# RG RUNNING FROM STEP-SCALING MATRICES IN $\chi$SF SCHEMES FOR $\Delta F = 2$ FOUR-FERMION OPERATORS

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in collaboration with

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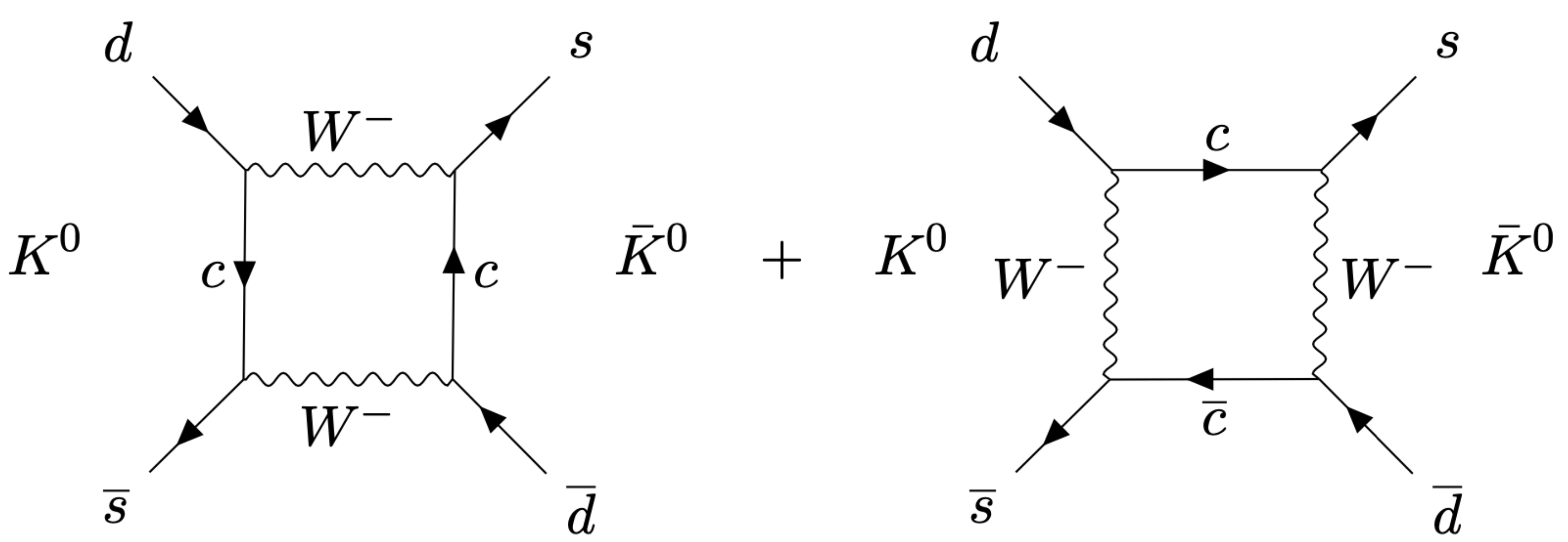
# MOTIVATIONS

- **Future goal: accurate evaluation of the CP-violating angle $\delta$ of the CKM matrix**
- **$K^0 - \bar{K}^0$ oscillations in the SM are sensitive to loop effects, and so to BSM contributions**



- **Indirect investigation of CP violation: $\varepsilon$ parameter**

$$\varepsilon^{\mathrm{theor}}(\delta) = \frac{A(K_L \to (\pi\pi)_{I=0})}{A(K_S \to (\pi\pi)_{I=0})} \propto \tilde{U}(\mu)\langle \bar{K}^0 | \mathbf{Q}(\mu) | K^0 \rangle F(\delta)$$

**To be evaluated non-perturbatively**

- **Comparing $\varepsilon^{\mathrm{theor}}$ with its experimental estimate we obtain**

**in the SM:**

1. **new estimate of the phase $\delta$**
2. **non-perturbative uncertainties**

**beyond the SM:**

1. **$\delta$ kept to the current estimate**
2. **bounds to BSM contributions**

# $K^0 - \bar{K}^0$ OSCILLATIONS

- **Effective Hamiltonian for K oscillations:**

**SM:** $H_{\text{eff}}^{\Delta S=2} = \tilde{U}_1 \mathbf{Q}_1$

**Only one relevant operator**

**BSM:** $H_{\text{eff}}^{\Delta S=2} = \sum_{i=1}^{5} \tilde{U}_i \mathbf{Q}_i + \sum_{i=1}^{3} \tilde{U}'_i \tilde{\mathbf{Q}}_i$

**An operator basis** $\mathbf{Q}_i$

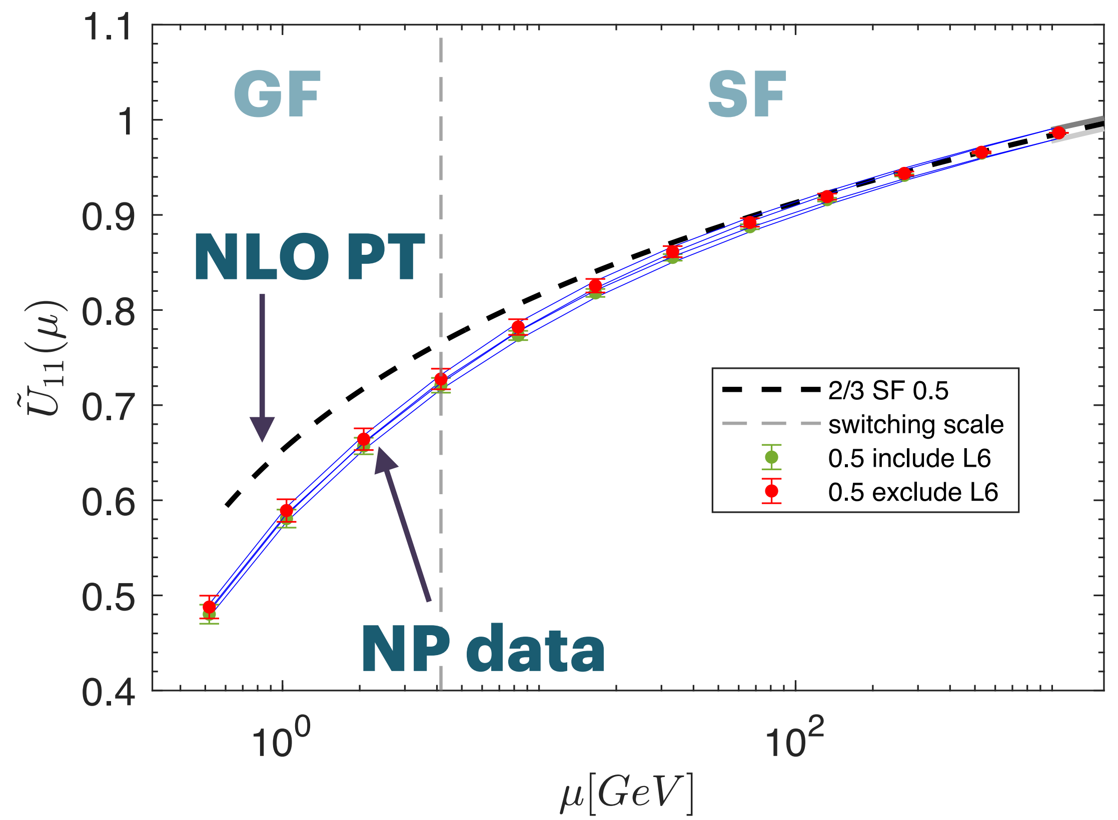
- **Transition amplitudes are calculated with** $\langle \bar{K}^0 | H_{\text{eff}}^{\Delta S=2} | K^0 \rangle$

- **The renormalisation introduces an energy-scale in the matrix elements and in the Wilson coefficients:**

$$\langle \bar{K}^0 | H_{\text{eff}}^{\Delta S=2} | K^0 \rangle_{\text{conservation}}^{\text{parity}} = \tilde{U}_i(\mu) \langle \bar{K}^0 | \mathbf{Q}_i(\mu) | K^0 \rangle$$

# FEATURES

- **Running evaluation with 3 quark flavours in the sea down to $\sim 4\text{GeV}$ with SF coupling;**
- **Running evaluation down to $\sim 500 MeV$ with Gradient Flow (GF) coupling;**
- **New theoretical formulation of the operator running and mixing in the perturbative regime for $N_f = 3$.**




**A difference often observed between PT and non-PT results at 3GeV (the scale at which matrix elements in FLAG are renormalised), could be relevant in the estimate of quantities like $\varepsilon(\delta)$**

[FLAG2021]

$$B_i \propto \langle \mathbf{Q}_i(\mu = 3\mathrm{GeV})\rangle$$

| Collaboration | Ref. | $N_f$ | publication status | continuum extrapolation | chiral extrapolation | finite volume | renormalization | running | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ETM 15 | [55] | 2+1+1 | A | ★ | ○ | ○ | ★ | *a* | 0.46(1)(3) | 0.79(2)(5) | 0.78(2)(4) | 0.49(3)(3) |
| RBC/UKQCD 16 | [60] | 2+1 | A | ○ | ○ | ○ | ★ | *b* | 0.488(7)(17) | 0.743(14)(65) | 0.920(12)(16) | 0.707(8)(44) |
| SWME 15A | [58] | 2+1 | A | ★ | ○ | ★ | ○† | – | 0.525(1)(23) | 0.773(6)(35) | 0.981(3)(62) | 0.751(7)(68) |
| SWME 14C | [508] | 2+1 | C | ★ | ○ | ★ | ○† | – | 0.525(1)(23) | 0.774(6)(64) | 0.981(3)(61) | 0.748(9)(79) |
| SWME 13A‡ | [495] | 2+1 | A | ★ | ○ | ★ | ○† | – | 0.549(3)(28) | 0.790(30) | 1.033(6)(46) | 0.855(6)(43) |
| RBC/ UKQCD 12E | [502] | 2+1 | A | ■ | ○ | ★ | ★ | *b* | 0.43(1)(5) | 0.75(2)(9) | 0.69(1)(7) | 0.47(1)(6) |
| ETM 12D | [59] | 2 | A | ★ | ○ | ○ | ★ | *c* | 0.47(2)(1) | 0.78(4)(2) | 0.76(2)(2) | 0.58(2)(2) |

- **Inconsistencies between different estimates**
- **Some results refer to perturbative renormalisation**

# THE XSF

- **In the continuum we map the SF into the χSF with a chiral rotation:**

$$\psi' = R\left(\frac{\pi}{2}\right)\psi\,, \quad \bar{\psi}' = \bar{\psi}R\left(\frac{\pi}{2}\right)\,, \quad R(\alpha) = e^{\frac{i}{2}\alpha\gamma_5\tau^3}$$

- **Correspondence between correlation functions in the SF and χSF:**

$$\langle O[\psi,\bar{\psi}]\rangle_{\rm SF}^{\rm cont} = \left\langle O\left[R\left(\pi/2\right)\psi, \bar{\psi}R\left(\pi/2\right)\right]\right\rangle_{\chi{\rm SF}}^{\rm cont}$$

- **The boundary rotation removes $\mathcal{O}(a)$ effects in the observables!**

$$\left\langle O_{\rm even}\right\rangle_{\rm c} = \left\langle O_{\rm even}\right\rangle_{\rm c}^{\rm cont} + \mathcal{O}(a^2)$$

# FOUR-FERMION OPERATORS RENORMALISATION

- **Four-Fermion Operators (FFO):**

$$\mathcal{O}^{\pm}_{[\Gamma_1\Gamma_2\pm\Gamma_2\Gamma_1]} := \mathcal{O}^{\pm}_{[\Gamma_1\Gamma_2]} \pm \mathcal{O}^{\pm}_{[\Gamma_2\Gamma_1]}\,,$$

$$\mathcal{O}^{\pm}_{[\Gamma_1\Gamma_2]} := \frac{1}{2}\Big[\big(\bar\psi_1\Gamma_1\psi_2\big)\big(\bar\psi_3\Gamma_2\psi_4\big) \pm \big(\bar\psi_1\Gamma_1\psi_4\big)\big(\bar\psi_3\Gamma_2\psi_2\big)\Big]$$

- **Parity-odd operators:**

$$\mathcal{Q}^{\pm}_1 = \mathcal{O}^{\pm}_{[VA+AV]} \quad \mathcal{Q}^{\pm}_3 = \mathcal{O}^{\pm}_{[PS-SP]} \quad \mathcal{Q}^{\pm}_5 = -2\mathcal{O}^{\pm}_{[T\tilde{T}]}$$

$$\mathcal{Q}^{\pm}_2 = \mathcal{O}^{\pm}_{[VA-AV]} \quad \mathcal{Q}^{\pm}_4 = \mathcal{O}^{\pm}_{[PS+SP]}$$

- **Behaviour under renormalisation as in a regularisation with exact chiral symmetry:**

$$\begin{pmatrix} \bar{\mathcal{Q}}^{\pm}_1 \\ \bar{\mathcal{Q}}^{\pm}_2 \\ \bar{\mathcal{Q}}^{\pm}_3 \\ \bar{\mathcal{Q}}^{\pm}_4 \\ \bar{\mathcal{Q}}^{\pm}_5 \end{pmatrix} = \begin{pmatrix} \mathcal{Z}_{11} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{Z}_{22} & \mathcal{Z}_{23} & 0 & 0 \\ 0 & \mathcal{Z}_{32} & \mathcal{Z}_{33} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{Z}_{44} & \mathcal{Z}_{45} \\ 0 & 0 & 0 & \mathcal{Z}_{54} & \mathcal{Z}_{55} \end{pmatrix}^{\pm} \begin{pmatrix} \mathcal{Q}^{\pm}_1 \\ \mathcal{Q}^{\pm}_2 \\ \mathcal{Q}^{\pm}_3 \\ \mathcal{Q}^{\pm}_4 \\ \mathcal{Q}^{\pm}_5 \end{pmatrix}$$

# EVOLUTION MATRICES

- **Evolution matrices between two scales:** $\bar{\mathcal{Q}}_i(\mu_2) = U_{ij}(\mu_2,\mu_1)\bar{\mathcal{Q}}_j(\mu_1)$
- **Evolution matrices down to a scale $\hat{U}(\mu)$:** $\mathbf{U}(\mu_2,\mu_1) =: \left[\hat{\mathbf{U}}(\mu_2)\right]^{-1}\hat{\mathbf{U}}(\mu_1)$
- **Problem: for $N_{\rm f} = 3$ (and $30$) two eigenvalues of $\gamma_0/\beta_0$ accidentally satisfy the resonance condition $\lambda_i - \lambda_j = 2$, making it impossible to adopt the usual definition**

$$\tilde{\mathbf{U}}(\mu) = \left[\frac{\bar{g}^2(\mu)}{4\pi}\right]^{-\frac{\gamma_0}{2b_0}}\mathbf{W}(\mu)$$

- **Problem solved in the past for the two-scale operator $U(\mu_2,\mu_1)$, but a single-scale evolution operator is needed to represent a Wilson coefficient**

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# WILSON COEFFICIENTS FROM THE POINCARÉ-DULAC THEOREM

- **The connection $A(g) = \frac{\gamma(g)}{\beta(g)}$ can be set [2013.16220v3] in its canonical form**

$$\mathbf{A}^{\text{can}}(g) = \frac{1}{g}\left(\mathbf{\Lambda} + g^2 \mathbf{N}_2\right)$$

$\mathbf{\Lambda}$: **diagonal**; $\mathbf{N}_2$: **upper-diagonal**

**through a change of operator basis** $\mathbf{S}(g) \simeq \left(\mathbb{1} + \sum_{k=1}^{n} \mathbf{H}_{2k} g^{2k}\right)\mathbf{S}_{\mathrm{D}} \equiv \mathbf{s}_n(g)\mathbf{S}_{\mathrm{D}}$

- **The evolution operator can be evaluated and then rotated back to the original operator basis:**

$$\hat{\mathrm{U}}(u) = \mathbf{S}_{\mathrm{D}}^{-1} \exp\left(-\frac{1}{2}\mathbf{\Lambda}\ln(u)\right) \exp\left(-\frac{1}{2}\mathbf{N}_2 \ln(u)\right) \mathbf{s}_n(u)\mathbf{S}_{\mathrm{D}}$$

# STEP-SCALING FUNCTIONS

- **Non-perturbative evolution from the step-scaling functions (SSF):**

$$\boldsymbol{\sigma}(u) := \mathbf{U}(\mu/2,\mu)\Big|_{\bar{g}^2(\mu)=u} \longrightarrow \mathbf{U}(u_{\rm had},u_{\rm pt}) = \boldsymbol{\sigma}(u_1)\cdots\boldsymbol{\sigma}(u_N)$$

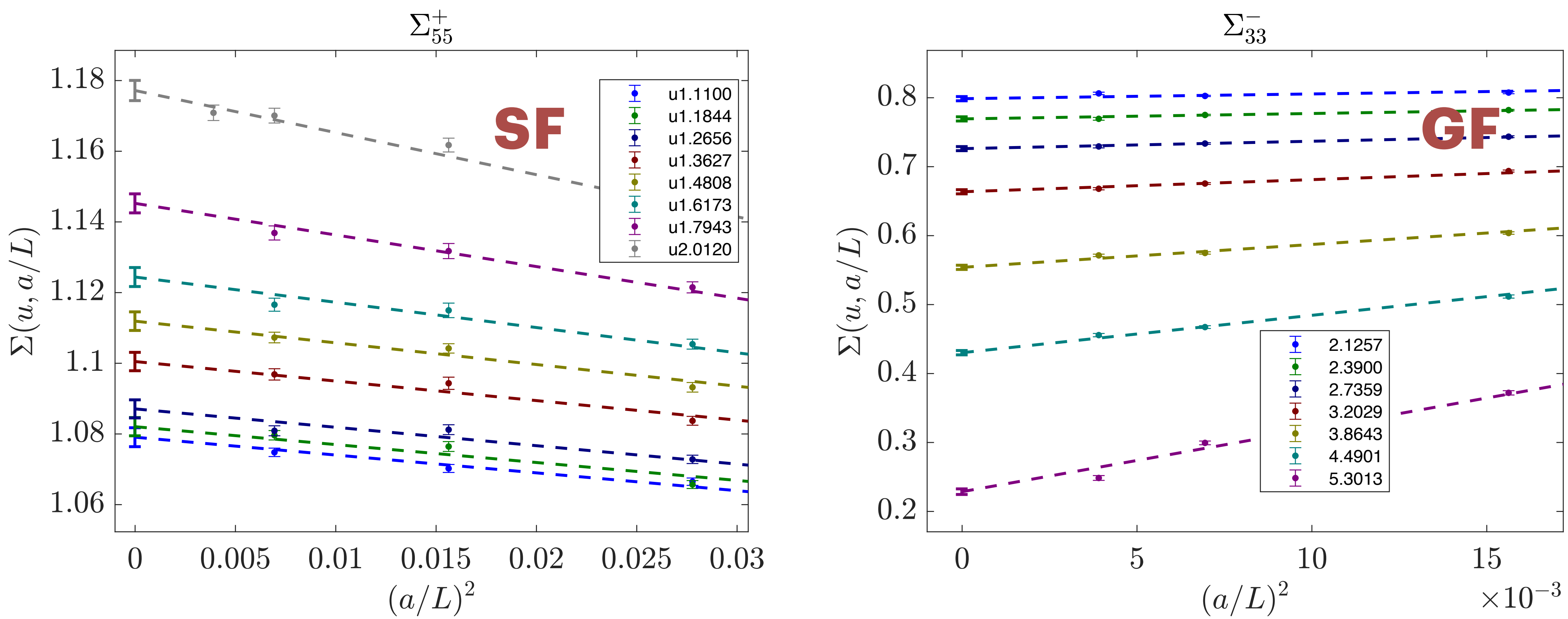
- **Discrete step-scaling functions:** $\boldsymbol{\Sigma}\Big(g_0^2,\frac{a}{L}\Big) := \boldsymbol{\mathcal{Z}}\Big(g_0^2,\frac{a}{2L}\Big)\Big[\boldsymbol{\mathcal{Z}}\Big(g_0^2,\frac{a}{L}\Big)\Big]^{-1}$

- $\mathcal{O}(g^2)$ **lattice artefacts in the SF energy region removed using subtracted SSF [2112.10606] :**

$$\tilde{\boldsymbol{\Sigma}}\Big(u,\frac{a}{L}\Big) := \boldsymbol{\Sigma}\Big(u,\frac{a}{L}\Big)[\mathbf{1} + u\log(2)\boldsymbol{\delta}_k(a/L)\boldsymbol{\gamma}_0]^{-1}$$

# SSF CONTINUUM EXTRAPOLATION

- **Global fits are performed with the ansatz** $$\left[\tilde{\boldsymbol{\Sigma}}\left(u_n, \frac{a}{L}\right)\right]_{ij} = [\boldsymbol{\sigma}(u_n)]_{ij} + \left(\frac{a}{L}\right)^2 \sum_{m=0}^{2} [\boldsymbol{\rho}_m]_{ij} u_n^m$$
- **Parameters found by $\chi^2$ minimisation**
- **L = 6,8,12 for SF coupling, L = 8,12,16 for GF coupling**



# EVOLUTION MATRICES FROM SSF

- **Continuum fit in the SF region:**

$$\boldsymbol{\sigma}(u)_{\mathrm{SF}} = \mathbf{1} + \mathbf{r}_1 u + \mathbf{r}_2 u^2 + \mathbf{r}_3 u^3 + \mathbf{r}_4 u^4$$

**($\mathbf{r}_1$ and $\mathbf{r}_2$ fixed by PT)**

- **Continuum fit in the GF region:**

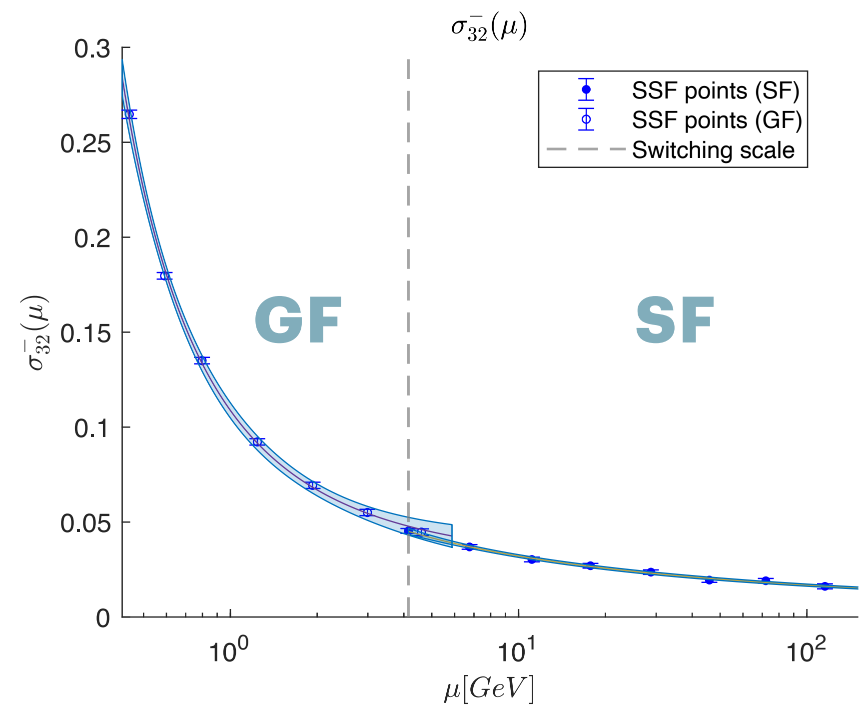
$$\boldsymbol{\sigma}(u)_{\mathrm{GF}} = \mathbf{R}_0 + \mathbf{R}_1 u + \mathbf{R}_2 u^2$$

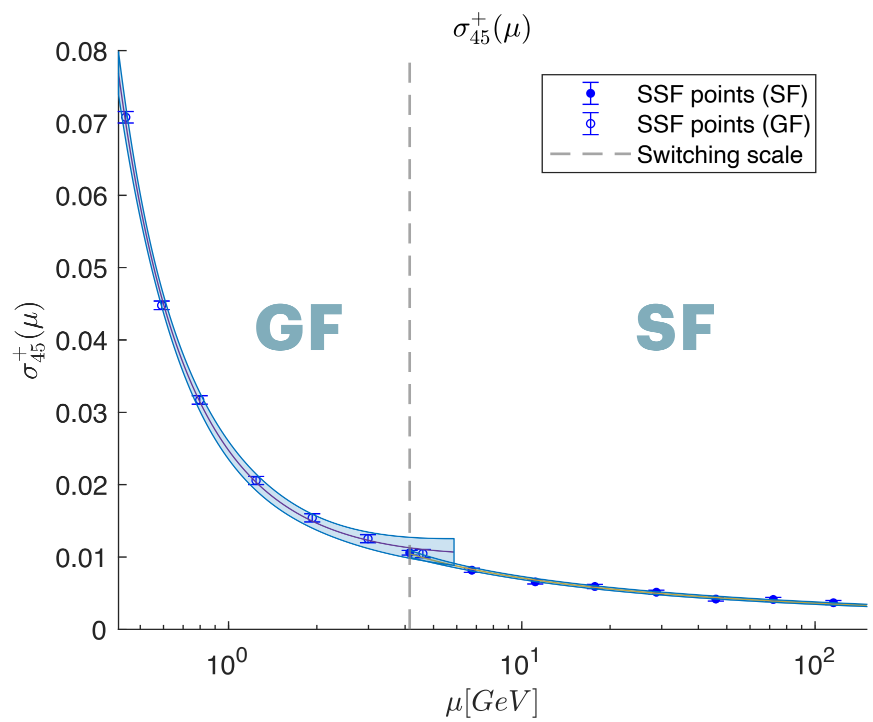
- $N$ **couplings evaluated as**

$$u_n = \sigma^{-1}_{\mathrm{coupling}}(u_{n-1})$$

- **Evolution operator between $u_1$ and $u_N$:**

$$\mathbf{U}(u_1, u_N) = \boldsymbol{\sigma}(u_1)\ldots\boldsymbol{\sigma}(u_N)$$



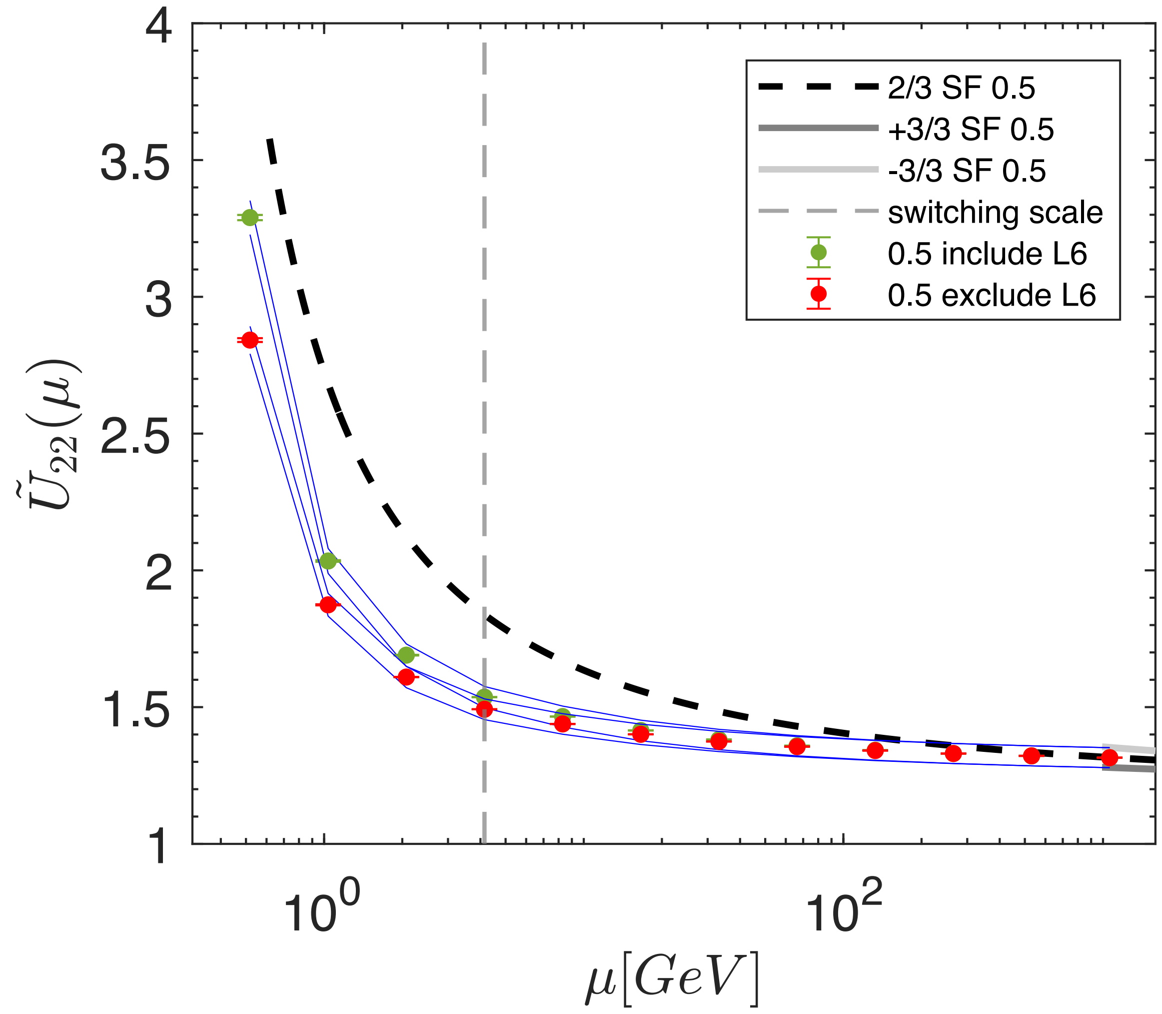
# ERROR ESTIMATES

- The non-perturbative running is finally given by

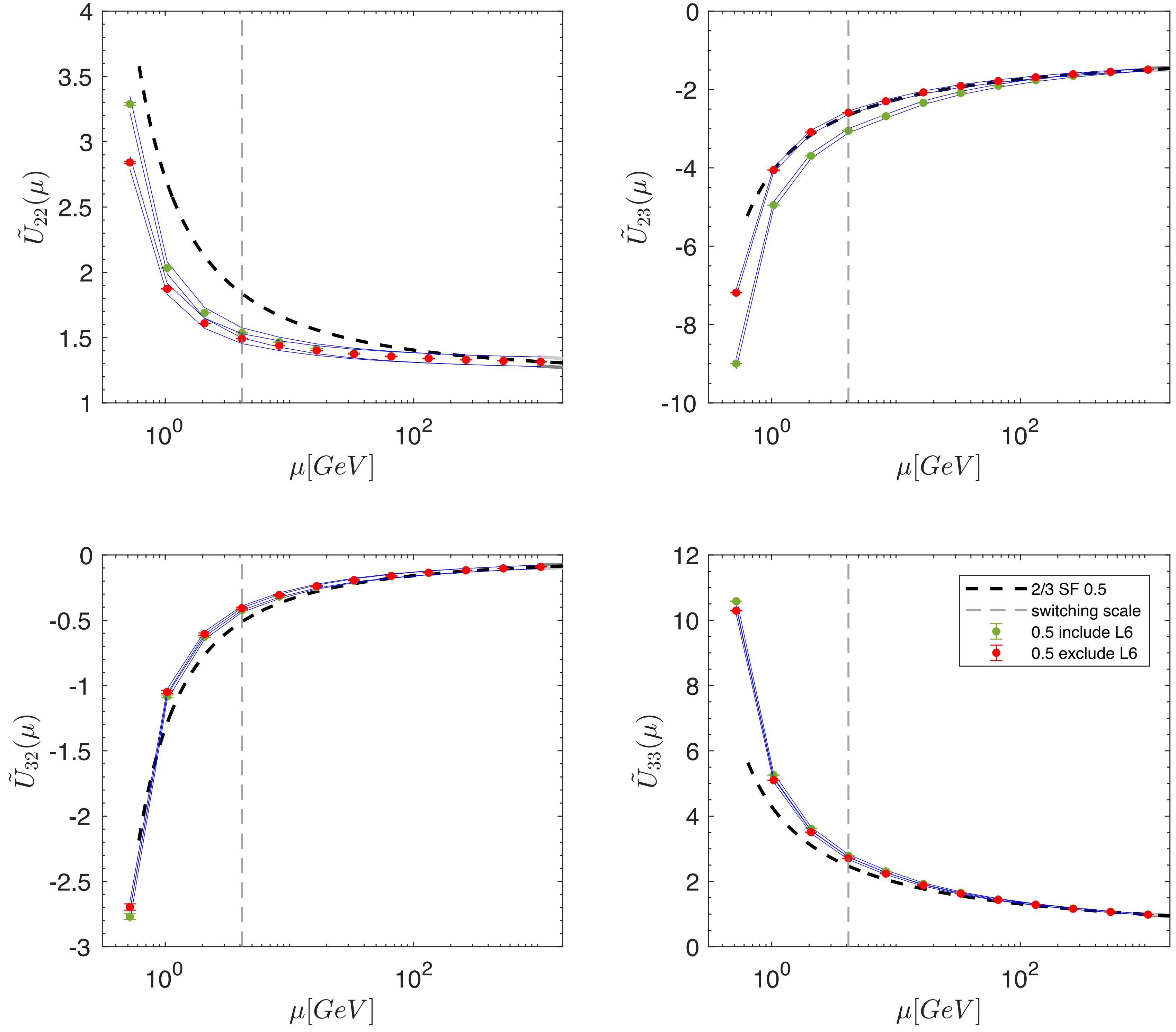
$$\hat{\mathbf{U}}(u) = \mathbf{S}_{\mathrm{D}}^{-1} \exp\left(-\frac{\mathbf{\Lambda}}{2} \ln u_{\mathrm{pt}}\right) \exp\left(-\frac{\mathbf{N}_2}{2} \ln u_{pt}\right) \mathbf{s}_n(g)\, \mathbf{S}_{\mathrm{D}} [\mathbf{U}(u, u_{\mathrm{pt}})]^{-1}$$

- **Statistical errors**: propagation from the fits

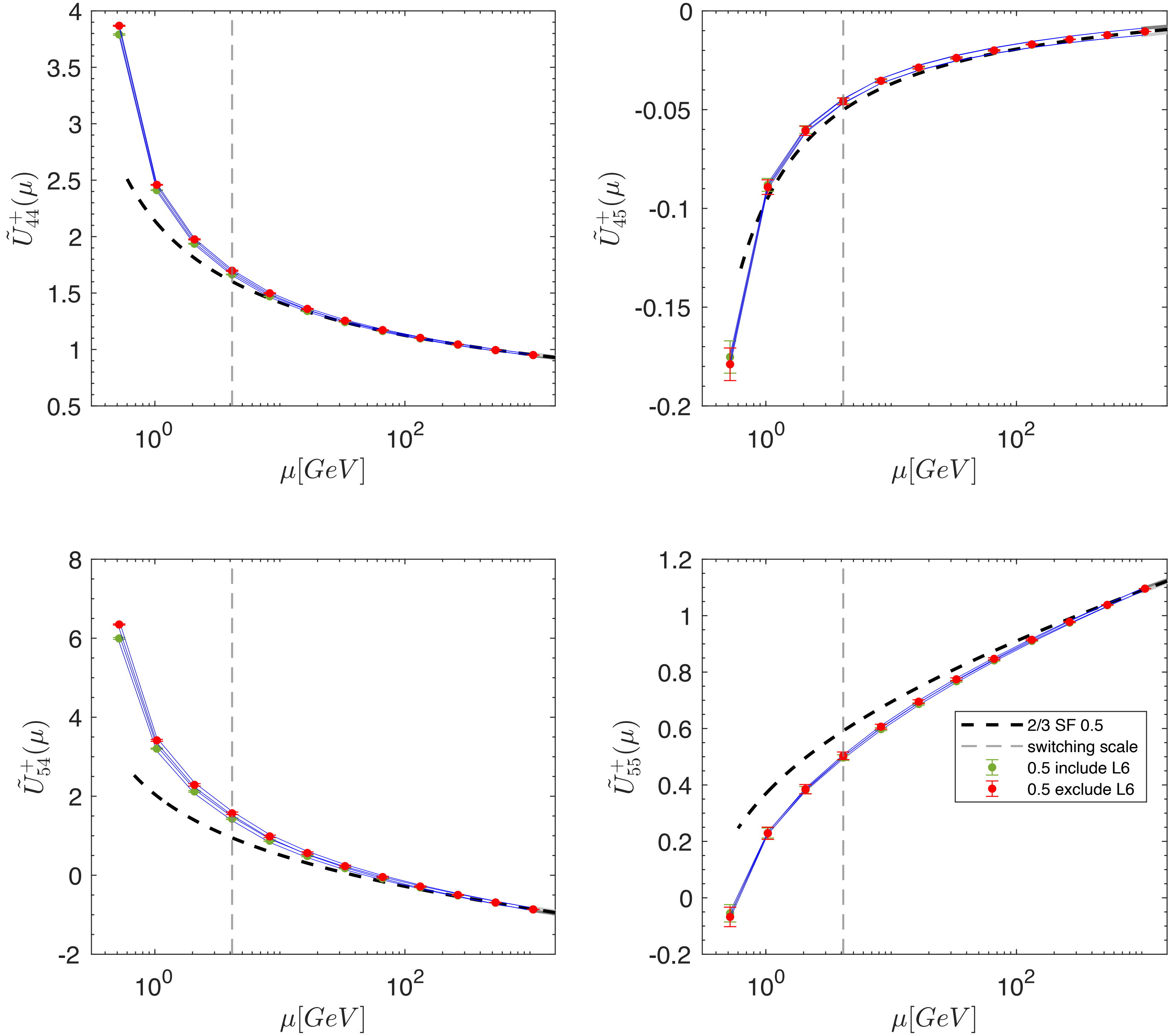
- **Systematic errors (guess)** :
  - Lack of knowledge on higher orders of the anomalous dimension
  - Differences arising if L=6 is included or not



# NP RUNNING: BSM 2|3 INDICES



# NP RUNNING: BSM 4|5 INDICES



# FUTURE DEVELOPMENTS

- **We have conducted a preliminary analysis computing non-perturbatively the running down to a scale $\mathcal{O}(500\,\mathrm{MeV})$ incorporating the NLO in the perturbative part of the study and solving the problem that appears for $N_f = 3$ .**

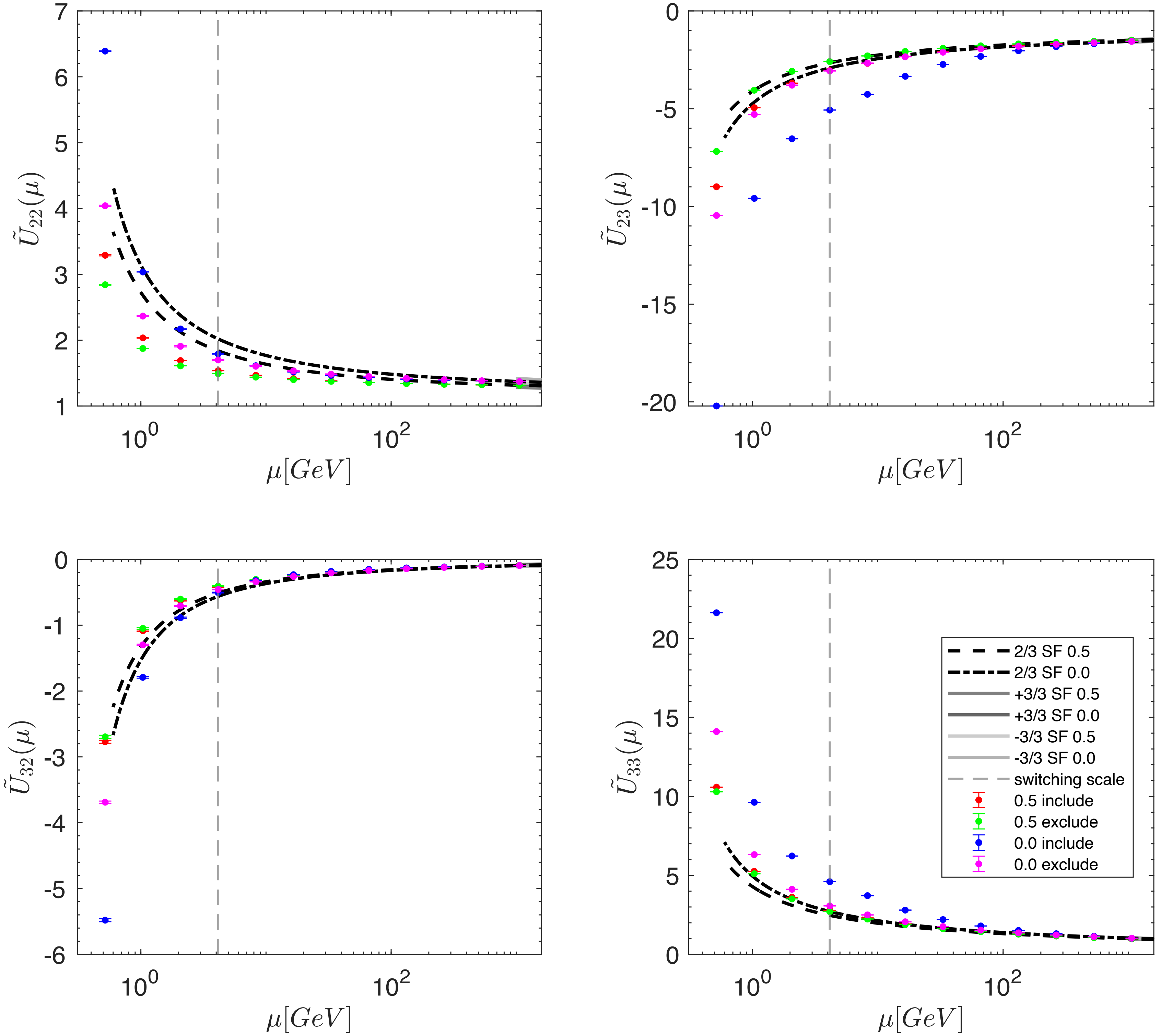
**In order to evaluate the value of $\varepsilon^{\mathrm{theor}}$ we are planning to perform the following computations:**

- **bare tm-QCD matrix elements estimated on Wilson gauge configurations (CLS);**
- **a non-perturbative evaluation of the renormalisation constants in the χSF at the hadronic scale ( $\sim 500$MeV) at the lattice spacings of the CLS ensembles.**

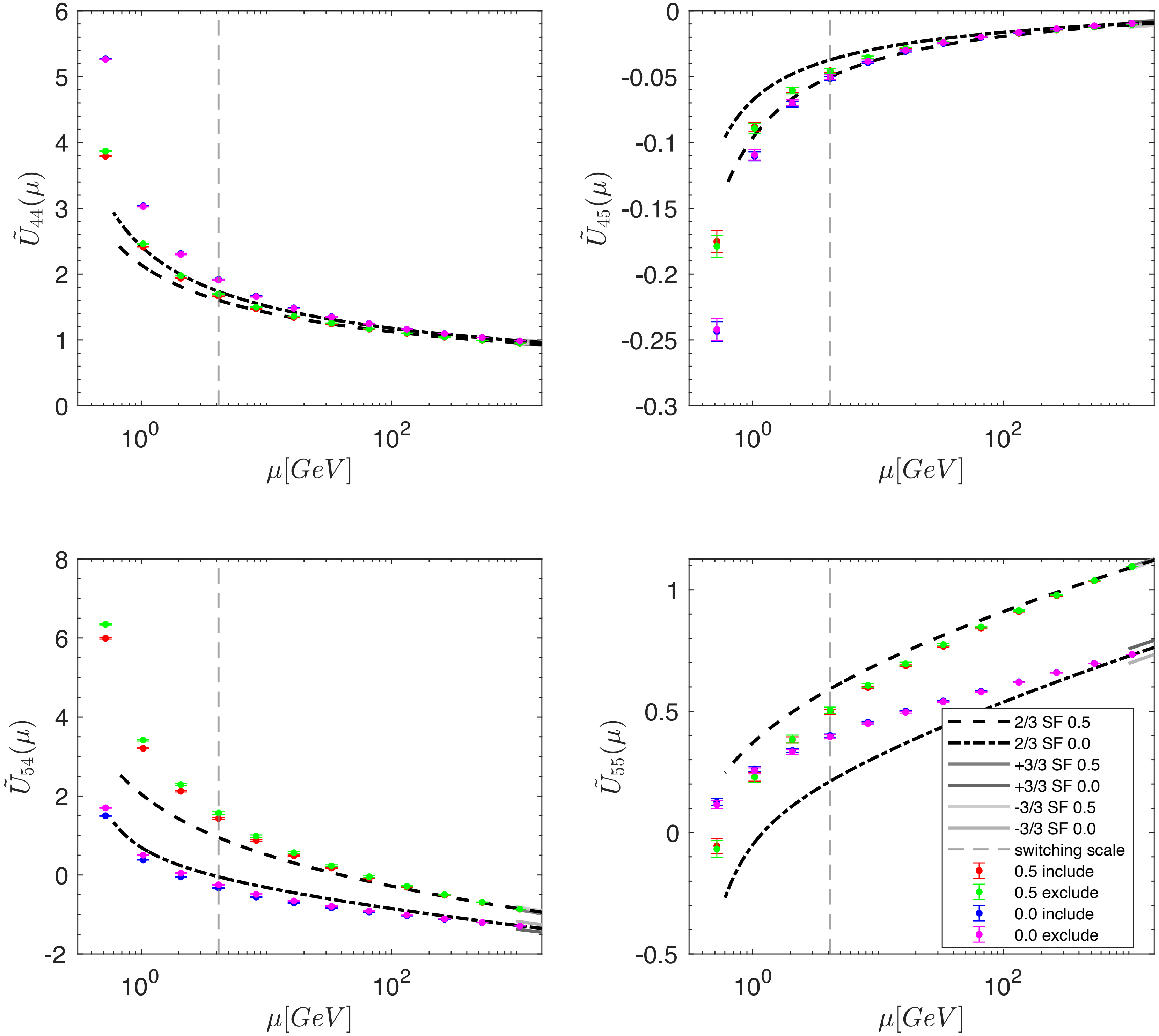
**Thank you**

# BACKUP

# NORMALISATION SCHEMES COMPARISON - 2|3



# NORMALISATION SCHEMES COMPARISON - 4|5



# W($\mu$) DEFINITION ISSUES

- **W(μ) is the solution of the equation**

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\mathbf{W}(\mu) = [\gamma[\bar{g}(\mu)], \mathbf{W}(\mu)] - \beta[\bar{g}(\mu)]\left(\frac{\gamma[\bar{g}(\mu)]}{\beta[\bar{g}(\mu)]} - \frac{\gamma^{(0)}}{\bar{g}(\mu)b_0}\right)\mathbf{W}(\mu)$$

**admitting the perturbative expansion**

$$\mathbf{W}(\mu) = \mathbf{1} + \bar{g}^2(\mu)\mathbf{J}_1 + \bar{g}^4(\mu)\mathbf{J}_2 + \bar{g}^6(\mu)\mathbf{J}_3 + \dots$$

**that implies**

$$2\mathbf{J}_1 - \left[\frac{\gamma_0}{b_0}, \mathbf{J}_1\right] = \frac{b_1}{b_0}\frac{\gamma_0}{b_0} - \frac{\gamma_1}{b_0}$$

- **Non-invertible system of equations if $N_\mathrm{f} = 3$ (or $30$) !**