Progress on the infinite volume based gradient flow for high precision determination of the $\Lambda_{\overline{MS}}$ scale of QCD.

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On behalf of LatHC collaboration

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LATTICE 2024

Determination of $\alpha_S(m_Z)$

- Goal: High precision determination of the strong coupling $\alpha_S(m_Z)$ at the Z-pole in QCD with massless $N_f = 3$ fermions.
- The determination of $\alpha_S(m_Z)$, equivalently $\mu^{-1}\Lambda_{\overline{MS}}$, requires integration of the inverse β-function from perturbative regime to the scale of hadron physics at strong coupling.

$$\mu^{-1} \cdot \Lambda_{\overline{MS}} = (b_0 \bar{g}^2)^{-b_1/2b_0^2} \cdot \exp(-1/2b_0\bar{g}^2) \cdot \exp\left(-\int_0^{\bar{g}} dx[1/\beta(x) + 1/b_0x^3 - b_1/b_0^2 x]\right)$$

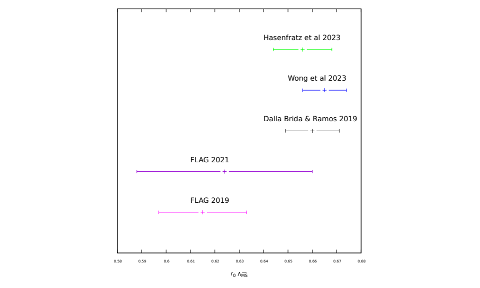
- The integration beyond the perturbative regime requires non-perturbative calculation of the β-function, which can be done on lattice
- On the lattice:
  - g^2 and β-function are not defined in $\overline{MS}$ scheme, but it is not an issue as long as the conversion to $\Lambda_{\overline{MS}}$ is known
  - μ^{-1} is typically r_0 or $\sqrt{8t_0}$ where $\bar{g}^2 = g^2(\mu)$ is known

Determination of $\alpha_S(m_Z)$

- Since 2017, we developed a variety of high-precision, gradient-flow based strategies utilizing β-function defined in infinite physical volume, applied in the sextet model, $N_f = 10$ and $N_f = 12$ models.
- Our goal is to apply this method on $N_f = 3$ model. In Lattice 2022, we applied this to $N_f = 0$ and $N_f = 10$ as pilot studies.
- It turns out $N_f = 0$, supposedly the simplest case, is already very elaborated

$N_f = 0$ model

- The FLAG 2019 world average summarizes studies prior to GF-based determinations
- The GF-based determination by Dalla Brida & Ramos 2019 significantly deviates from FLAG 2019 world average, creating tension with previous studies
- Our results, Wong et al 2023(PoS 2022), agrees with Dalla Brida & Ramos 2019
- Hasenfratz et al 2023, using the same approach as us, got very consistent results with us and worsens the tension



- (Related topic in Alberto Ramos' talk in this session)

$N_f = 0$ model

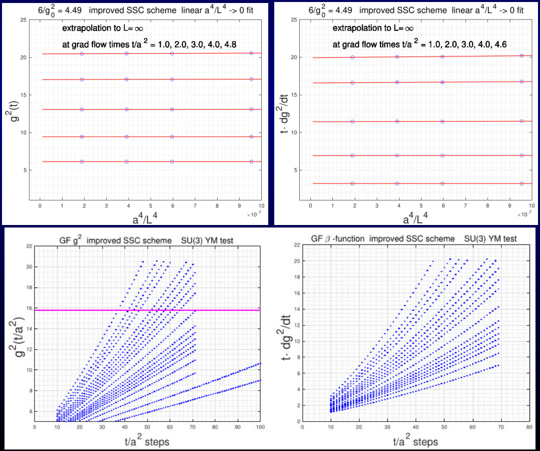
- The error in our preliminary result reported in Wong et al 2023: $\approx 1.4\%$
- FLAG 2024(to be published) decides to add a dedicated section about GF-based determinations
- Their criticism towards our work:
  "We conclude that at this reference coupling a determination of the Λ-parameter to better than 5 % seems impossible"
- This motivates us to improve our dataset and analysis to check whether this is the case
- In this talk, we discuss the progress on such investigation

LATTICE 2022 Results

- Simulation details of $N_f = 0$:
  - Symanzik improved $SU(3)$ gauge action in periodic boundary conditions
  - Heat-bath + over relaxation ($N_{\text{overrelax}} = 3$)
  - Only improved SSC scheme (definitions in [2203.15847]) is discussed

LATTICE 2022 Results

- Methodology:
  - $g^2(t)$ is defined in infinite physical volume wtih Gradient Flow schemes at flow time t
  - $\beta(g^2(t)) = t \cdot dg^2(t)/dt$ is obtained, approximated by 5-point stencil
  - Infinite volume limit $\Rightarrow$ Target g^2 $\Rightarrow$ Continuum limit
  - Scan g^2 values from weak to strong coupling regimes up to $g^2(t_0)$



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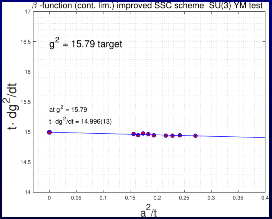
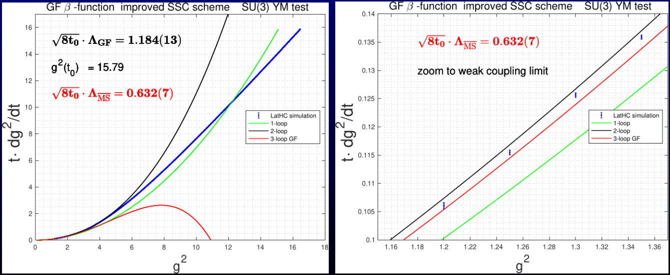


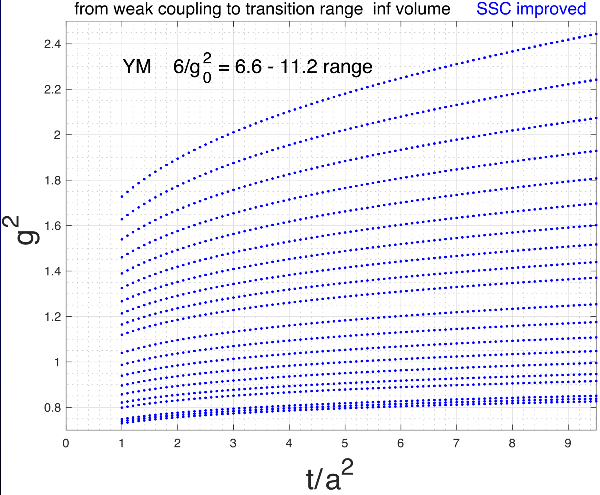
LATTICE 2022 Results

- Numerical data is not available for $g^2 < g^2_{\rm PT} = 1.2$
- Assume that β-function is well-described by the known 3-loop perturbative GF β-function($\beta^{\text{3-loop}}_{\rm GF}$)[JHEP06(2016)161] at $g^2 < g^2_{\rm PT}$
- $\sqrt{8t_0}\,\Lambda_{\rm GF}$ is obtained by :
  - Integrating $\beta^{\text{3-loop}}_{\rm GF}$ from 0 to $g^2_{\rm PT}$
  - Integrate a spline-fit using data between $g^2_{\rm PT}$ and $g^2(t_0)$
- Convert $\Lambda_{\rm GF}$ into $\Lambda_{\rm MS}$ using the well known 1-loop calculation
- Convert $\sqrt{8t_0}$ into r_0 with $\sqrt{8t_0}/r_0 = 0.948(7)$[JHEP08(2010)071]
- $r_0\,\Lambda_{\overline{MS}} = 0.665(9)$ is obtained



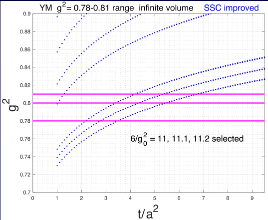
Improvements

- In this talk, we focus on the weak coupling regime $g^2 \leq 1.8$
- We have improvements in both dataset and analysis
- The dataset extended by ≈ 20 more ensembles aiming at the weak coupling regime

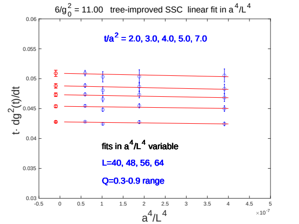


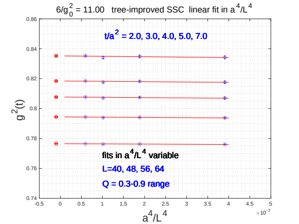
Improvements

- In the LAT 22 analysis, there are very few intersections between targeted g^2 and available data unless $6/g_0^2$ is sampled densely and carefully chosen in the right ranges



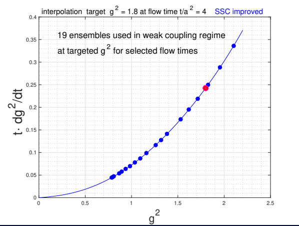
- $\Rightarrow$ We modify our analysis as follows:
  - At each t/a^2 and $6/g_0^2$, take infinite volume limit of g^2 and β

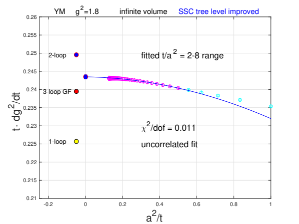




Improvements

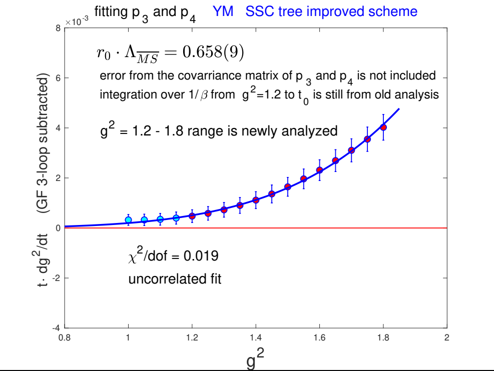
- At each t/a^2 and $6/g_0^2$, take infinite volume limit of g^2 and β
- At each t/a^2, for a set of target g^2 values, interpolate g^2 and β among $6/g_0^2$ values in infinite volume limit
- Take continuum limit of β at each targeted g^2 values





Improvements

- At $g^2 = 1.2$, the difference between 2-loop and 3-loop perturbative β-functions is large and 4-loop is not available
- $\Rightarrow$ Replacement with 3-loop perturbative β-function below g^2_{PT} may introduce systematic errors coming from unknown higher-order contributions
- The effect can be put under control by fitting β a polynomial with p_3 and p_4: $\beta(g^2) = g^4(b_0 + b_1\, g^2 + b_2\, g^4 + p_3\, g^6 + p_4\, g^8)$, then integrate using the fitted polynomial



Conclusion

- Result of replacing the three-loop β function with fitted polynomial within $g^2 = 0$ to 1.2 : $r_0 \cdot \Lambda_{\overline{MS}} = 0.658(9)$
- LATTICE 2022: $0.665(9)$ $\Leftarrow 1-\sigma$ difference
- The data at different g^2 are correlated. Correlation fits are required
- The consistency among different schemes, different fit ansatz and different orders of limits remains to be checked
- We will extend the modified analysis into the range $g^2 > 1.2$
- The limit $g^2_{\mathrm{PT}} \to 0$ needs to be taken