The Cabibbo Angle from Inclusive τ Decays

Giuseppe Gagliardi, INFN & Università di Roma Tre

On behalf of the ETM Collaboration

Lattice 2024, Liverpool

The talk is based on:

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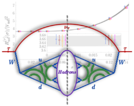
Inclusive Hadronic Decay Rate of the τ Lepton from Lattice QCD: The $\bar{u}s$ Flavor Channel and the Cabibbo Angle

Constantia Alexandrou, Simone Bacchio, Alessandro De Santis, Antonio Evangelista, Jacob Finkenrath, Roberto Frezzotti, Giuseppe Gagliardi, Marco Garofalo, Bartosz Kostrzewa, Vittorio Lubicz, Simone Romiti, Francesco Sanfilippo, Silvano Simula, Nazario Tantalo, Carsten Urbach, and Urs Wenger (Extended Twisted Mass Collaboration)

Phys. Rev. Lett. **132**, 261901 – Published 25 June 2024



EDITORS' SUGGESTION

Inclusive hadronic decay rate of the τ lepton from lattice QCD

The authors express the inclusive hadronic decay rate of the tau lepton as an integral over the spectral density of the two-point correlator of the weak $V - A$ hadronic current which they compute fully nonperturbatively in lattice QCD. In a lattice QCD computation with all systematic errors except for isospin breaking effects under control, they then obtain the CKM matrix element V_{ud} with subpercent errors showing that their nonperturbative method can become a viable alternative to superallowed nuclear beta decays for obtaining V_{ud}.

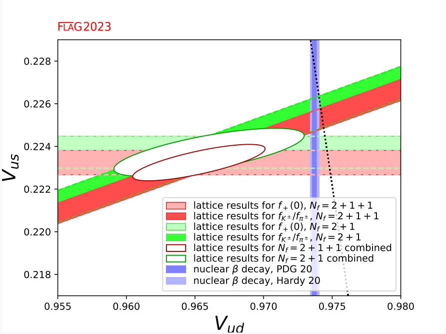
A. Evangelista *et al.*

Phys. Rev. D **108**, 074513 (2023)

Unitarity tests of the CKM matrix V_{CKM}

Checking the unitarity of V_{CKM} is an important test of the SM. E.g. for the first row:

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 \ (?)$$



Most precise determinations of $V_{ud(s)}$ from:

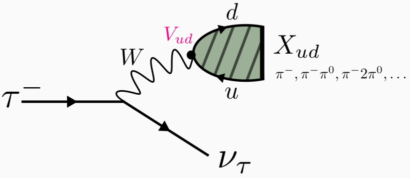
- $|V_{ud}|$: superallowed $\beta-$decay (0.3 ppt), semileptonic decay $\pi^+ \to \pi^0 e \nu_e$ (0.3%).
- $|V_{us}|$: semileptonic decays $K \to \pi \ell \nu_\ell$ (0.3%).
- $|V_{us}|/|V_{ud}|$: ratio of leptonic $K^\pm$ and $\pi^\pm$ decays $K/\pi \to \ell \nu_\ell(\gamma)$ (0.2%).

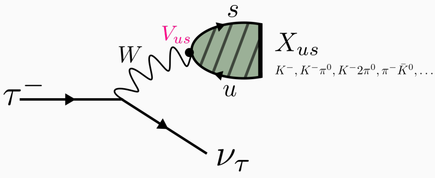
$$\Gamma[K^+ \to \pi \ell \nu_\ell] \propto |V_{us}|^2 |f_+(0)|^2 \cdot (1 + \delta^\ell_{K\pi}), \qquad \frac{\Gamma[K^\pm \to \ell \nu_\ell(\gamma)]}{\Gamma[\pi^\pm \to \ell \nu_\ell(\gamma)]} \propto \frac{|V_{us}|^2}{|V_{ud}|^2} \left(\frac{f_K^+}{f_\pi^+} \right)^2 \cdot (1 + \delta_{K\pi})$$

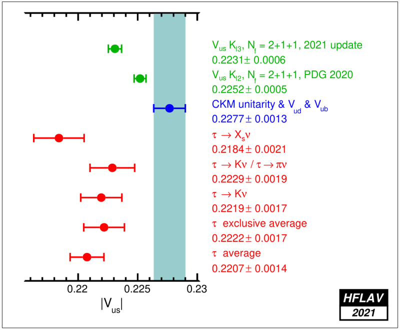
$\boxed{\text{Tension between } |V_{us}| \text{ from leptonic and semileptonic decays \& } |V_{us}^{\mathrm{uni}}|}$

$\tau \to X_{ud/us}\nu_\tau$ enters the game...

The hadronic decays of the τ lepton, i.e. $\tau \to X_{ud}\nu_\tau$, $\tau \to X_{us}\nu_\tau$ provide an alternative way to determine $|V_{ud}|$ and $|V_{us}|$:



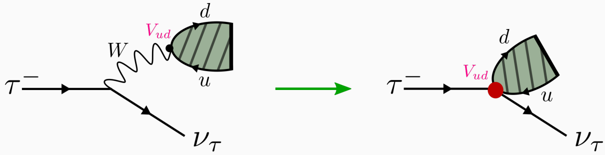




- The lower value of $|V_{us}|$ from $\bar{u}s$ inclusive τ decays is the so-called τ-puzzle.
- Inclusive $\tau \to X_{us}\nu_\tau$ result on the left plot obained using truncated operator-product-expansion (OPE).
- Exclusive-channel $\tau^\pm \to K^\pm\nu_\tau$ determination of $|V_{us}|$ larger than inclusive (but $\lesssim |V_{us}^{\mathrm{uni}}|$).

Calculation of the inclusive τ-decay rate

At lowest-order in the Fermi effective theory one has (e.g. for ud channel):



$$H_{ud}^{\text{eff}}(x) = \frac{G_F}{\sqrt{2}} V_{ud}\, S_{\text{EW}}^{1/2} \; \underbrace{\bar{\nu}_\tau(x)\gamma^\alpha(1-\gamma^5)\tau(x)}_{J^\alpha_{\nu_\tau\tau}(x)} \times \underbrace{\bar{d}(x)\gamma^\alpha(1-\gamma^5)u(x)}_{(J^\alpha_{ud})^\dagger(x)} + \text{h.c.}$$

G_F is Fermi constant, $S_{\text{EW}} = 1 + \mathcal{O}(\alpha_{\text{em}}) = 1.0201(3)$ a SD EW correction.

The amplitude $\mathcal{A}(\tau \to X_{ud}\nu_\tau) \equiv \langle \nu_\tau X_{ud} | \tau \rangle_{\text{SM}}$ is given in the EFT by:

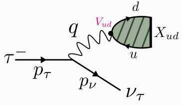
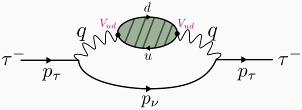
$$\begin{aligned}
\mathcal{A}(\tau \to X_{ud}\nu_\tau) &= \frac{G_F}{\sqrt{2}} V_{ud}\, S_{\text{EW}}^{1/2} \; \langle X_{ud}\, \nu_\tau |\, J^\alpha_{\nu_\tau\tau}(0) J^\alpha_{ud}(0)^\dagger \,| \tau \rangle \\
&= \frac{G_F}{\sqrt{2}} V_{ud}\, S_{\text{EW}}^{1/2} \; \langle \nu_\tau |\, J^\alpha_{\nu_\tau\tau}(0) \,| \tau \rangle \; \langle X_{ud} |\, J^\alpha_{ud}(0)^\dagger \,| 0 \rangle
\end{aligned}$$

For the inclusive rate we need $|\mathcal{A}|^2 = \sum_{X_{ud}} |\mathcal{A}(\tau \to X_{ud}\,\nu_\tau)|^2 \ldots$

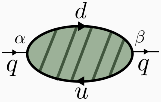
Inclusive rate from the optical theorem

$$\boxed{\Gamma[\tau \to X_{ud}\nu_\tau] = \frac{1}{2\,m_\tau}\int_\Phi \frac{\mathrm{d}^3 p_\nu}{(2\pi)^3\,2E_\nu}|\mathcal{A}|^2 = \frac{1}{2m_\tau}2\mathrm{Im}[\Gamma_{\tau\tau}]}$$

$\Gamma_{\tau\tau} = \langle\tau|T|\tau\rangle$ is the forward amplitude ($\mathrm{S} = 1 + iT$) due to interactions with flavoured $ud(us)$ states only.

$\sum_{X_{ud}} \Big|$  $\Big|^2 = \frac{1}{2m_\tau}\,2\mathrm{Im}$ 

Källén-Lehmann representation of hadronic-vacuum-polarization (HVP):



$$\equiv \Pi^{\alpha\beta}_{ud}(q) = i\int \mathrm{d}^4x\, e^{iqx}\,\langle 0|\mathrm{T}\left\{J^\alpha_{ud}(x)J^\beta_{ud}(0)^\dagger\right\}|0\rangle$$

$$= -\int_0^\infty \frac{\mathrm{d}M^2}{2\pi}\frac{\rho^{\alpha\beta}_{ud}(q_M)}{q^2 - M^2 + i\varepsilon}\,, \qquad q_M = (\sqrt{\boldsymbol{q}^2 + M^2}, \boldsymbol{q})$$

$$\boxed{\rho^{\alpha\beta}_{ud}(q) = (2\pi)^4\langle 0|J^\alpha_{ud}(0)\delta^4(\mathcal{P} - q)J^\beta_{ud}(0)^\dagger|0\rangle \qquad \text{spectral density}}$$

The inclusive decay rate

Exploiting Lorentz invariance $\rho_{ud}^{\alpha\beta}(q)$ decomposed into a longitudinal (L) and a transverse component (T)

$$\rho_{ud}^{\alpha\beta}(q) = \left(q^\alpha q^\beta - q^2 g^{\alpha\beta}\right) \rho_{\mathrm{T}}(q^2) + q^\alpha q^\beta \rho_{\mathrm{L}}(q^2)$$

The decomposition allows to evaluate the inclusive rate $\Gamma[\tau \to X_{ud}\nu_\tau]$ as

$$\Gamma[\tau \to X_{ud}\nu_\tau] = \frac{G_F^2 \, |V_{ud}|^2 S_{\mathrm{EW}} \, m_\tau^5}{32\pi^2} \int_0^1 \mathrm{d}s \, (1-s)^2 \left[(1+2\,s) \overbrace{\rho_{\mathrm{T}}(s) + \rho_{\mathrm{L}}(s)}^{s=q^2/m_\tau^2}\right]$$

It is convenient to normalize the inclusive hadronic decay rate of the τ over

$$\Gamma[\tau \to e\bar{\nu}_e\nu_\tau] = \frac{G_F^2 \, m_\tau^5}{192\,\pi^3}$$, obtaining the elegant relation

$$\boxed{R_{ud}^{(\tau)} = \frac{\Gamma(\tau \to X_{ud}\,\nu_\tau)}{\Gamma(\tau \to e\,\bar{\nu}_e\,\nu_\tau)} = 6\pi \, S_{\mathrm{EW}} \, |V_{ud}|^2 \int_0^1 \mathrm{d}s \, (1-s)^2 \left[(1+2\,s)\,\rho_{\mathrm{T}}(s) + \rho_{\mathrm{L}}(s)\right]}$$

The inclusive decay rate on the lattice

The quantity we can directly access in lattice QCD are Euclidean time-dependent correlation functions. The primary input for e.g. the ud channel is

$$\boxed{C^{\alpha\beta}\,(t,\boldsymbol{q}) = \int d^3x\, e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}\, \langle 0|\; T\left(J^{\alpha}_{ud}(-it,\boldsymbol{x})\, J^{\beta}_{ud}(0)^{\dagger}\right)\; |0\rangle}$$

Q : How is the spectral density $\rho^{\alpha\beta}_{ud}(q)$ related to the Euclidean correlator $C^{\alpha\beta}(t,\boldsymbol{q})$?

A : $\rho^{\alpha\beta}_{ud}$ is related to $C^{\alpha\beta}$ through an inverse Laplace transform (LT) [BACKUP]

$$\boxed{C^{\alpha\beta}(t,\boldsymbol{q}) \overset{t\geq 0}{=} \int_0^{\infty} \frac{\mathrm{d}E}{2\pi}\, e^{-Et}\, \rho^{\alpha\beta}_{ud}(E,\boldsymbol{q})}$$

⚠ Finding the inverse LT of a function affected by uncertainty and known on a finite set of points, as in a typical lattice calculation, is an ill-posed problem.

Do we really need $\rho_{ud}^{\alpha\beta}(E, q)$?

To compute:

$$R_{ud}^{(\tau)} = 6\pi S_{\mathrm{EW}} |V_{ud}|^2 \int_0^1 \mathrm{d}s \, (1-s)^2 \left[(1 + 2\,s)\, \rho_{\mathrm{T}}(s) + \rho_{\mathrm{L}}(s) \right]$$

$$= 12\pi S_{\mathrm{EW}} \frac{|V_{ud}|^2}{m_\tau^3} \int_0^\infty \mathrm{d}E \left[K_{\mathrm{T}} \left(\frac{E}{m_\tau} \right) E^2 \rho_{\mathrm{T}}(E^2) + K_{\mathrm{L}} \left(\frac{E}{m_\tau} \right) E^2 \rho_L(E^2) \right]$$

we actually "only" need the convolution of the longitudinal and transverse spectral densities $E^2 \rho_L(E^2)$ and $E^2 \rho_T(E^2)$ with the kernel functions

$$K_{\mathrm{L}}(x) \equiv \frac{1}{x} \left(1 - x^2 \right)^2 \theta(1-x) \,, \qquad K_{\mathrm{T}}(x) \equiv \left(1 + 2x^2 \right) K_L(x)$$

θ-function needed to implement the closure of the phase-space at $E = m_\tau$.

$$\boxed{C_{\mathrm{L}}(t) \equiv C^{00}(t, \mathbf{0}) = \int_0^\infty \frac{dE}{2\pi} e^{-Et} \rho_{\mathrm{L}}(E^2) E^2}$$

$$\boxed{C_{\mathrm{T}}(t) \equiv \frac{1}{3} C^{ii}(t, \mathbf{0}) = \int_0^\infty \frac{dE}{2\pi} e^{-Et} \rho_{\mathrm{T}}(E^2) E^2}$$

Evaluating the convolution integral ($\mathrm{I} = \{T, L\}$)

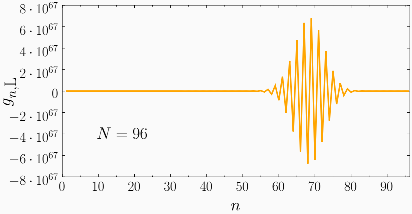
To evaluate directly R^{τ}_{ud} we can approximate $K_{\mathrm{I}}(\frac{E}{m_\tau}) \simeq \sum_{n=1}^{N} g_{n,\mathrm{I}} e^{-naE}$, and then evaluate $R^{(\tau)}_{ud} = R^{(\tau,\mathrm{L})}_{ud} + R^{(\tau,\mathrm{T})}_{ud}$ using $\left(B = 12\pi S_{\mathrm{EW}} \frac{|V_{ud}|^2}{m_\tau^3}\right)$:

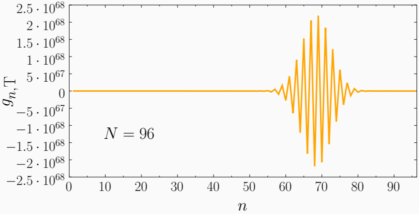
$$R^{(\tau,\mathrm{I})}_{ud} \simeq B \sum_{n=1}^{N} g_{n,\mathrm{I}}\; C_{\mathrm{I}}(na) = B \int_0^\infty \frac{\mathrm{d}E}{2\pi} \underbrace{\left(\sum_{n=1}^{N} g_{n,\mathrm{I}} e^{-naE} \right)}_{\simeq K_{\mathrm{I}}(E/m_\tau)} E^2 \rho_{\mathrm{I}}(E^2)$$

Coefficients $g_{n,I}$ can be obtained minimizing:

$$A_{\mathrm{I}}[\boldsymbol{g}] = \int_{E_0}^\infty \frac{\mathrm{d}E}{2\pi} \left| K_{\mathrm{I}}\left(\frac{E}{m_\tau}\right) - \sum_{n=1}^{N} g_n e^{-naE} \right|^2, \qquad E_0 > 0 \text{ to ensure convergence}$$

However they are strongly oscillating...



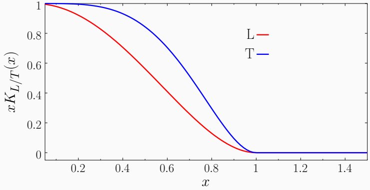


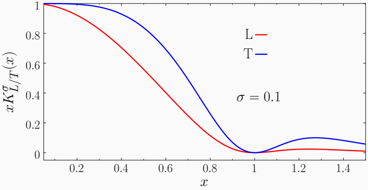
The smeared-ratio $R_{ud}^{(\tau)}(\sigma)$

- What makes the coefficients $g_{n,\mathrm{I}}$ bad-behaved is the presence in the kernel functions K_L and K_T of the θ-function, which is non-smooth.
- To make a step-forward, we follow the approach developed in [Gambino et al., PRL 125 (2020)] in the context of inclusive semileptonic B decays.

We first introduce the smeared kernel functions $K_\mathrm{I}^\sigma(x)$ through the replacement

$$\theta(x) \to \Theta_\sigma(x) \equiv \frac{1}{1+e^{-x/\sigma}}\ , \qquad \lim_{\sigma \mapsto 0} \Theta_\sigma(x) = \theta(x)$$





and evaluate on the lattice, for several σ, the corresponding smeared-ratio

$$R_{ud}^{(\tau,\mathrm{I})}(\sigma) = 12\pi S_\mathrm{EW} \frac{|V_{ud}|^2}{m_\tau^3} \int_0^\infty \mathrm{d}E\, K_\mathrm{I}^\sigma\left(\frac{E}{m_\tau}\right) E^2 \rho_\mathrm{I}(E^2)$$

$L \to \infty$ before $\sigma \to 0$ avoids power-like finite-size effects (FSEs).

The smeared-ratio from a Backus-Gilbert-like approach

We however still need a regularization mechanism to tame the oscillations of the $\boldsymbol{g}_{\mathrm{I}}$ coefficients (that would blow up our uncertainties).

The Hansen-Lupo-Tantalo (HLT) method provides the coefficients $g_{\mathrm{I}}(\sigma)$ minimizing a functional $W_{\mathrm{I}}^{\alpha}[g]$ which balances syst. and stat. errors of reconstructed $R_{ud}^{(\tau,\mathrm{I})}(\sigma)$

$$\boxed{W_{\mathrm{I}}^{\alpha}[\boldsymbol{g}] = \frac{A_{\mathrm{I}}^{\alpha}[\boldsymbol{g}]}{A_{\mathrm{I}}^{\alpha}[\boldsymbol{0}]} + \lambda B_{\mathrm{I}}[\boldsymbol{g}] \ , \qquad \left.\frac{\partial W_n[\boldsymbol{g}]}{\partial g}\right|_{\boldsymbol{g}=\boldsymbol{g}_{\mathrm{I}}} = 0}$$

$$A_{\mathrm{I}}^{\alpha}[\boldsymbol{g}] = \int_{E_{\min}}^{r_{\max}/a} \mathrm{d}E \ e^{aE\alpha} \left| K_{\mathrm{I}}^{\sigma}\left(\frac{E}{m_\tau}\right) - \sum_{n=1}^{N} g_n e^{-naE} \right|^2 \ \Longleftarrow \ (\text{syst.})^2 \text{ error due to reconstruction}$$

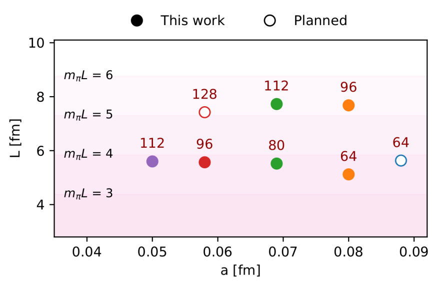
$$B_{\mathrm{I}}[\boldsymbol{g}] \propto \sum_{n_1,n_2=1}^{N} g_{n_1} g_{n_2} \,\mathrm{Cov}\left(C_{\mathrm{I}}(an_1), C_{\mathrm{I}}(an_2)\right) \ \Longleftarrow \ (\text{stat.})^2 \text{ error of reconstructed } R_{ud}^{(\tau,\mathrm{I})}(\sigma)$$

- λ is trade-off parameter $\implies$ tuned for optimal balance of syst. and stat. errors. $\{\alpha, E_{\min}, r_{\max}\}$ algorithmic params. to tune for optimal performance.

Numerical results for $\bar{u}s$ channel

Simulation details [only isoQCD: $\alpha_{\text{em}} = m_d - m_u = 0$]

Six physical-point $N_f = 2+1+1$ ensembles, with $a \in [0.049 \text{ fm} - 0.080 \text{ fm}]$. $L \sim 5.1$ fm and $L \sim 7.6$ fm to control Finite Size Effects (FSEs).



ID	L/a	a fm	L fm
B64	64	0.07951(4)	5.09
B96	96	0.07951(4)	7.63
C80	80	0.06816(8)	5.45
C112	112	0.06816(8)	7.63
D96	96	0.05688(6)	5.46
E112	112	0.04891(6)	5.48

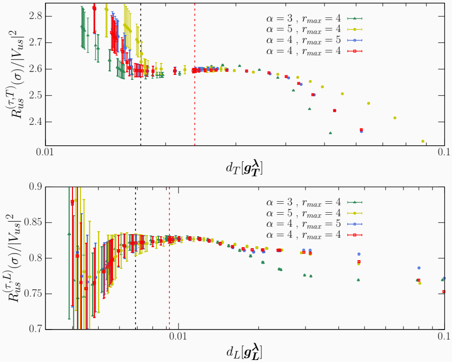
- We use two distinct lattice regularizations of the weak current, the so-called TM and OS currents, which must produce exact same results in the continuum limit.

- Iwasaki action for gluons.
- Wilson-clover twisted mass fermions at maximal twist for quarks (automatic $\mathcal{O}(a)$ improvement).
- Scale set using $f_\pi = 130.5$ MeV, $m_\pi = 135$ MeV, $m_K = 494.6$ MeV, $m_{D_s} = 1967$ MeV.

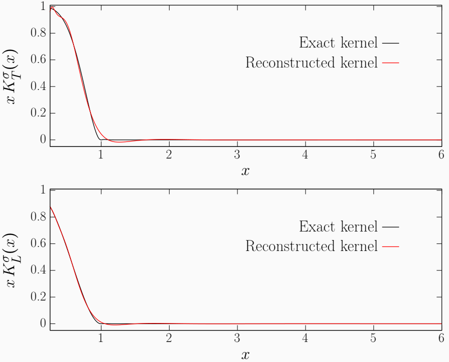
Stability analysis [Bulava et al, JHEP07 (2022)] ($\sigma = 0.02$)

For each contribution and σ, perform a scan in λ to find the region where stat. errors dominate over systematics due to incorrect reconstruction of kernel functions.

- Goodness of reconstruction *measured* by $d_{\mathrm{I}}[\boldsymbol{g}_{\mathbf{I}}^{\boldsymbol{\lambda}}] \equiv \sqrt{A_{\mathrm{I}}^{0}[\boldsymbol{g}_{\mathbf{I}}^{\boldsymbol{\lambda}}]/A_{\mathrm{I}}^{0}[\mathbf{0}]}$



Optimal λ (rightmost vertical line) chosen in the region of small $d_{\mathrm{I}}[\boldsymbol{g}_{\mathbf{I}}^{\boldsymbol{\lambda}}]$ where a plateaux is visible.

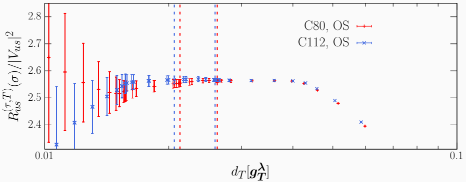


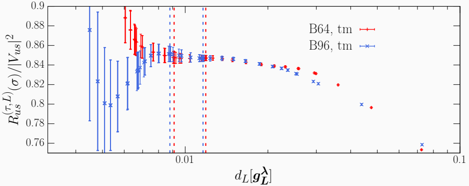
Comparison between exact and reconstructed kernel at optimal λ.

Exponential penalty $\exp(\alpha a E)$ for errors at large E drastically improves stability.

Data-driven estimate of FSEs ($\sigma = 0.02$)

FSEs estimated from observed spread on B64/B96 and C80/C112 ensembles.





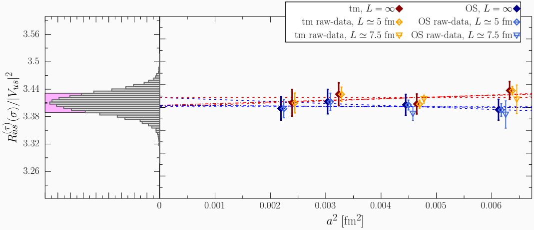
- FSEs typically very tiny...larger than $2\sigma_{\rm stat}$ in only 1% of the cases.
- We associate to our results at $L \sim 5.5$ fm a systematic error due to FSEs estimated as

$$\Sigma_{\rm I}^{\rm FSE}(\sigma) = \max_{\rm r=\{tm,OS\}} \left\{ \Delta_{\rm I}^{\rm r}(\sigma)\, {\rm erf}\left(\frac{1}{\sqrt{2}\sigma_{\Delta_{\rm I}^{\rm r}(\sigma)}} \right) \right\}$$

$$\Delta_{\rm I}^{\rm r}(\sigma) = \left| R_{us}^{(\tau,{\rm I}),{\rm r}}(\sigma, {\rm C80}) - R_{us}^{(\tau,{\rm I}),{\rm r}}(\sigma, {\rm C112}) \right|, \quad \sigma_{\Delta_I^r(\sigma)} \text{ is relative uncertainty of } \Delta_{\rm I}^{\rm r}(\sigma)$$

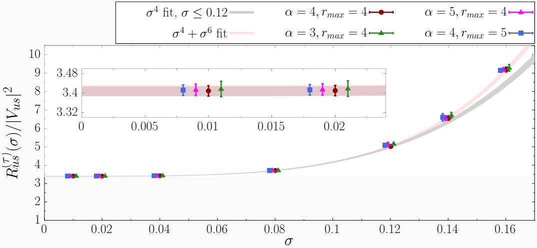
Continuum-limit and $\sigma \to 0$ extrapolation

$a \to 0$ at fixed σ (example for $\sigma = 0.02$)



- Performed combined (TM and OS) constant, linear and quadratic (only for $\sigma \geq 0.12$) extrapolations in a^2.
- We observe extremely small UV cutoff effects at small σ.
- Results combined using Akaike Information Criterion (AIC).

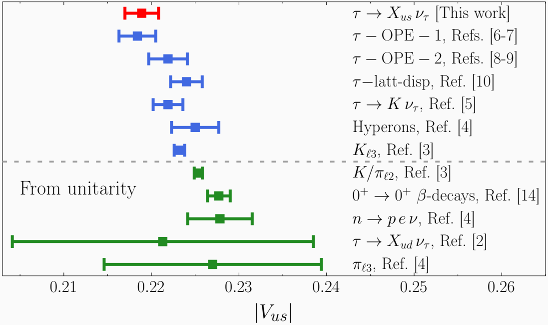
$\sigma \to 0$ after $a \to 0$. On theoretical grounds one expects $R_{us}^{(\tau)}(\sigma) = R_{us}^{(\tau)} + \mathcal{O}(\sigma^4)$



- σ^6 corrections subleading for $\sigma \leq 0.12$.
- σ^4 corrections subleading for $\sigma \leq 0.04$ (flat behaviour).
- Well controlled $\sigma \to 0$ extrapolation ✓.

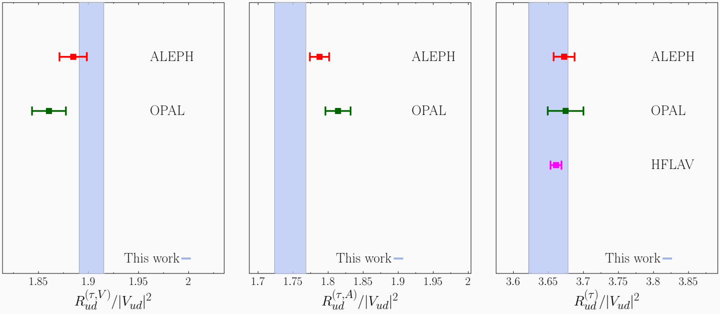
The Cabibbo angle from inclusive τ-decays

Our final determination is $R^{\tau}_{\text{us}}/|V_{\text{us}}|^2 = 3.407\ (22)$ [0.6% uncertainty]



- Using the exp. value $R^{\text{exp}}_{\text{us}} = 0.1632(27)$, we get $|V_{\text{us}}| = 0.2189(7)_{\text{th}}(18)_{\text{exp}}$
- Our result has $< 1\%$ uncertainty, and agrees with OPE results.
- Our results is 3.2σ (2.2σ) smaller than determination from leptonic (semileptonic) decays.
- At present accuracy level, QED+strong isospin-breaking corrections needed.

Determination of $|V_{ud}|$ [Evangelista et al., PRD 108 (2023)], if I have time



- Good agreement with ALEPH data (both for total and A/V channels), while we observe some difference w.r.t. OPAL data in A/V channels. For this comparison we used $|V_{\rm ud}|$ from superallowed β-decay.
- Alternatively, we can use the HFLAV average value $R_{\rm ud}^{(\tau)}(\rm HFLAV) = 3.471\ (7)$ to obtain

$$|V_{\rm ud}| = 0.9752\ (39)$$

  in good agreement with $|V_{\rm ud}| = 0.97373(31)$ from superallowed β-decay.
- Our result shows that a permille precision extraction of $|V_{\rm ud}|$ from inclusive τ-decay can be obtained in the near future!

Conclusions

- We have presented a first principles determination of $|V_{\rm ud}|$ and $|V_{\rm us}|$ from inclusive $\tau-$decays.
- Our current uncertainties are $\simeq 0.4\%$ for $|V_{\rm ud}|$ and $\simeq 0.9\%$ for $|V_{\rm us}|$. The latter uncertainty is dominated by the experimental uncertainty on the inclusive rate in the $\bar{u}s$ channel.
- It is however important to keep in mind that our calculations are still missing of the QED and strong-isospin-breaking (SIB) corrections which are parametrically expected to give a $\mathcal{O}(\alpha_{\rm em}) \simeq \mathcal{O}(\frac{\delta m_{ud}}{\Lambda_{QCD}}) \simeq 1\%$ contribution, and which MUST be determined at this level of precision.
- The calculation of the leading isospin-breaking corrections is underway!

Stay tuned and thank you very much for the attention 🙂

Backup slides

Relation between spectral density and Euclidean correlator

$$\boxed{C^{\alpha\beta}\left(t, \boldsymbol{q}\right) = \int d^3x\, e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \langle 0|\ T\left(J^{\alpha}_{ud}(-it, \boldsymbol{x})\, J^{\beta}_{ud}(0)^{\dagger}\right)\ |0\rangle}$$

Let's find the relation between $C^{\alpha\beta}(t, \boldsymbol{q})$ and the spectral density $\rho^{\alpha,\beta}(E, \boldsymbol{q})$:

$$C^{\alpha\beta}(t, \boldsymbol{q}) \overset{t\geq 0}{=} \int d^3x e^{-i\boldsymbol{q}\boldsymbol{x}} \langle 0|J^{\alpha}_{ud}(0) e^{-\mathcal{H}t + i\boldsymbol{\mathcal{P}}\boldsymbol{x}} J^{\beta}_{ud}(0)^{\dagger}|0\rangle$$

$$= \langle 0|J^{\alpha}_{ud}(0) e^{-\mathcal{H}t} (2\pi)^3 \delta^3(\boldsymbol{\mathcal{P}} - \boldsymbol{q}) J^{\beta}_{ud}(0)^{\dagger}|0\rangle$$

$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}E}{2\pi} e^{-Et}\, \langle 0|J^{\alpha}_{ud}(0)(2\pi)^4 \underbrace{\delta(\mathcal{H} - E)\, \delta^3(\boldsymbol{\mathcal{P}} - \boldsymbol{q})}_{\delta^4(\mathcal{P} - q_E)\ ,\ q_E = (E, \boldsymbol{q})} J^{\beta}_{ud}(0)^{\dagger}|0\rangle$$

where we just used the relation $e^{-\mathcal{H}t} = \int_{-\infty}^{\infty} \frac{\mathrm{d}E}{2\pi}\, e^{-Et}\, 2\pi\, \delta(\mathcal{H} - E)$

Recalling the definition of the spectral density one has

$$\boxed{C^{\alpha\beta}(t, \boldsymbol{q}) \overset{t\geq 0}{=} \int_0^{\infty} \frac{\mathrm{d}E}{2\pi} e^{-Et} \rho^{\alpha\beta}_{ud}(E, \boldsymbol{q})}$$