The hadronic contribution to the running of α and the electroweak mixing angle

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Lattice 2024 - University of Liverpool

August 1^{st}, 2024

Why we care

Relevant quantities for precision tests of Standard Model

- Electromagnetic coupling: [Zyla et al. 2020]

$$\alpha(q^2 = 0) = 1/137.035999084(21)$$
$$\alpha(-M_Z^2) = 1/127.951(9)$$

- Hadronic contribution as main source of uncertainty
- Observed tensions with phenomenological estimates

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Standard approach

- Experimental input
- Dispersion theory

Lattice determination

- First-principles calculation
- Exact flavour separation

[M. Cè *et al.* 2022]

Time Momentum Representation

- Electroweak couplings as a function of the momentum transfer q^2

$$\alpha(-q^2) = \alpha/(1 - \Delta\alpha(-q^2)), \qquad \sin^2\theta_W(-q^2) = \sin^2\theta_W(1 + \Delta\sin^2\theta_W(-q^2))$$

Leading hadronic contribution

$$\Delta\alpha_{\mathrm{had}}(-q^2) = 4\pi\alpha\bar{\Pi}^{\gamma\gamma}(-q^2), \qquad (\Delta\sin^2\theta_W)_{\mathrm{had}}(-q^2) = -4\pi\alpha/\sin^2\theta_W\bar{\Pi}^{Z\gamma}(-q^2)$$

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Time Momentum Representation (TMR) [Bernecker, Meyer 2011; Francis *et al.* 2013]

$$\bar{\Pi}(-q^2) = \int_0^\infty \mathrm{d}t G(t) K(t, q^2) \qquad G(t) = -\frac{1}{3}\int \mathrm{d}\vec{x}\sum_{k=1}^{3}\langle j_k^{\gamma(Z)}(x) j_k^{\gamma}(0)\rangle$$

- In the $SU(3)$-flavour basis $j_k^a = \bar{q}\gamma_k(\lambda_a/2)q$, $a = 3, 8, 0$

$$G_{\mu\nu}^{\gamma\gamma}(x) = G_{\mu\nu}^{33}(x) + \frac{1}{3}G_{\mu\nu}^{88}(x) + \frac{4}{9}G_{\mu\nu}^{cc}(x)$$

$$G_{\mu\nu}^{Z\gamma}(x) = \left(\frac{1}{2} - \sin^2\theta_W\right)G_{\mu\nu}^{\gamma\gamma}(x) - \frac{1}{6\sqrt{3}}G_{\mu\nu}^{08}(x) - \frac{1}{18}G_{\mu\nu}^{cc}(x)$$

Computational strategy

- Two discretisations of the vector current, the local (L) and point-split (C)

Set 1: Improvement coefficients from large-volume [1811.08209]

Set 2: Improvement coefficient from SF setup [1805.07401, 2010.09539]

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- Isovector contribution [S. Kuberski *et al.* 2024]

$$\Delta\alpha^{3,3}(Q^2) = (\Delta\alpha^{3,3})_{\text{sub}}(Q^2) + b^{(3,3)}(Q^2, Q_m^2)$$

where

$$b^{(3,3)}(Q^2, Q_m^2) = \left(\frac{Q}{2Q_m}\right)^2 \frac{\log(2)}{4\pi}, \qquad K(t, Q^2, Q_m^2)_{\text{sub}} = \frac{16}{Q^2}\sin^4\left(\frac{Qt}{4}\right) - \frac{Q^2}{Q_m^4}\sin^4\left(\frac{Q_m t}{2}\right)$$

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- Isoscalar contribution

$$\Delta\alpha^{8,8} = \Delta\alpha^{3,3} + \Delta_{\text{ls}}(\Delta\alpha)$$

where

$$\Delta_{\text{ls}}(\Delta\alpha) = G^{88} - G^{33} \propto \alpha_s(m_s^2 - m_l^2), \qquad K(t, Q^2) = \frac{16}{Q^2}\sin^4\left(\frac{Qt}{4}\right)$$

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$$\Delta\alpha^{8,8} = \Delta\alpha^{3,3} + \Delta_{\text{ls}}(\Delta\alpha)$$

- Charm connected contribution

$$\Delta\alpha^{c,c} = (\Delta\alpha^{c,c})_{\text{sub}}(Q^2) + 2b^{(3,3)}(Q^2, Q_m^2) + \Delta_{lc}b$$

where

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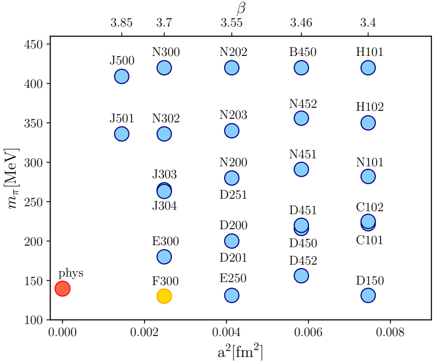
Lattice setup - CLS ensembles

[Lüscher and Schaefer, JHEP 1107 036 - JHEP 1502 043 - 1712.04884 - 2003.13359]

- Lüscher-Weisz tree-level improved gauge action
- $N_f = 2+1$ non-perturbatively $\mathcal{O}(a)$-improved Wilson fermions
- Open boundary conditions in time for fine values of the lattice spacings

  $\rightarrow$ Reliable error estimates



Lattice spacings :
$a = 0.087,\ 0.077,\ 0.065,\ 0.050,\ 0.039$ fm

Pion masses :
$130\ \text{MeV} \leq m_\pi \leq 420\ \text{MeV}$

Pushing to high Q^2

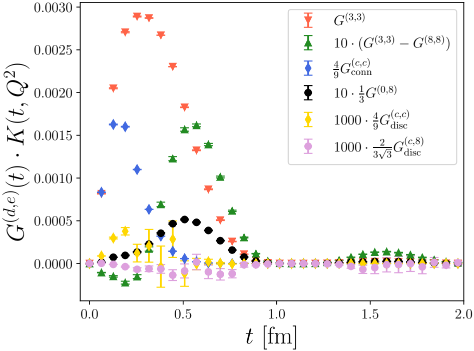
- Distinct separation of the various Euclidean distances

$$\bar{\Pi}(Q^2) = \Pi(Q^2) - \Pi(0) = \left[\Pi(Q^2) - \Pi(Q^2/4)\right] + \left[\Pi(Q^2/4) - \Pi(0)\right]$$

- Closely related with a_μ^{HVP}
[S. Kuberski, MON 12:35]

Preserve blinding in $g-2$ analysis

- E250: $a \approx 0.065$ fm, $m_\pi \approx 130$ MeV
- Set 1 impr. coefficients, $Q^2 = 9$ GeV^2



Isovector contribution: tree-level improvement

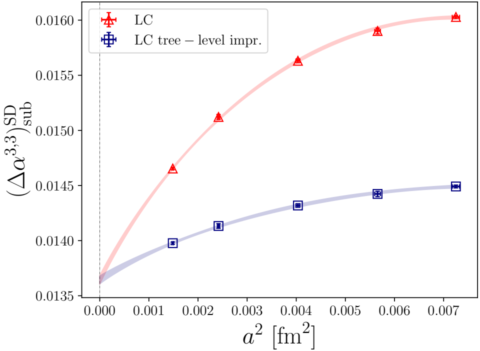
- Reduction of cutoff effects in the short Euclidean distance
  [ETM 2022; M. Cè *et al.* 2021; S. Kuberski *et al.* 2024]

- Continuum extrapolation of $(\Delta\alpha^{3,3})^{\mathrm{SD}}_{\mathrm{sub}}$ at the $SU(3)$-symmetric point $M_\pi = M_K \approx 415$ MeV

- Tree-level improvement based on massless perturbation theory

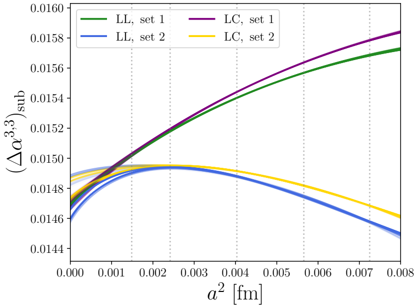
$$\mathcal{O}(a) \to \mathcal{O}(a)\frac{\mathcal{O}^{\mathrm{tl}}(0)}{\mathcal{O}^{\mathrm{tl}}(a)}$$

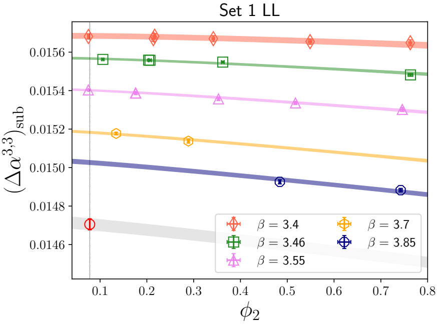
- Cutoff effects at $a = 0.087$ fm reduced from 19% to 7%



Isovector contribution

- $\Delta\alpha^{3,3} = (\Delta\alpha^{3,3})_{\rm sub} + b^{(3,3)}$ chiral-continuum extrapolation at $Q^2 = 9\ \mathrm{GeV}^2$
- No log-enhanced cutoff effects expected



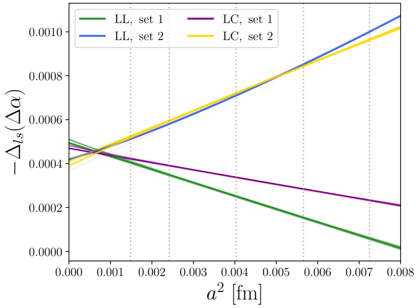


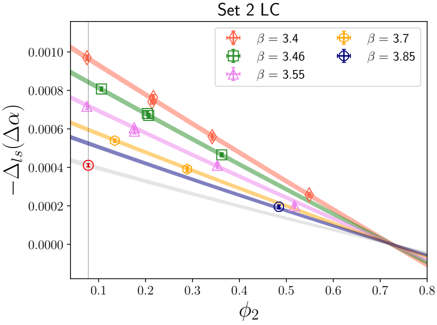
- Model average to assess the systematics arising from model selection

[W. I. Jay and E. T. Neil 2021; J. Frison 2023]

Isoscalar contribution

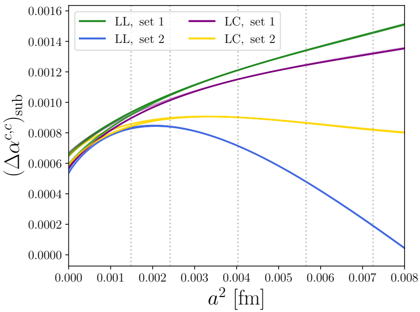
- $-\Delta_{\text{ls}}(\Delta\alpha) = \Delta\alpha^{3,3} - \Delta\alpha^{8,8}$ chiral-continuum extrapolation at $Q^2 = 9\ \text{GeV}^2$
- $SU(3)_f$ breaking $\to$ parametrically suppressed at short distance
- No help from perturbation theory required

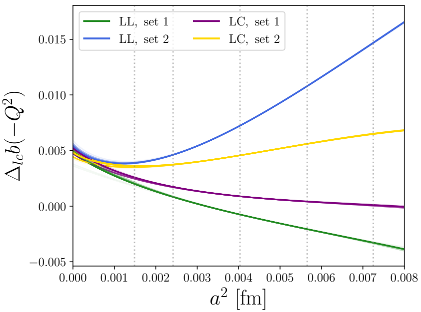




Charm connected contribution

- $\Delta\alpha^{c,c} = (\Delta\alpha^{c,c})_{\text{sub}} + b^{(c,c)}$ chiral-continuum extrapolation at $Q^2 = 9\ \text{GeV}^2$
- $b^{(c,c)}(Q^2) = 2b^{(3,3)}(Q^2) + \Delta_{\text{lc}} b$
- Very good agreement despite significantly different cutoff effects





Further, small contributions

- $\bar{\Pi}^{08}$ contribution entering $(\Delta \sin^2 \theta_W)^{0,8}$
  - → only CL discretization used
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  - $\rightarrow$ continuum results compatible with zero

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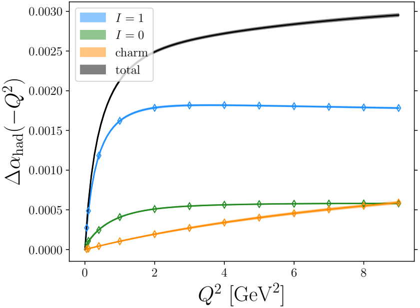
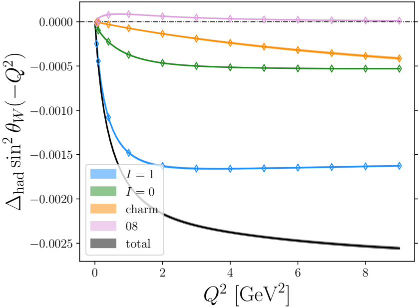
- Isospin-breaking corrections (ongoing) [J. Parrino, Thu 9:40; D. Erb, Thu 10:00]
  - → $\approx 0.3\%$ contribution to the total error [M. Cè *et al.* 2022; S. Kuberski *et al.* 2024]
  - → IB in scale setting, work in progress [A. Segner *et al.* 2023]
- Heavy quark contributions (ongoing)
  - → missing charm sea quark
  - → b-quark contribution

The running with energy [Preliminary]

- Results for $\bar{\Pi}(-Q^2) - \bar{\Pi}(-Q^2/4)$ in the range $0 \leq Q^2 \leq 9\ \mathrm{GeV}^2$
- Rational approximation of the running through a multi-points Padé Ansatz
  [Aubin *et al.* 2012; M. Cè *et al.* 2022]

$$\bar{\Pi}(-Q^2) \approx \frac{\sum_{j=0}^{M} a_j Q^{2j}}{1 + \sum_{k=1}^{N} b_k Q^{2k}}$$





Results and error budget [Preliminary]

- Results for the subtracted HVP $\bar{\Pi}(-Q^2) - \bar{\Pi}(-Q^2/4)$

  → statistical error from Lattice data

  → systematic error from model exploration

  → scale setting error [Bali *et al.* 2022]

Q^2 [GeV^2]	$\bar{\Pi}^{33}$	$\bar{\Pi}^{88}$
1.0	0.01768 (5) (9)(2)[10][0.6%]	0.004462(29)(32)(0)[43][1%]
5.0	0.019769(42)(58)(0)[72][0.4%]	0.006237(15)(22)(0)[27][0.5%]
9.0	0.019437(41)(68)(0)[79][0.4%]	0.006329(14)(25)(0)[29][0.5%]
Q^2 [GeV^2]	$\bar{\Pi}^{cc}$	$\bar{\Pi}^{08} \times 10^5$
1.0	0.001132(21)(32)(13)[41][3.6%]	24.63(76)(42)(19)[89][3.6%]
5.0	0.00438 (7)(10) (4)[13][2.9%]	7.55 (19)(24) (3)[31][4.1%]
9.0	0.00645 (9)(13) (5)[16][2.5%]	3.41 (12)(29) (0)[32][9.3%]

Conclusions and outlook

Summary

- Preliminary results of $(\Delta\alpha)_{\rm had}$ and $(\Delta \sin^2\theta_W)_{\rm had}$ in the range $0 \leq Q^2 \leq 9\ \mathrm{GeV}^2$
- High values of Q^2 reached by computing $\bar{\Pi}(-Q^2) - \bar{\Pi}(-Q^2/4)$
- Several improvements with respect to Mainz 2022 results [M. Cè *et al.* 2022]

Future

- Computation of the missing contribution $\bar{\Pi}(-Q^2/4) - \bar{\Pi}(0)$
- Full calculation of Isospin-breaking correction
- Connection between $(\Delta\alpha)_{\rm had}(-Q^2)$ and $(\Delta\alpha)_{\rm had}(M_Z^2)$

Thank You!

Related works of the Mainz group at Lattice 2024:

- HVP contribution to the muon $g-2$ [S. Kuberski, Mon 12:35]
- The timelike pion form factor [N. Miller, Tue 16:15]
- ML noise reduction strategies for $g-2$ [H. Wittig, Wed 11:35]
- UV-finite QED correction to $g-2$ [J. Parrino, Thu 9:40]
- The isospin violating part of the HVP [D. Erb, Thu 10:00]

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GSI

Lattice setup - CLS ensembles

[Lüscher and Schaefer, JHEP 1107 036 - JHEP 1502 043 - 1712.04884 - 2003.13359]

[Plot by J. Simeth]

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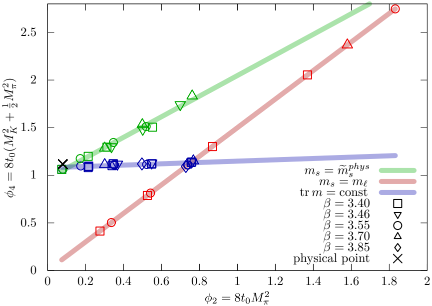
- $N_f = 2 + 1$ non-perturbatively $\mathcal{O}(a)$-improved Wilson fermions
- Open boundary conditions in time for fine values of the lattice spacings

  → Reliable error estimates

- Chiral trajectory $\Phi_4 \propto \mathrm{Tr}(M_q) = \Phi_4^{\text{phys}}$

  → 4 ensembles on $m_s \approx m_s^{\text{phys}}$ to account for small mistuning



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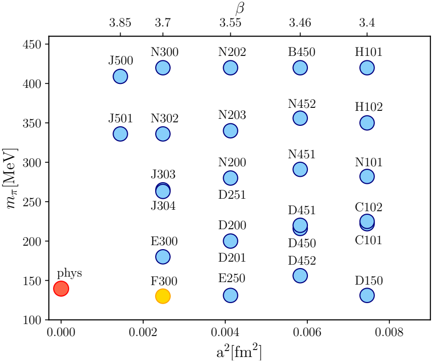

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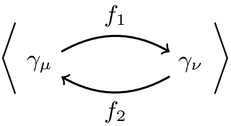
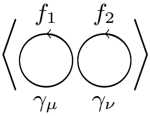
Lattice correlators

In the $SU(3)$ flavour basis and the isospin-symmetric limit:

- $I = 1$ contribution: $G^{33}_{\mu\nu}(x) = \frac{1}{2} C^{\ell\ell}_{\mu\nu}(x)$
- $I = 0$ contribution: $G^{88}_{\mu\nu}(x) = \frac{1}{6}\left[C^{\ell\ell}_{\mu\nu}(x) + 2C^{ss}_{\mu\nu}(x) + D^{\ell-s,\ell-s}_{\mu\nu}(x)\right]$
- Z-γ mixing: $G^{08}_{\mu\nu}(x) = \left[C^{\ell\ell}_{\mu\nu}(x) - C^{ss}_{\mu\nu}(x) + D^{2\ell+s,\ell-s}_{\mu\nu}(x)\right]$

where the connected and disconnected Wick's contractions read

$C^{f_1,f_2}_{\mu\nu} = -$ 

, $D^{f_1,f_2}_{\mu\nu} =$ 

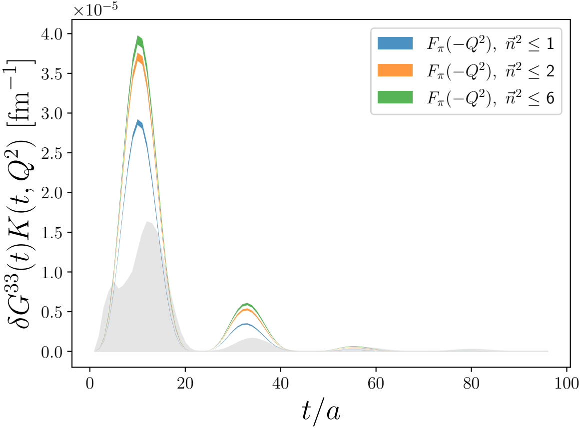
The relevant correlators are therefore given by

$$G^{\gamma\gamma}_{\mu\nu}(x) = G^{33}_{\mu\nu}(x) + \frac{1}{3} G^{88}_{\mu\nu}(x) + \frac{4}{9} G^{cc}_{\mu\nu}(x)$$

$$G^{Z\gamma}_{\mu\nu}(x) = \left(\frac{1}{2} - \sin^2\theta_W\right) G^{\gamma\gamma}_{\mu\nu}(x) - \frac{1}{6\sqrt{3}} G^{08}_{\mu\nu}(x) - \frac{1}{18} G^{cc}_{\mu\nu}(x)$$

Correction of finite-size effects

- Hansen-Patella (HP) method [Hansen, Patella 2019;2020]
- Pion and Kaon FVC included



$\bar{\Pi}^{08}$ contribution

- $(\Delta \sin^2 \theta_W)^{0,8}$ chiral-continuum extrapolation at $Q^2 = 9$ GeV
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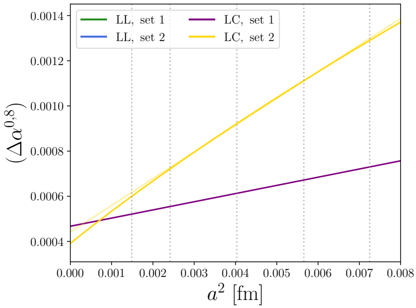
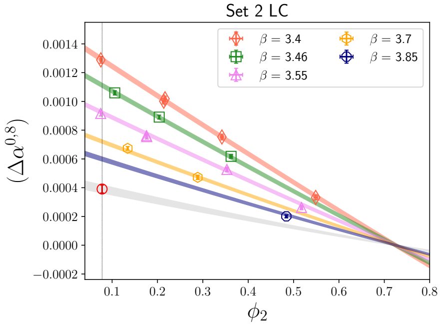




Chiral-continuum extrapolations

Functional forms for isovector and charm connected contributions

- Our general ansatz for the chiral dependence reads

$$\mathcal{O}(\phi_2) = \mathcal{O}(\phi_2^{\text{phys}}) + \gamma_1(\phi_2 - \phi_2^{\text{phys}}) + +\gamma_2(f(\phi_2) - f(\phi_2^{\text{phys}}))$$

where

$$f(\phi_1) \in \{\phi_2 \log(\phi_2),\ \phi_2^2\}.$$

- To account for a small mistuning from m_s^{phys}

$$\mathcal{O}(\phi_4) = \mathcal{O}(\phi_4^{\text{phys}}) + \gamma_0\big(\phi_4 - \phi_4^{\text{phys}}\big).$$

- Cutoff effects are described by the general form

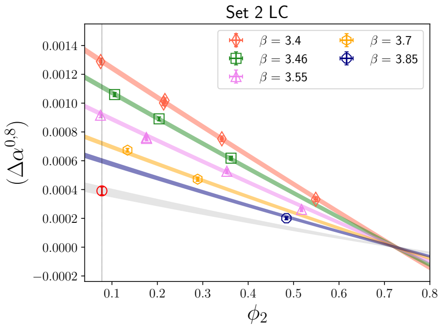
$$\mathcal{O}(a) = \beta_2 \frac{a^2}{8t_0} + \beta_3 \left(\frac{a^2}{8t_0}\right)^{3/2} + \beta_4 \left(\frac{a^2}{8t_0}\right)^2 + \delta_2 \frac{a^2}{8t_0}\big(\phi_2 - \phi_2^{\text{phys}}\big)$$
$$+ \delta_3 \left(\frac{a^2}{8t_0}\right)^{3/2}\big(\phi_2 - \phi_2^{\text{phys}}\big) + \epsilon_2 \frac{a^2}{8t_0}\big(\phi_4 - \phi_4^{\text{phys}}\big)$$

Chiral-continuum extrapolations

Functional forms for $\Delta_{\rm ls}$ and $\bar{\Pi}^{08}$, $SU(3)$-flavour breaking quantities.

- Expected to depend at leading order on $m_l - m_s$
- Defining $\Phi_\delta = \Phi_4 - \frac{3}{2}\Phi_2$, our general ansatz reads

$$\mathcal{O}(\Phi_\delta, \phi_2, a) = \Phi_\delta\left(\gamma_1 + \gamma_2\Phi_\delta + \beta_2\frac{a^2}{8t_0} + \beta_3\left(\frac{a^2}{8t_0}\right)^{3/2} + \gamma_0\Phi_4\right)$$



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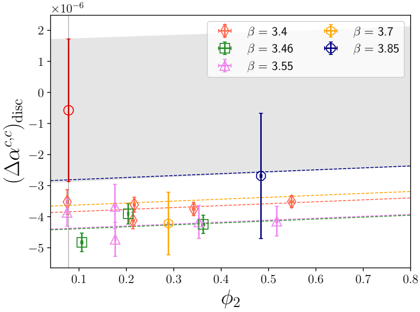

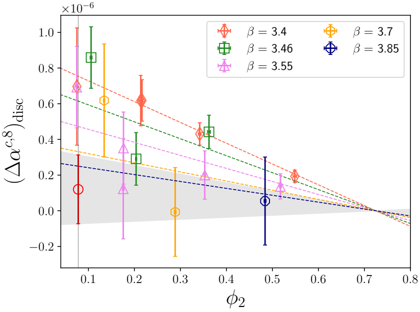

Data Analysis and Model average [HVPObs.jl]

- Γ-method for error estimation [U. Wolff, hep-lat/03060174, A. Ramos 2012.11183]
- χ^2_{exp} for correlated fits [M. Bruno, R. Sommer, 2209.14188]
- Takeuchi Information Criterion (TIC) as measure for the best fit [J. Frison, 2302.06550]

$$\text{TIC} = \chi^2 - 2\chi^2_{\text{exp}}, \qquad w_i \propto \exp\left(-\frac{1}{2}\text{TIC}(m_i)\right)$$

Model average [Jay, Neil: Phys.Rev.D 103 (2021) 114502]

$$\langle \mathcal{O} \rangle = \sum_{i=1}^{M} w_i \langle \mathcal{O} \rangle_i$$

Estimate the systematics

$$\sigma^2_{\mathcal{O}} = \sum_{i=1}^{M} w_i \langle \mathcal{O} \rangle_i^2 - \left(\sum_{i=1}^{M} w_i \langle \mathcal{O} \rangle_i \right)^2$$

Definition of the isosymmetric QCD world

- The scale setting is performed with [Bali *et al.* 2022]

$$\sqrt{8t_0^{\mathrm{ph}}} = 0.4081(19)\ \mathrm{fm}$$

- We define our scheme for isosymmetric QCD via the conditions

$$m_\pi = 134.9768(5)\ \mathrm{MeV}, \qquad m_K = 495.011(10)\ \mathrm{MeV}$$

- Valence charm quark mass tuned to reproduce the physical D_s meson mass