Nested Sampling for U(1) in $2+1$ dimensions

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Outline

Introduction

Nested sampling

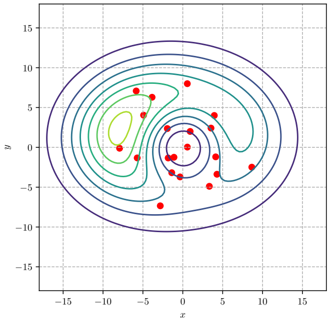
$U(1)$ gauge theory in 2+1 dimensions

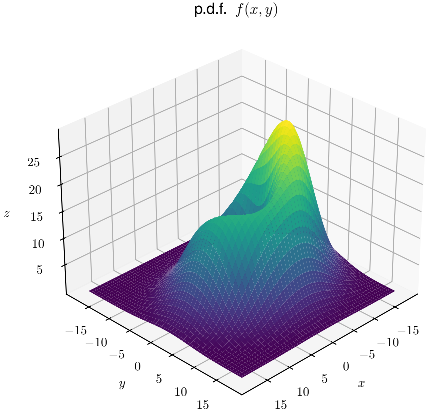
Results and Discussion

Conclusion

Introduction

- What is nested sampling?
- Strenghts and limitations
- Application to a $U(1)$ gauge theory





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Nested Sampling algorithm (part 1/4)

Origin: Bayesian statistics, calculation of the evidence p.d.f.

Evidence p.d.f.

$$Z = \int \mathcal{D}\phi \mathcal{L}[\phi] = \int d^n\omega \, \pi(\omega)\mathcal{L}(\omega) \approx \lim_{\Delta\omega \to 0} \sum_i \Delta\omega \pi(\omega_i)\mathcal{L}(\omega_i)$$

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Multi-dimensional integral: change of variable

Naive idea:

- We draw the hyper-contours where

$$P(\omega_i) = \pi(\omega_i)\mathcal{L}(\omega_i) = \text{const.}$$

- The hyper-volumes separating the contours are ΔX_i

$$Z \approx \sum_i \Delta X_i P(X_i)$$

Nested Sampling algorithm (part 2/4)

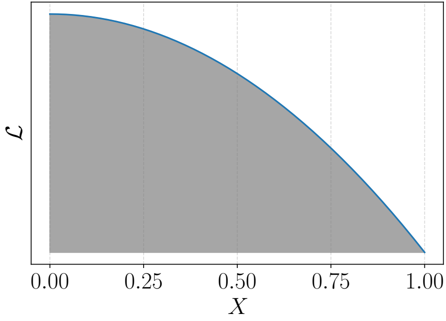
Multi-dimensional integral: change of variable

Rigorously:

$$X(L^*) = \int_{L^* > L} \pi(\omega) d\omega$$

The evidence reads:

$$Z = \int_0^1 dX \mathcal{L}(X)$$



Remarks:

- By construction,

$$L(X = 1) = 0$$

- $\mathcal{L}$ is monotonically decreasing with X

Nested Sampling algorithm (part 2/4)

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Application to lattice field theories

- The Likelyhood $\mathcal{L}$ is the Boltzmann factor
- The prior is the d.o.f. measure (e.g. Haar)
- Z is the partition function

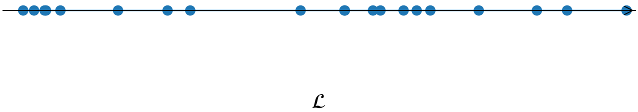
$$\langle \mathcal{O} \rangle = \int \mathcal{D}\phi \, \mathcal{O}[\phi] e^{-\beta S[\phi]}$$

Nested Sampling algorithm (part 3/4)

Steps of the algorithm

1. Draw n_{live} points randomly in the parameter space
2. Find the one corresponding to $L^* = \min\{\mathcal{L}\}$
3. Save L^*, remove from the n_{live} points, and draw another one from the constrained prior:

$$\pi^*(\omega) = \Theta\left(L(\omega) > L^*\right)\, \pi(\omega)$$



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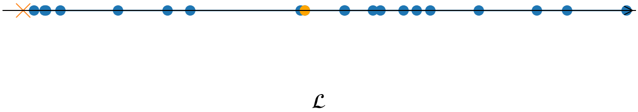

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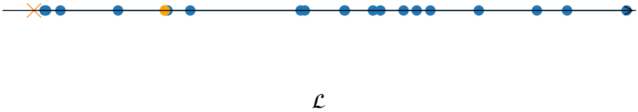

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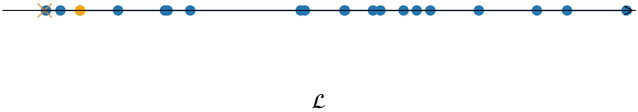

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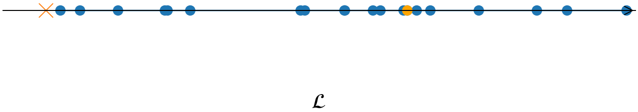

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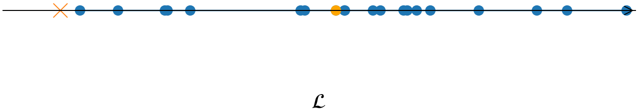

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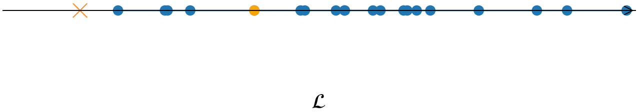

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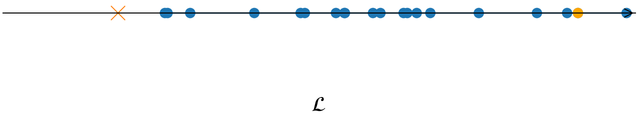

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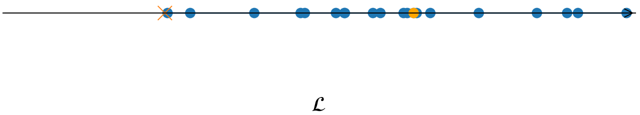

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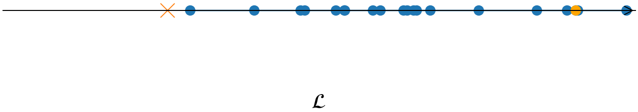

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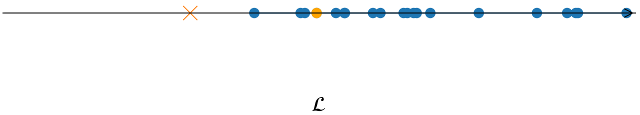

Nested Sampling algorithm (part 3/4)

Steps of the algorithm

1. Draw $n_{\rm live}$ points randomly in the parameter space
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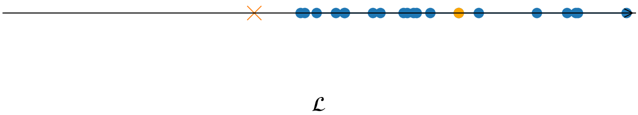

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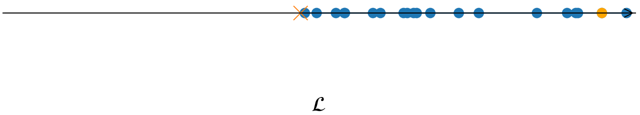

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$\mathcal{L}$

Nested Sampling algorithm (part 4/4)

At each step we restrict the phase space:

$$X_i = t_i X_{i-1} \, , \, X_0 = 1$$

How do we estimate the volume elements?
$\rightarrow$ compression factor t follows a β distribution:

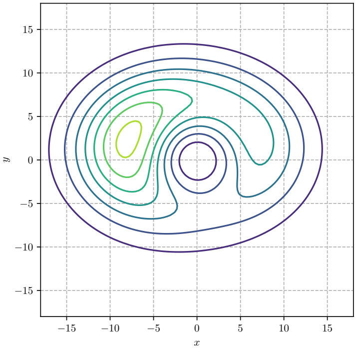
$$\beta(t) \sim t^{n_{\text{live}}-1}$$

Volume elements

$$\Delta X_i = \frac{1}{2} \left(X_{i-1} - X_{i+1} \right)$$

We can estimate it as:

- $\log t \approx \langle \log t \rangle = -1/n_{\text{live}}$
- t drawn from a β p.d.f.



Applications to field theory

Recap on Monte Carlo integration

- Path integral is approximated by Monte Carlo (ϕ_i sampled according to $P[\phi]$):

$$\int \mathcal{D}\phi \, P[\phi] O[\phi] \approx \frac{1}{N} \sum_{\phi_i} O[\phi_i]$$

- Statistical error scaling as $1/\sqrt{N}$

Note: We sample only the values of the action $S[\phi]$.

Generic observable: Bad approximation if $O[\phi]$ level do not overlap with $P[\phi]$

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Sampling configurations with NS

- Control variable: X
- Overlap can be controlled: Likelyhood VS observable

Sampling configurations with NS

$$Z = \int dX \mathcal{L}(X) = \int dS \rho(S) \mathcal{L}(S)$$

- $\rho(S) = dX/dS$
- $\mathcal{L}(S) = e^{-\beta S}$

Remark: $\rho(S)$ is determined independently of β!

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Applications

- Phase transitions
- Topological unfreezing

For further discussion, see talks by U. Wenger and G. Kanwar

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$U(1)$ gauge theory in 2+1 dimensions

$$\beta S = \beta \sum_{x} \sum_{\mu<\nu} \mathrm{Re}\left(1 - P_{\mu\nu}(x)\right) \tag{1}$$

$$P_{\mu\nu}(x) = U_\mu(x) U_\nu(x+\mu) U^\dagger(x+\nu) U^\dagger(x)$$

Main features

- Confinement at all couplings (Dirac monopoles)
- Static quark potential

$$V(r) = a + b\log(r) + \sigma R$$

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Geometry considered here

$$L/a = 4 \quad , \quad T/a = 8 \quad , \quad d = 3$$

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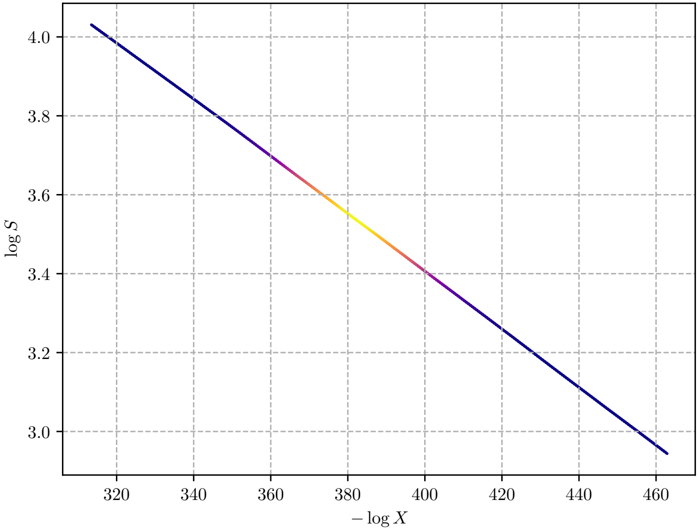
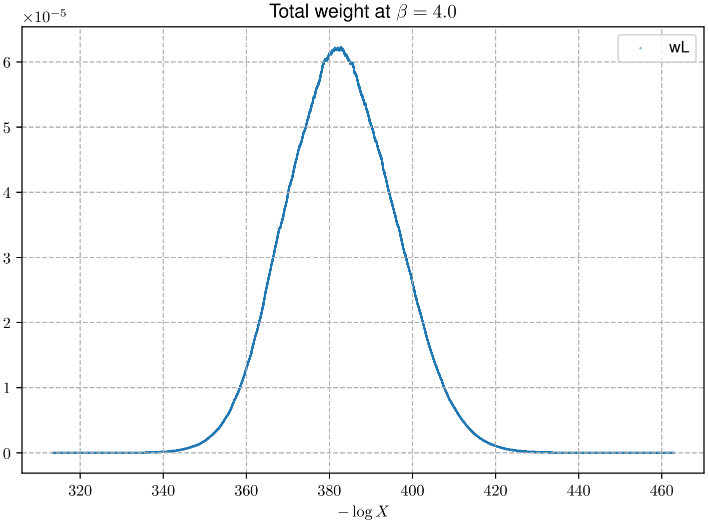
Gauge action through the phase space

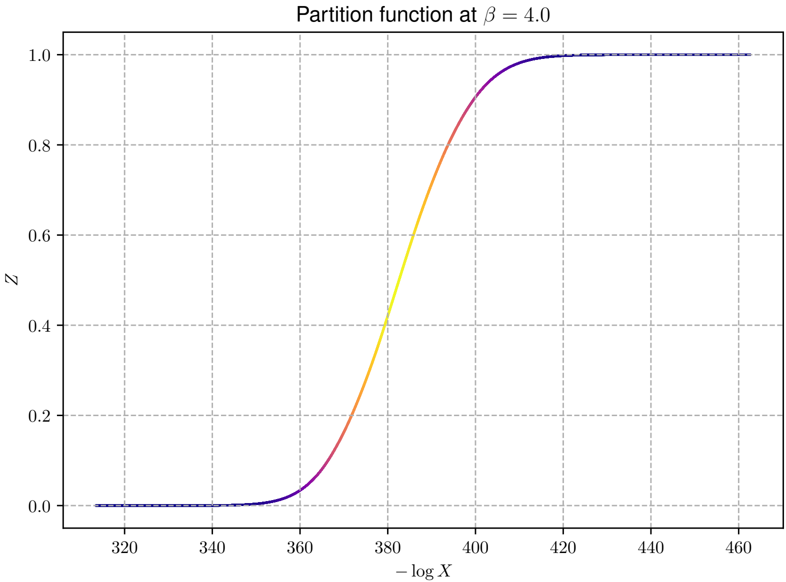


Total weights

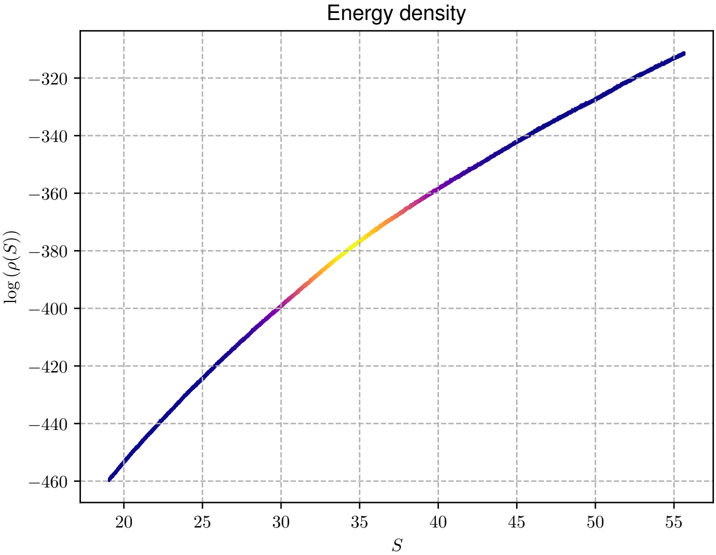
$$Z = \sum_i w_i \mathcal{L}_i$$



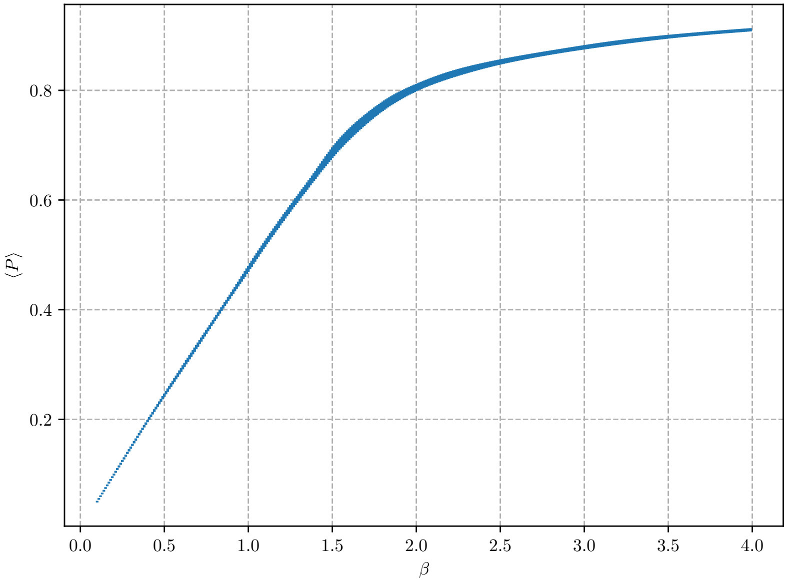
Partition function



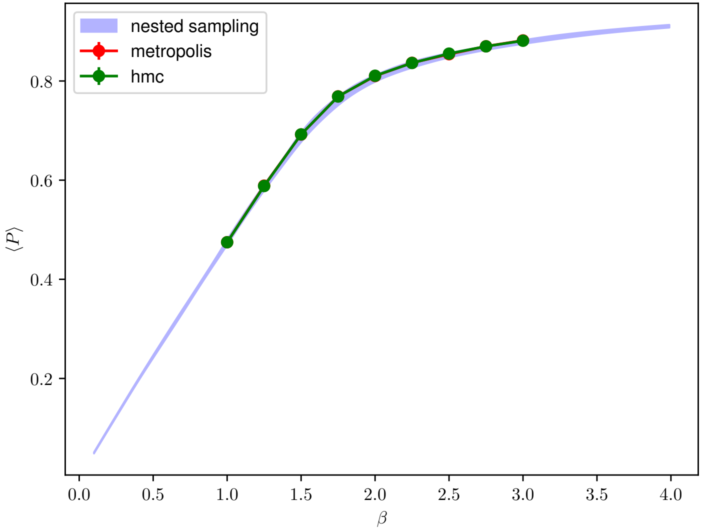
Density of states



Plaquette expectation value



Comparison with Metropolis and HMC



Conclusion

- Application of NS to pure gauge $U(1)_{2+1}$
- Compatibililty with Metropolis and HMC

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Future directions

- Anisotropic actions:

$$\beta S = \frac{\beta}{\xi_0} \sum_{x,i} \mathrm{Re}\,(1 - P_{0i}(x)) + \beta \xi_0 \sum_{x,i>j} \mathrm{Re}\,(1 - P_{ij}(x))$$

  scan β dependence at fixed ξ_0
- U(1) in $3+1$ dimensions: bulk phase transition
- SU(N) gauge theories

Thank you for your attention!