G_{ij}(\pi(x))\, \partial_\mu\pi^i\partial_\mu\pi^j$$
in which $G_{ij}(\pi)$ is the well known metric tensor on the sphere $G_{ij} = \delta_{ij} + \frac{\pi_i\pi_j}{1-\pi^2}$
- We regulate infrared divergences by adding a term that breaks the non-linear symmetry to the action
$$S \to S - h\int_\Sigma d^d x\, \sigma(x)$$
- To obtain a generating functional for π field correlation functions we add sources $\boldsymbol{J}$ for the field π and H for the composite operator σ. It is then given by
$$\mathcal{Z}(J) = \mathcal{Z}(J, H)_{H(x)=h}$$

Renormalization of the energy momentum tensor (EMT)

- The energy momentum tensor characterizes the conformal anomaly, central charge and equation of state of the theory.
- It can be decomposed into three parts transforming in the respective irreps of symmetry of discretized spacetime D_4, given by (no sum over μ, ν)
$$O^1_{\mu\nu}(\phi) = \delta_{\mu\nu}\hat\partial_\eta\phi(x)\hat\partial_\eta\phi(x) \quad \text{(scalar)} \qquad O^2_{\mu\nu}(\phi) = \delta_{\mu\nu}(\hat\partial_\mu\phi(x)\hat\partial_\nu\phi(x) - \frac{1}{2}\hat\partial_\eta\phi(x)\hat\partial_\eta\phi(x)) \quad \text{(traceless diagonal)}$$
$$O^3_{\mu\nu}(\phi) = \hat\partial_\mu\phi(x)\hat\partial_\nu\phi(x) - \delta_{\mu\nu}\hat\partial_\mu\phi(x)\hat\partial_\nu\phi(x) \quad \text{(off-diagonal).}$$
- Quite generally, if $\mathcal{O}(\phi, x)$ is an operator of canonical dimension $[\mathcal{O}(\phi)] = D$, then it renormalizes as
$$\{\mathcal{O}(\phi, x)\}_r = \sum_{\alpha: D_\alpha = [\mathcal{O}_\alpha] \le D} Z^\alpha\mathcal{O}^\alpha(\phi, x),$$
where Z_α are renormalization constants. The operators $\mathcal{O}_\alpha(\phi, x)$ are all those that are compatible with the symmetries of $\mathcal{O}(\phi, x)$.
- We have a non-linearly realized coset group $O(3)/O(2)$ symmetry acting infinitesimally as $\quad \delta_\omega\pi_i = \omega_i(1 - \boldsymbol{\pi}^2(x))^{1/2} \quad \delta_\omega(1 - \boldsymbol{\pi}^2(x))^{1/2} = -\boldsymbol{\omega}\cdot\boldsymbol{\pi}(x).$
- This symmetry implies non-linear Ward identities, i.e. a Zinn-Justin equation that the effective action Γ is constrained by [1]. Finding a general solution to this equation will constrain operators that show up in the renormalization. This calculation must be done with the source $H(x)$, and after finding a solution one can take the limit $H(x) = h \to 0$.
- Solving the Zinn-Justin equation, we find that the renormalized EMT is of the form
$$T^r_{\mu\nu}(\phi) = Z\delta_{\mu\nu} + Z'\delta_{\mu\nu}\frac{1}{\sigma(x)}\check\partial_\mu\hat\partial_\mu\sigma(x) + Z^1O^1_{\mu\nu}(\phi) + Z^2O^2_{\mu\nu}(\phi) + Z^3O^3_{\mu\nu}(\phi).$$
- Evidently, there is an extra nontrivial operator mixing with the scalar part. Interestingly, it is not mainfestly $O(3)$ invariant but depends on the choice of parametrization.
- On the lattice, we are only able to probe expectation values of manifestly $O(3)$ invariant operators. This is however not a problem as in $O(3)$ invariant expectation values, we can exchange non-invariant operators and invariant ones. We employ Schwinger-Dyson equations and use local Ward identities to find a perturbation expansion independent form of the extra operator for any insertion of a manifestly $O(N)$ invariant operator O,
$$\frac{2}{3}\left\langle \phi_i(y)\frac{\delta O(\phi(y))}{\delta\phi_i(x)}\Big|_{y=x}\right\rangle + \langle O(\phi(x))\phi_i(x)\check\partial_\mu\hat\partial_\mu\phi_i(x)\rangle + \frac{2}{a^2}\langle O(\phi(x))\rangle = \left\langle O(\phi(x))\frac{1}{\sigma(x)}\check\partial_\mu\hat\partial_\mu\sigma(x)\right\rangle.$$
- We want to construct the non-perturbatively renormalized EMT by imposing Ward identities at fixed lattice spacing that hold up to cutoff effects which vanish in the continuum limit. For this, we use numerical simulations.

Simulations

- Simulations are on the way where we extract mixing coefficients by calculating one point EMT functions with shifted boundary conditions [3].
- We use a Wolff cluster algorithm with an optimized constraint action
$$S_{\text{con}}[\phi] = \sum_{x,\mu}\begin{cases}\beta\,(1 - \phi(x)^i\phi(x+\hat\mu)^i) & \text{for } \phi(x)^i\phi(x+\hat\mu)^i > -0.345\\ \infty & \text{else}\end{cases}$$
to reduce cutoff effects [4].

Outlook

- Carry out a detailed comparison of different methods for the non-perturbative renormalization. The use of shifted boundary conditions, Ward identities with probes at positive flowtime and the small flowtime expansion [5].
- Investigate the trace anomaly and calculate the central charge in the $O(2)$ model (conformal).
- Generalize the program to supersymmetric target spaces. [6].

References

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