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Matching Curved Lattices to Anisotropic Tangent Planes

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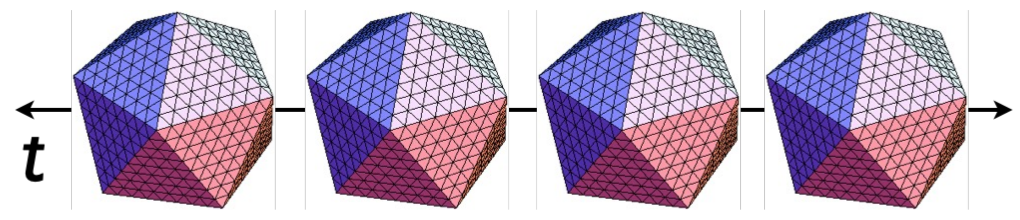
Near-Conformal Field Theories

- Strongly-coupled near-conformal field theories could be important for BSM physics.
- Example: composite Higgs boson $H \sim \bar{Q}Q$, $v \sim \langle \bar{Q}Q \rangle$.
- This implies a composite Yukawa mechanism to give mass to SM fermions: $y_f \langle \bar{Q}Q \rangle \bar{f}f / \Lambda^2$
- But, this also leads to flavor changing neutral currents $(\bar{f}f)(\bar{f}f)/\Lambda^2$ which requires $\Lambda > 1000\ TeV$.
- So, composite Higgs theory must be strongly-coupled over a range of 0.1 – 1000 TeV.
- Very hard to study on hypercubic flat torus. See talks by A. Hasenfratz and O. Witzel on Friday.

Radial Quantization



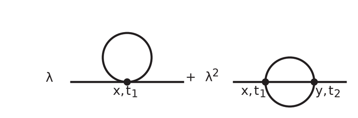
- Eigenstates of **Dilatation operator** defined on surfaces of constant radius.
- Eigenstates labeled by angular momenta (ℓ, m_ℓ) due to rotational invariance.
- Dynamical dispersion relation (conformal):
$$\Delta_{\mathcal{O},\ell} = \Delta_{\mathcal{O},0} + \ell$$
- Correlations (conformal):
$$C(\ell, t, t') = \sum_{\mathcal{O}} B(\Delta_{\mathcal{O}}, \ell)\, e^{-\Delta_{\mathcal{O},\ell}|t-t'|}$$
- Near-conformal would modify integer spacing and t-dependence.
- **Challenge**: How to define action on irregular spherical lattice that has rotational symmetry in continuum limit?

Quantum Finite Elements

- **Limited Solution**: Finite Element Method (FEM) gives classically perfect action. QFE adds perturbative counterterms.
- Method worked for critical 3D ϕ^4 theory but very slow convergence to continuum limit.
- Also, discovered a novel coupling to local curvature density, $Ric(x)\phi^2(x)$, which further slowed convergence $\sim \mathcal{O}(a^{0.41})$.
- **Lesson 1**: Adjust lattice so curvature density is uniform a la Regge calculus.
- **Lesson 2**: Need a better method to define lattice action which is closer to strongly coupled IR fixed point.

$$S = \frac{1}{2} \sum_{y \in \langle x,y \rangle} \frac{l^*_{xy}}{l_{xy}} \left(\tilde{\phi}_{t,x} - \tilde{\phi}_{t,y}\right)^2 + \frac{a^2}{4R^2}\sqrt{\tilde{g}_x}\tilde{\phi}^2_{t,x} + \sqrt{\tilde{g}_x}\left[\frac{a^2}{a_t^2}\left(\tilde{\phi}_{t,x} - \tilde{\phi}_{t+1,x}\right)^2 + m_0^2\tilde{\phi}^2_{t,x} + \lambda_0\tilde{\phi}^4_{t,x}\right]$$

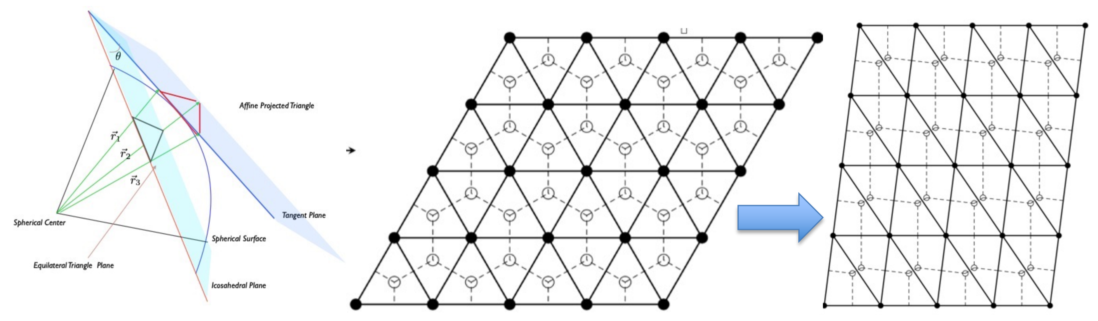
$$S_{QFE} = S - \sum_{t,x} \sqrt{\tilde{g}_x}\left[6\lambda_0 \delta G_x - 24\lambda_0^2 \delta G_x^{(3)}\right]\tilde{\phi}^2_{t,x}.$$



Beyond QFE: Affine Conjecture

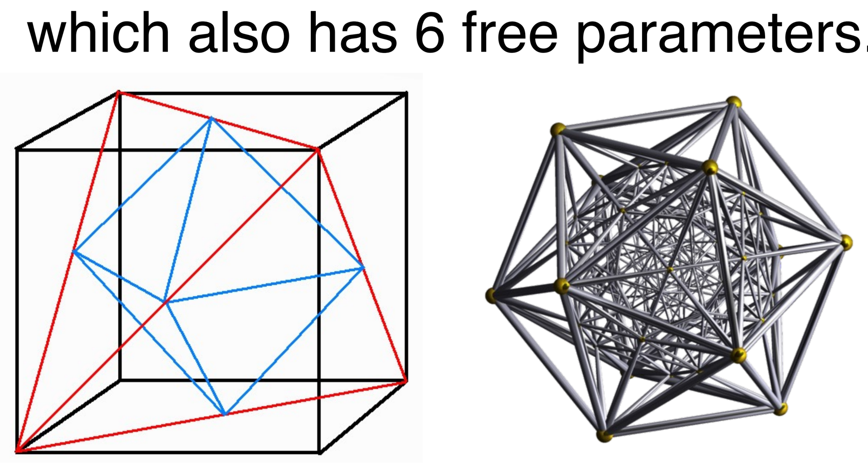
See Talk by R. Brower on Tues.

- Start with uniform simplicial graph on a refined regular (D+1)-polytope (e.g. Icosahedron or 600-cell)
- Project vertices to $\mathbb{S}^D$. Optimize vertex positions to uniform curvature density (Regge calculus) while preserving graph structure and isometries of polytope.
- Each D-simplex no longer has uniform edge lengths but still defines a "tangent" plane.
- Tesselate each tangent plane with an asymmetric simplicial honeycomb (A_D root lattice) using edge lengths of associated D-simplex.
- **Challenge**: In the tangent plane, find the anisotropic bare lattice action that dynamically produces the desired ratios of edge lengths.
- That tangent plane action is the lattice action for the associated D-simplex on the $\mathbb{S}^D$.
- Proof of principle: critical Ising model on $\mathbb{S}^2$, E. Owen and R. Brower, 2023.



Specific Goal for This Work

- Solve the critical D=3 Ising model on a general anisotropic face-centered cubic (FCC, aka A_3root lattice).
- The isotropic FCC case has been solved many times: P.H.Lundow et al 2009, U. Yu 2015.
- Under the *affine conjecture*, a general solution would enable critical Ising model calculations on discretized $\mathbb{S}^3$ starting from 600-cell (higher-dimensional icosahedron) and tessellating each regular tetrahedral cell with an FCC lattice.
- Note a general anisotropic FCC lattice has 6 unique lengths and any lattice can be transformed to the isotropic FCC lattice by affine transformation which also has 6 free parameters.

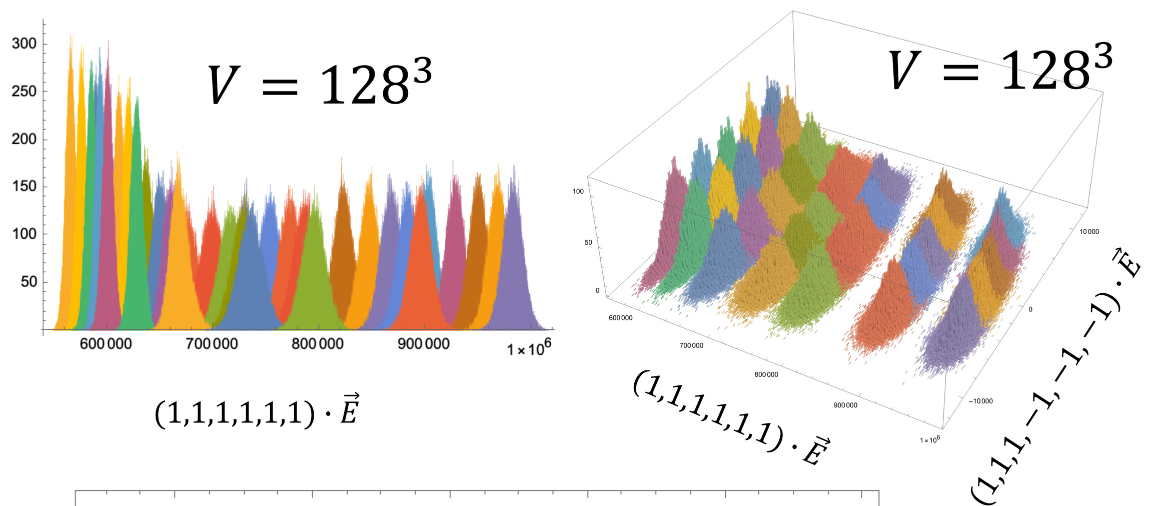


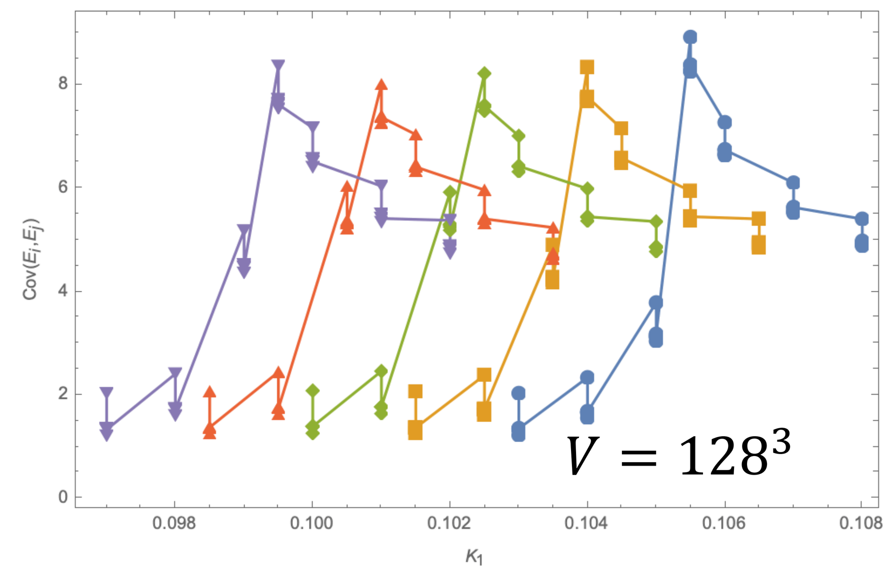
Generalized FCC Ising model

- Affine FCC partition function
$$Z(K_1, \cdots, K_6) = \sum_{s_n = \pm 1} e^{K_1 s_n s_{n+\hat{1}} + \cdots + K_6 s_n s_{n+\hat{6}}},$$
$$E_{n,\hat{e}} = -s_n s_{n+\hat{e}}$$
- Multihistogram master equation (solved iteratively)
$$Z_k = \sum_{r=1}^{R} \sum_{i=1}^{N_r} \frac{1}{\sum_{j=1}^{R} N_j\, Z_j^{-1}\, e^{(\vec{K}_k - \vec{K}_j)\cdot \vec{E}_{r,i}}}$$
where R is number of runs, N_r is length of run r, $\vec{E}_{r,i}$ are energies on i-th configuration of run r.
- Observables for any other nearby $\vec{K}$:
$$\langle \mathcal{O}(\vec{K}) \rangle = \frac{1}{Z(\vec{K})} \sum_{r=1}^{R} \sum_{i=1}^{N_r} \frac{\mathcal{O}_{r,i}}{\sum_{j=1}^{R} N_j\, Z_j^{-1}\, e^{(\vec{K} - \vec{K}_j)\cdot \vec{E}_{r,i}}}$$
$$Z(\vec{K}) = \sum_{r=1}^{R} \sum_{i=1}^{N_r} \frac{1}{\sum_{j=1}^{R} N_j\, Z_j^{-1}\, e^{(\vec{K} - \vec{K}_j)\cdot \vec{E}_{r,i}}}$$

First Test

- First test: $\frac{K_2}{K_1} = \frac{K_3}{K_1} = 1$, $\frac{K_4}{K_1} = \frac{K_5}{K_1} = \frac{K_6}{K_1} \in \{0.94, 0.97, 1.00, 1.03, 1.06\}$
- K_1 is tuned close to critical point.
- Solve multihistogram consistency condition for all 35 runs, each run $N_r = 50{,}000$ configs.





- **Important**: all susceptibilities peak at same critical coupling.
- Using multihistogram reweighting, find critical surface $\vec{K}_{crit}$ by identifying peak in $Cov(\vec{E}, \vec{E})$. In general, it is a 5-d surface with permutation symmetry.
- Then, along critical surface compute two-point function:
$$\langle s(\vec{x}) s(0) \rangle = \frac{1}{\left(x_i\, G_{ij}(\vec{K}_{crit})\, x_j\right)^{\Delta_\sigma}}$$

References for Further Reading

Brower et al, PRD 104 (2021) 094502.
Glück et al, PRD 109 (2024) 114518.
Brower and Owen, PRD 108 (2023) 014511.
Brower and Owen, arXiv:2407.00459 [hep-lat].
Ferrenberg-Swendsen, PRL 63 (1989) 1195.

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