Gradient Flow for Quark Mass Determination

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Work in progress

Heavy quark mass

Precision of *heavy quark* (b, c) $\overline{\text{MS}}$ masses plays a crucial role in Higgs coupling measurement.

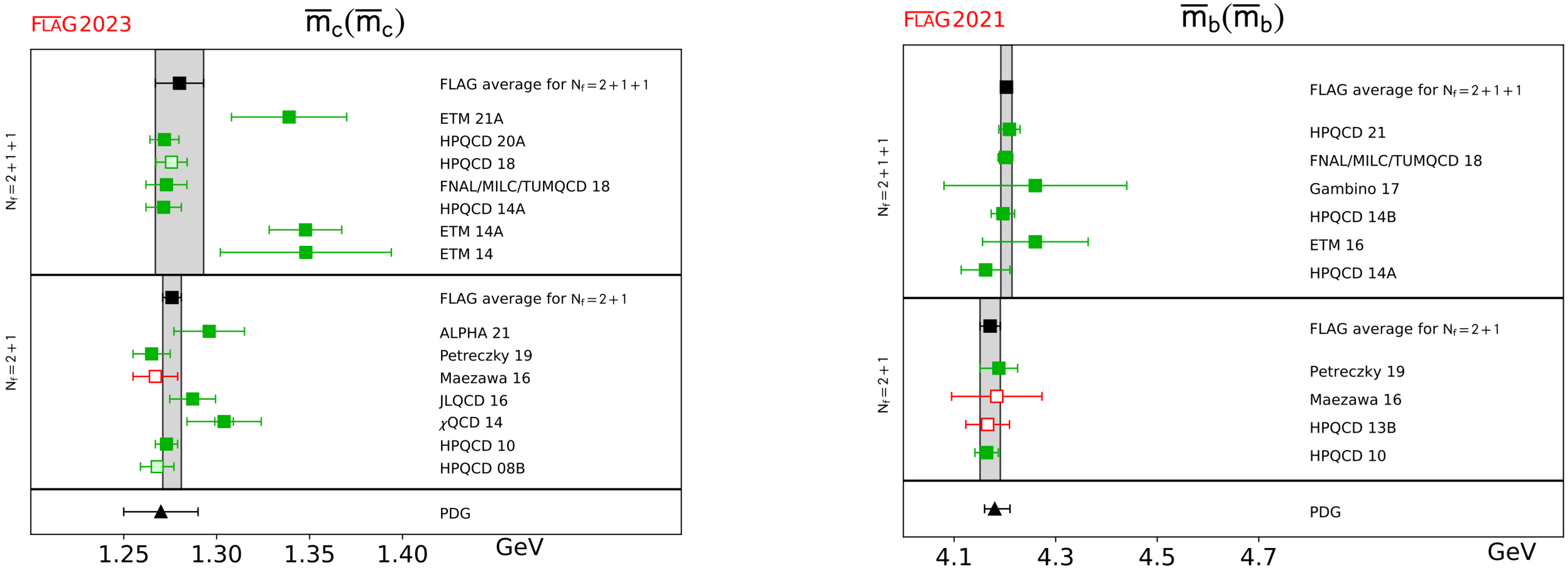
e.g. Bottom Yukawa coupling

Experimental measurement (future colliders)	SM prediction
$\Delta_{\rm exp} y_b / y_b \sim$ (subpercent level)	$y_b \sim \overline{m}_b / v_{\rm EW}$

$\overline{\text{MS}}$ masses need to be determined below subpercent level.

Also important widely for heavy quark physics:
Decay of heavy-light mesons, CKM matrix determinations, Quarkonium spectrum,...

The current status



Averaged values have subpercent precision!

Most of determinations are based on RI/(S)MOM or quark current correlator

Further examinations/other precise determinations should be welcome.

Gradient flow

I propose to utilize quark condensates in the gradient flow such as

$$\langle \bar{\chi}_h(t,x)\chi_h(t,x)\rangle / \langle \bar{\chi}_h(t,x) \overleftrightarrow{\not{D}} \chi_h(t,x)\rangle$$

*The wave function renormalization is canceled in the ratio. 2013 Lüscher, 2014 Makino, Suzuki

$\chi_h(t,x)$: Flowed field of heavy quark

Flow equations:

2013 Lüscher

$$\partial_t \chi_h(t,x) = (D_\mu D_\mu - \alpha_0 \partial_\mu B_\mu(t,x))\chi_h(t,x) \qquad [\chi_h(t=0,x) = \psi_h(x)]$$

$$\partial_t \bar{\chi}_h(t,x) = \bar{\chi}_h(t,x)(\overleftarrow{D}_\mu \overleftarrow{D}_\mu + \alpha_0 \partial_\mu B_\mu(t,x)) \qquad [\bar{\chi}_h(t=0,x) = \bar{\psi}_h(x)]$$

and

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x) \qquad [B_\mu(t=0,x) = A_\mu(x)]$$

2010 Lüscher, 2011 Lüscher, Weisz

α_0 : gauge parameter

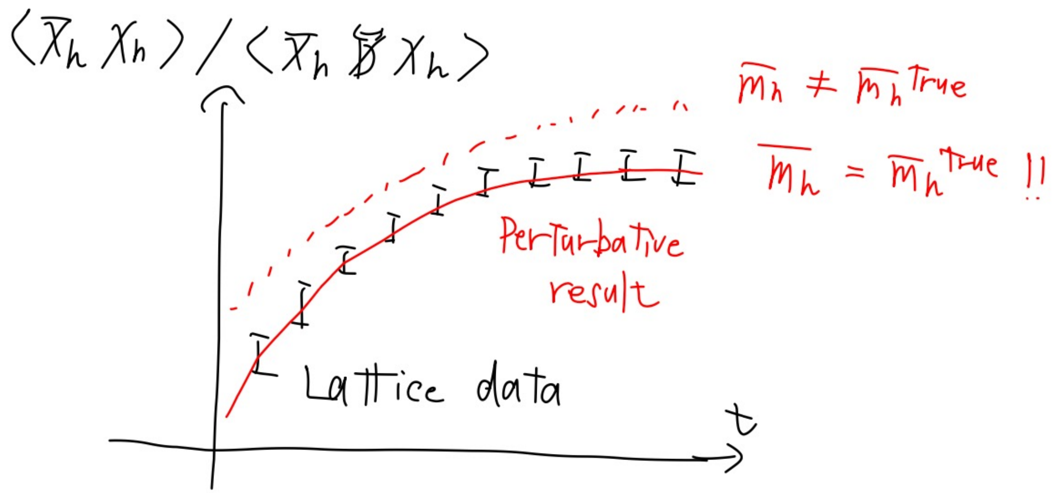
How to determine mass

$$\langle \bar{\chi}_h(t,x)\chi_h(t,x)\rangle / \langle \bar{\chi}_h(t,x) \overleftrightarrow{\not{D}} \chi_h(t,x)\rangle$$

Calculate it both in continuum spacetime and on the lattice

Continuum: Perturbation theory gives a function of $t, \overline{m}_h, \alpha_s$

Lattice: Gives a physical result once the lattice quark mass is determined properly (to reproduce the mass of hadrons consisting of heavy quark *h)*



Important aspects for precision

(Uncertainty)=$\sqrt{(\text{perturbative error})^2+(\text{lattice error})^2}$

perturbative error: Order of perturbative series,
Convergence of perturbative series

lattice error

Important aspects for precision

(Uncertainty)=$\sqrt{(\text{perturbative error})^2+(\text{lattice error})^2}$

perturbative error: Order of perturbative series, $\langle\bar{\chi}_h(t,x)\chi_h(t,x)\rangle/\langle\bar{\chi}_h(t,x)\overleftrightarrow{\not{D}}\chi_h(t,x)\rangle$

Convergence of perturbative series

Gauge invariance implies better perturbative convergence than gauge *non*-invariant method such as RI/(S)MOM [cf. renormalon argument]

lattice error: Good precision is expected from

-Gradient flow suppresses noise

-Heavy quark *one-point function*

-The only required extrapolation is $a \to 0$

I thank M. Kitazawa for discussion

Range of t

Lattice simulation: $a^2 \ll 8t \ll L^2$

Perturbativity: $8t \ll \Lambda_{\overline{\rm MS}}^{-2}$

Assuming $a^{-1} = 4\ \mathrm{GeV}$ as a reference (and $\Lambda_{\overline{\rm MS}} = 0.3\ \mathrm{GeV}$ and $L^2 \gtrsim \Lambda_{\overline{\rm MS}}^{-2}$ as well)

$$0.06\ \mathrm{GeV}^{-2} \ll 8t \ll 10\ \mathrm{GeV}^{-2}$$

Size of $\overline{m}_h^2 t$

Charm: $0.1 \ll 8\overline{m}_c^2 t \ll 6.8$

Bottom: $1.0 \ll 8\overline{m}_b^2 t \ll 70$

$\overline{m}_h^2 t$ spans a wide range

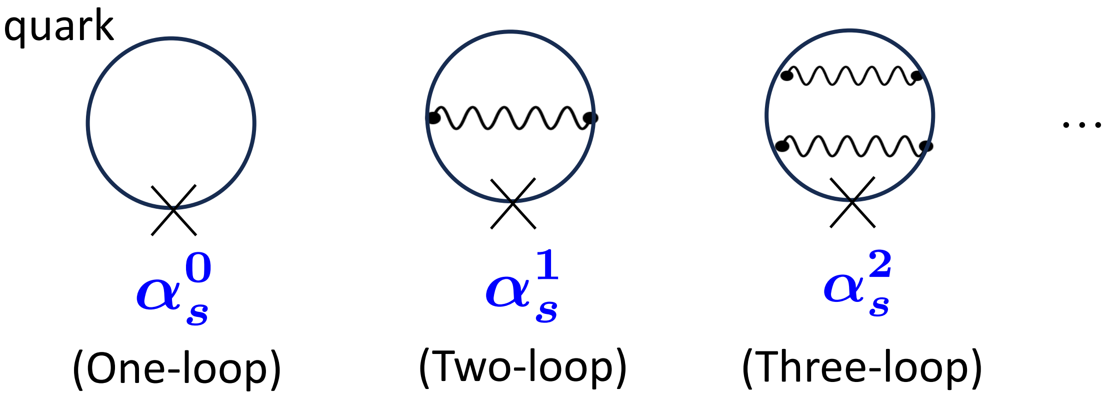
Perturbative calculation

So far a result for approximately massless quarks is available

NLO: 2013 Lüscher, 2014 Makino, Suzuki
NNLO: 2019 Artz, Harlander, Lange, Neumann, and Prausa

$$\langle \bar{\chi}_\ell(t,x)\chi_\ell(t,x)\rangle / \langle \bar{\chi}_\ell(t,x) \overleftrightarrow{\not{D}} \chi_\ell(t,x)\rangle$$
$$= \overline{m}t[(\underline{c_0} + \underline{c_1}\alpha_s + \underline{c_2}\alpha_s^2 + \underline{\mathcal{O}(\alpha_s^3)}) + \underline{\mathcal{O}(\overline{m}^2 t)}]$$

Known Missing



The known result is not sufficient for heavy quark mass determination

Our work

We aim to calculate

- Higher orders in $(\overline{m}^2 t)$-expansion (analytic)
- 1/ $(\overline{m}^2 t)$-expansion (analytic)
- full mass dependence (numerical)

at O(α_s) as a first step towards precise mass determination along this line.

Work in progress in collaboration with Robert Harlander and Fabian Lange

Two-loop integrals

$\langle \bar{\chi}_h(t,x)\chi_h(t,x)\rangle$ at O(α_s) is given by a linear combination of

$$I_1 = \int_0^t ds \int_{p,k} \frac{m}{k^2(p^2+m^2)} e^{-2tp^2-2sk^2},$$

$$I_2 = \int_0^t ds \int_0^s ds' \int_{p,k} \frac{mp^2}{k^2(p^2+m^2)} e^{-(2t-s+s')p^2-(s+s')k^2-(s-s')(k-p)^2},$$

$$I_3 = \int_0^t ds \int_{p,k} \frac{m}{k^2(p^2+m^2)} e^{-(2t-s)p^2-sk^2-s(k-p)^2},$$

$$I_4 = \int_0^t ds \int_0^t ds' \int_{p,k} \frac{m(p-k)^2}{k^2((p-k)^2+m^2)} e^{-(2t-s-s')p^2-(s+s')k^2-(s+s')(k-p)}$$

$$I_5 = \int_{p,k} \frac{m}{k^2(p^2+m^2)((p-k)^2+m^2)} e^{-tp^2-tk^2-t(k-p)^2},$$

$$I_6 = \int_0^t ds \int_{p,k} \frac{m}{k^2((p-k)^2+m^2)} e^{-(2t-s)p^2-sk^2-s(k-p)^2},$$

$$I_7 = \int_0^t ds \int_{p,k} \frac{m^3}{k^2(p^2+m^2)((p-k)^2+m^2)} e^{-(2t-s)p^2-sk^2-s(k-p)^2},$$

$$I_8 = \int_{p,k} \frac{m}{k^2(p^2+m^2)((p-k)^2+m^2)} e^{-2tp^2},$$

$$I_9 = \int_{p,k} \frac{m}{k^2(p^2+m^2)^2} e^{-2tp^2},$$

$$I_{10} = \int_{p,k} \frac{m}{(p^2+m^2)^2((p-k)^2+m^2)} e^{-2tp^2},$$

$$I_{11} = \int_{p,k} \frac{m^3}{k^2(p^2+m^2)^2((p-k)^2+m^2)} e^{-2tp^2}.$$

Method of expansion

Expansion by regions is the standard technique to expand loop integrals under certain hierarchy

1998 Beneke, Smirnov
2021 Harlander [application to gradient flow]

New method based on the Laplace transform is used in this work

$$\tilde{I}(v,t) \equiv \int_0^\infty d(m^2)(m^2)^{-v-1} I(m^2,t)$$

Looking at poles in the *v*-plane, I can obtain both small-$\overline{m}^2 t$ and large- $\overline{m}^2 t$ expansions.

Details will be explained in our paper.

$(\overline{m}^2 t)$-expansion of $\langle\bar{\chi}_h(t,x)\chi_h(t,x)\rangle$

$$
\begin{aligned}
&Z_\chi\langle\bar{\chi}_h(t,x)\chi_h(t,x)\rangle \\
&= -\frac{N_c}{8\pi^2}\frac{\overline{m}_h}{t} \\
&\times\Big\{1 + 0.759581 C_F\alpha_s \\
&\quad + \overline{m}_h^2 t\big[2\log 2 + 2\gamma_E + 2\log(\overline{m}_h^2 t) + \underline{C_F\alpha_s(2.3334 - 2.16804\log(\overline{m}_h^2/\mu^2)} \\
&\qquad \underline{+ \log(\overline{m}_h^2 t)(2.22817 - 0.95493\log(\overline{m}_h^2/\mu^2))}\big] \\
&\quad + \mathcal{O}((\overline{m}_h^2 t)^2)\Big\} \qquad \text{(L.1)}
\end{aligned}
$$

new

The other terms 2014 Makino Suzuki, 2021 Harlander [2111.14376]

$1/(\overline{m}^2 t)$-expansion of $\langle \bar{\chi}_h(t,x)\chi_h(t,x)\rangle$

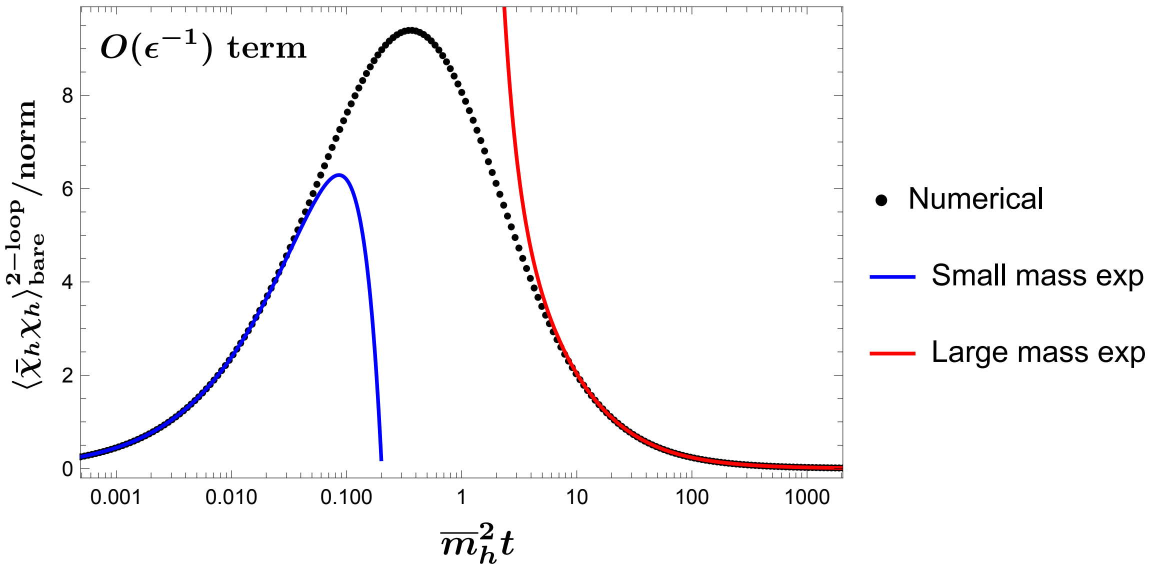
$$
\begin{aligned}
&Z_\chi \langle \bar{\chi}_h(t,x)\chi_h(t,x)\rangle \\
&= -\frac{N_c}{16\pi^2 t^2 \overline{m}_h} \\
&\times \Bigg\{ 1 + \frac{C_F\alpha_s}{4\pi}\left(-2 - 3\gamma_E - 9\log 2 + 9\log 3 - 3\log(\overline{m}^2 t) + 6\log(\overline{m}_h^2/\mu^2)\right) \\
&+ \frac{1}{\overline{m}_h^2 t}\left[-1 + \frac{C_F\alpha_s}{16\pi}\left(22 + 21\gamma_E + 63\log 2 - 51\log 3 + 21\log(\overline{m}^2 t) - 48\log(\overline{m}_h^2/\mu^2)\right)\right] \\
&+ \frac{1}{(\overline{m}_h^2 t)^2}\left[\frac{3}{2} + \frac{C_F\alpha_s}{64\pi}\left(-3(68 + 57\gamma_E) + 513\log 2 + 381\log 3 - 171\log(\overline{m}^2 t) + 432\log(\overline{m}_h^2/\mu^2)\right)\right] \\
&+ \mathcal{O}\left(\frac{1}{(\overline{m}_h^2 t)^3}\right) \Bigg\} \qquad \text{(L.2)}
\end{aligned}
$$

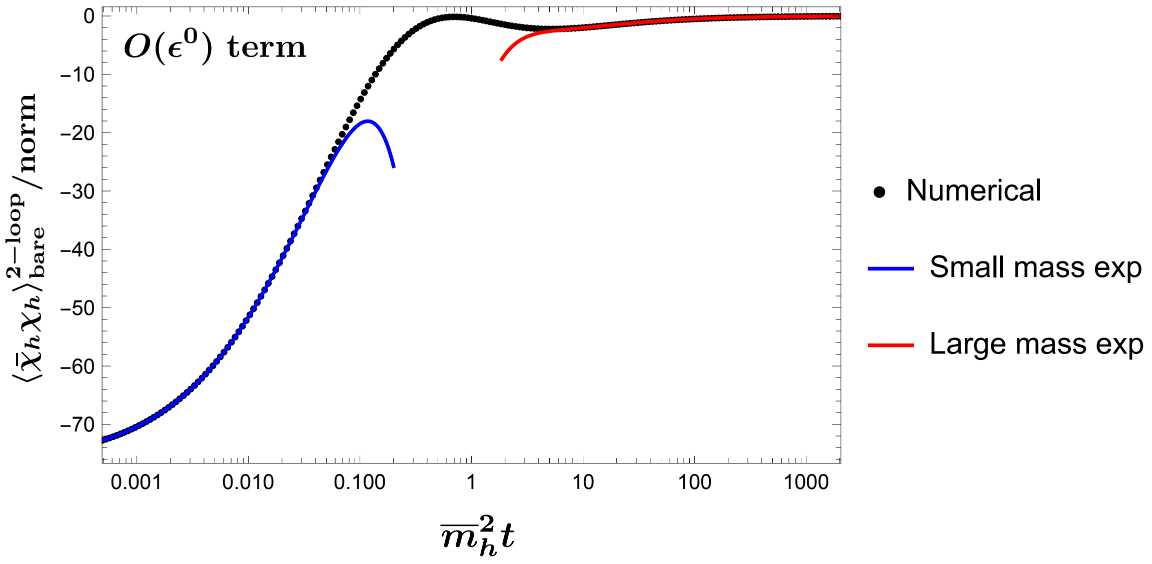
new

The other terms 2021 Harlander [2111.14376]

Full mass dependence of $\langle \bar{\chi}_h(t,x)\chi_h(t,x)\rangle$

Using "ftint" [2024 Harlander, Nellopoulos, Olsson, Wesle]





Numerical two-loop calculation can be performed for a wide range of $\overline{m}_h^2 t$, which agrees with the expansions for large- or small- $\overline{m}_h^2 t$.

Perturbative result for a physical quantity

$\langle\bar{\chi}_h(t,x)\chi_h(t,x)\rangle/\underline{\langle\bar{\chi}_\ell(t,x)\overleftrightarrow{\not{D}}\chi_\ell(t,x)\rangle}$: UV finite quantity

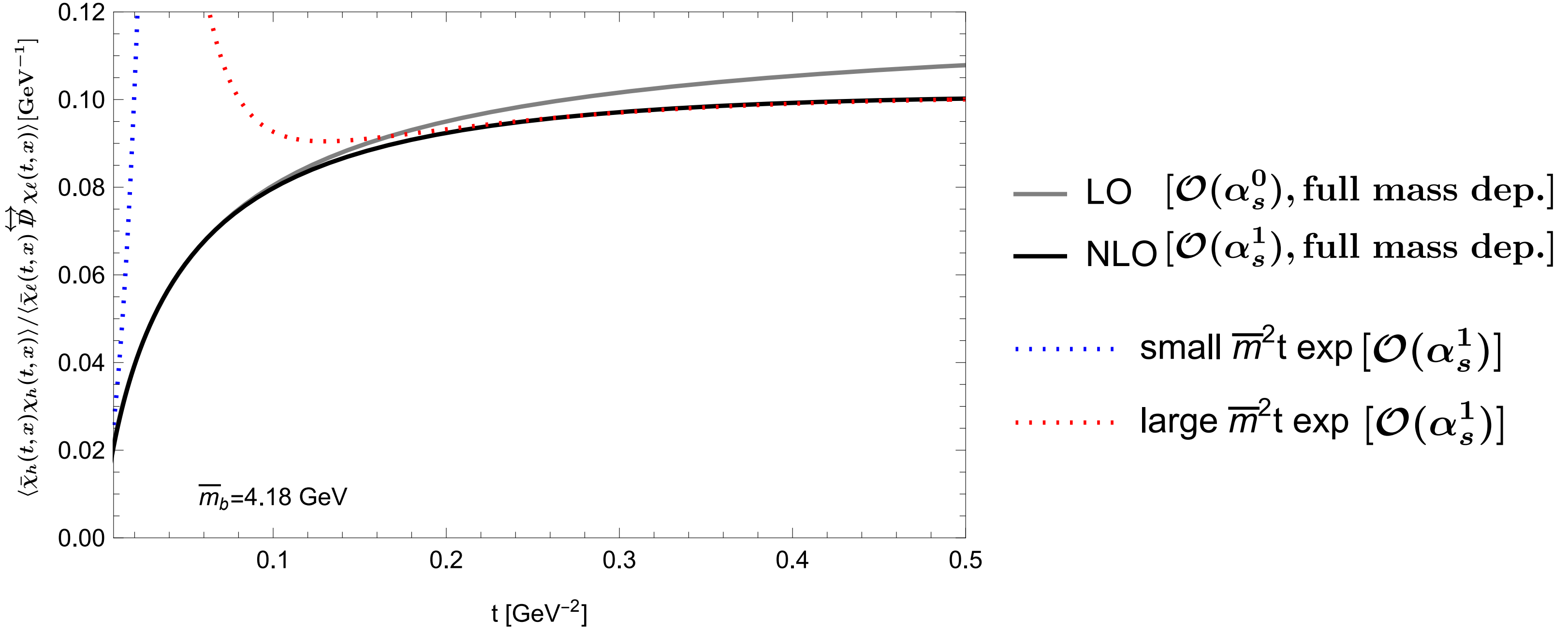
Massless quark

Perturbative result for a physical quantity

$\langle\bar{\chi}_h(t,x)\chi_h(t,x)\rangle/\langle\bar{\chi}_\ell(t,x)\overleftrightarrow{\not{D}}\chi_\ell(t,x)\rangle$: UV finite quantity

Massless quark

Bottom quark case



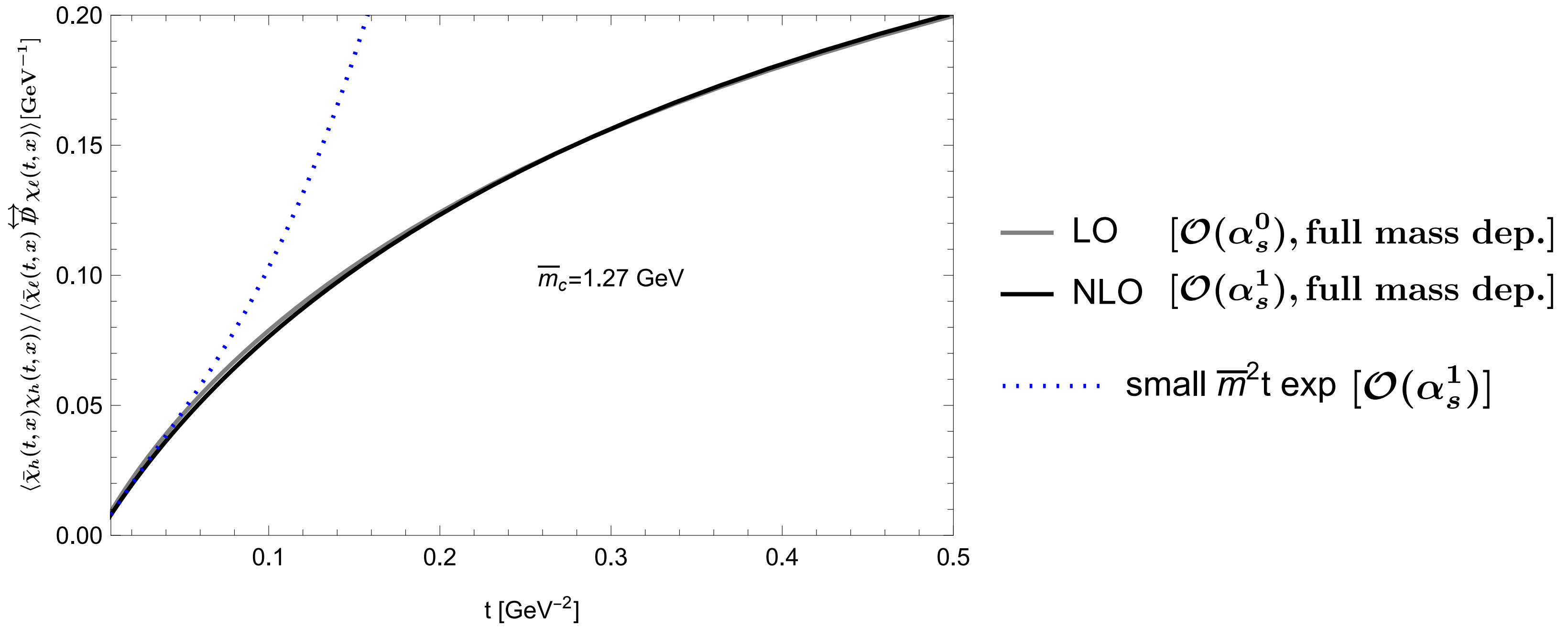
Input parameters: $\overline{m}_b = 4.18$ GeV , $\alpha_s(M_Z) = 0.1179$

Perturbative result for a physical quantity

$\langle\bar{\chi}_h(t,x)\chi_h(t,x)\rangle / \langle\bar{\chi}_\ell(t,x)\overleftrightarrow{\not{D}}\chi_\ell(t,x)\rangle$: UV finite quantity

Massless quark

Charm quark case



Input parameters: $\overline{m}_c = 1.27$ GeV , $\alpha_s(M_Z) = 0.1179$

Summary

- Heavy quark $\overline{\text{MS}}$ masses need to be known precisely in particle physics, especially in Higgs coupling measurement.

- I propose to use the gradient-flow quark condensates for precise heavy quark mass determination.

  -Gauge invariant

  -One-point function

- For this purpose, we perform perturbative calculation of a flowed *heavy quark* condensate $\langle \bar{\chi}_h(t,x)\chi_h(t,x)\rangle$ at two loops $[\mathcal{O}(\alpha_s^1)]$.

Future directions

Theory side

- It will be possible to extend the expansions in $\overline{m}_h^2 t$ or $(\overline{m}_h^2 t)^{-1}$ to higher orders.
- We are working on a perturbative calculation of $\langle\bar{\chi}_h(t,x)\overleftrightarrow{\not{D}}\chi_h(t,x)\rangle$ as well.

Two-loop results will be available for (at least) three quantities:

$$\frac{\langle\bar{\chi}_h(t,x)\chi_h(t,x)\rangle}{\langle\bar{\chi}_h(t,x)\overleftrightarrow{\not{D}}\chi_h(t,x)\rangle}, \quad \frac{\langle\bar{\chi}_h(t,x)\chi_h(t,x)\rangle}{\langle\bar{\chi}_\ell(t,x)\overleftrightarrow{\not{D}}\chi_\ell(t,x)\rangle}, \quad 1-\frac{\langle\bar{\chi}_h(t,x)\overleftrightarrow{\not{D}}\chi_h(t,x)\rangle}{\langle\bar{\chi}_\ell(t,x)\overleftrightarrow{\not{D}}\chi_\ell(t,x)\rangle}$$

High sensitivity is expected $\sim\mathcal{O}(\overline{m}_h^2 t)$

- Furthermore, three-loop calculations might be feasible.
- OPE structure? (convergence of perturbation theory and size of nonperturbative effects)

Lattice side

- *Important question: precision of lattice simulation of these quantities?*