Scale setting from a combination of lattice QCD formulations with Wilson and Wilson twisted mass valence quarks

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and Alessandro Conigli, Gregorio Herdoiza, Carlos Pena

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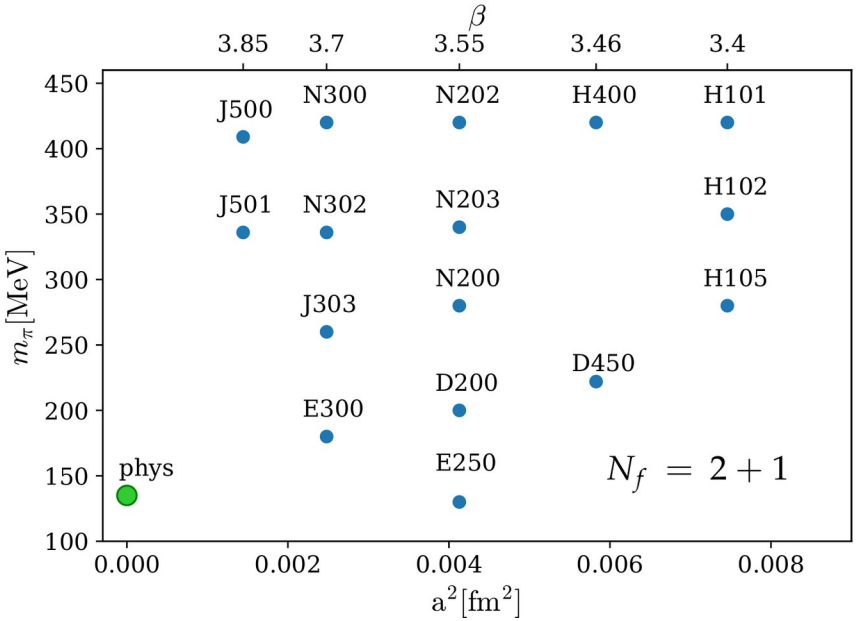
Instituto de Física Teórica IFT UAM-CSIC

Lattice24 @ Liverpool

Mixed Action: sea

- **Sea sector**: $\mathcal{O}(a)$ improved Wilson fermions, CLS, open boundary conditions
- **Chiral trajectory:**

$$\mathrm{tr}\left(M_q^{(s)}\right) = 2m_{q,l}^{(s)} + m_{q,s}^{(s)} = \mathrm{cnst.}$$



[Lüscher and Schaefer, JHEP,1107 036; Bruno et al. JHEP 1502 043 - 1712.04884 - 2003.13359], [Lüscher and Schaefer,1206.2809]
[ALPHA, hep-lat/0101001; Frezzotti and Rossi hep-lat/0306014, Pena et al., hep-lat/0405028], [S. Kuberski, 2306.02385]

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$$\mathrm{tr}\left(M_q^{(\mathrm{s})}\right) = 2m_{q,l}^{(\mathrm{s})} + m_{q,s}^{(\mathrm{s})} = \mathrm{const.}$$

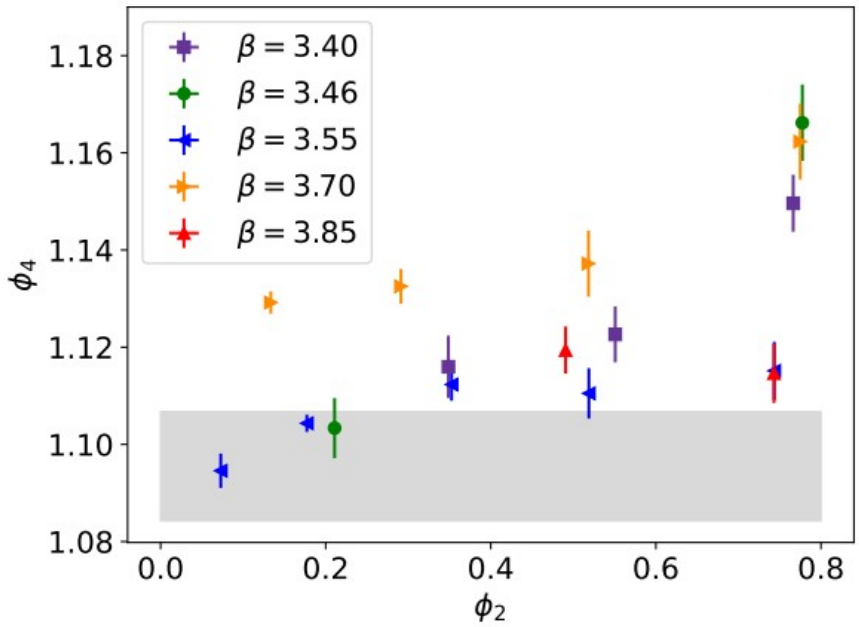
$$O\left(\phi_4^{(\mathrm{s})\prime} = \phi_4^{\mathrm{ph}}\right) = O\left(\phi_4^{(\mathrm{s})}\right) + \left(\phi_4^{\mathrm{ph}} - \phi_4^{(\mathrm{s})}\right)\frac{dO}{d\phi_4^{(\mathrm{s})}}$$

$$\frac{dO}{dm_q^{(\mathrm{s})}} = \sum_i \frac{\partial O}{\partial \langle P_i \rangle}\left[\left\langle \frac{\partial P_i}{\partial m_q^{(\mathrm{s})}}\right\rangle - \left\langle P_i \frac{\partial S}{\partial m_q^{(\mathrm{s})}}\right\rangle + \langle P_i\rangle \left\langle \frac{\partial S}{\partial m_q^{(\mathrm{s})}}\right\rangle\right]$$

2-pt funcs.

$$\phi_2 = 8t_0 m_\pi^2 \propto m_l^{\mathrm{R}},$$

$$\phi_4 = 8t_0\left(m_K^2 + \frac{1}{2}m_\pi^2\right) \propto 2m_l^{\mathrm{R}} + m_s^{\mathrm{R}} = \mathrm{tr}\left(M_q^{\mathrm{R}}\right)$$



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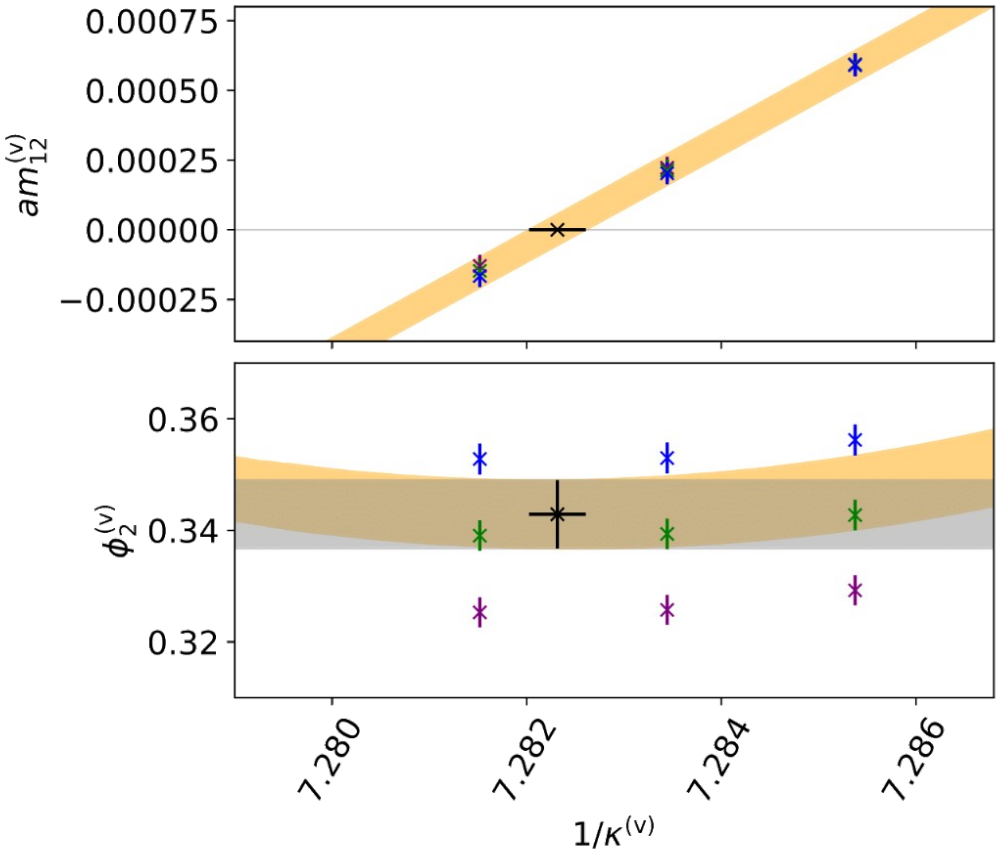
Mixed Action: valence

$$D_{\mathrm{W}} + \boldsymbol{m} + i\gamma_5\boldsymbol{\mu} + c_{\mathrm{sw}} a \frac{1}{2} \sum_{\mu<\nu} \sigma_{\mu\nu} \hat{F}_{\mu\nu}$$

- **Unitarity restoration**: matching sea and valence
- **Maximal twist**: automatic $\mathcal{O}(a)$ improvement up to residual effects $\mathcal{O}\left(a g_0^4 \mathrm{tr}\left(M_q^{(s)}\right)\right)$



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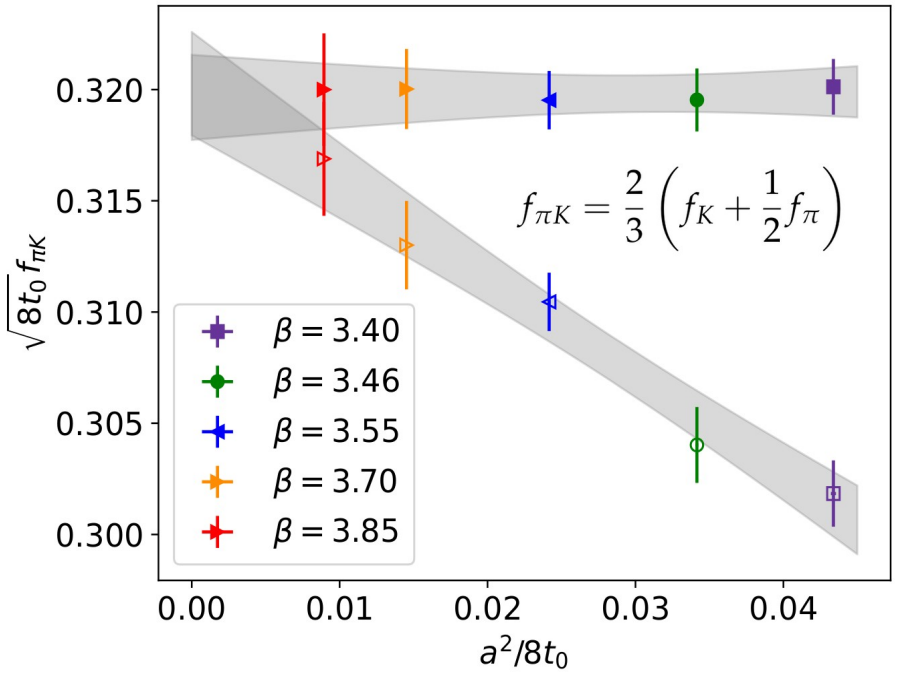
Universality

- **Continuum limit scaling:**
  - Wilson (empty)
  - Mixed Action (filled)
- **Symmetric point:**

$$m_\pi = m_K$$

$$\phi_2^{\mathrm{sym}} = \frac{2}{3}\phi_4^{\mathrm{ph}}$$



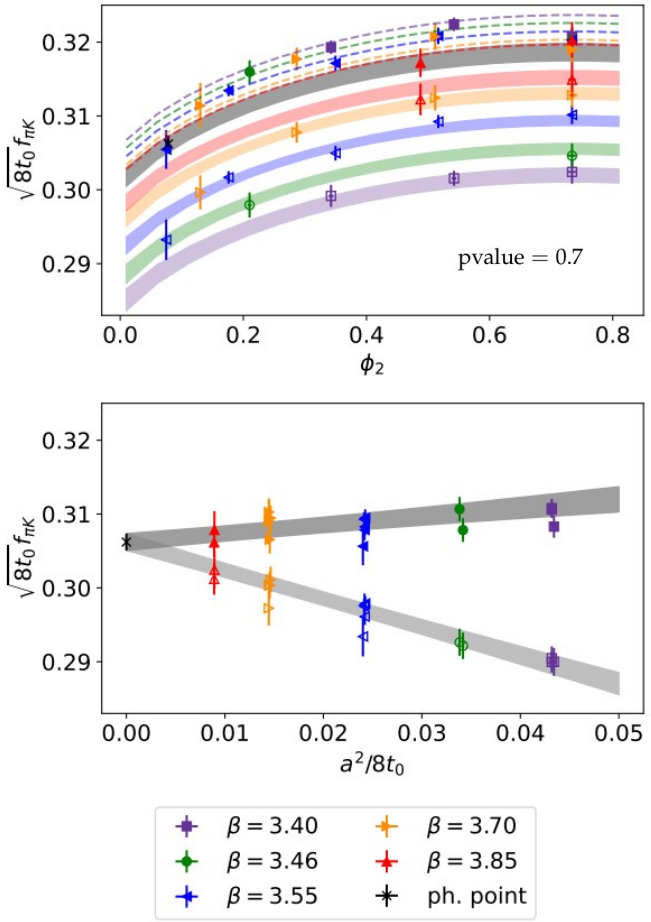
Scale Setting

- Set the scale with $f_{\pi K} = \frac{2}{3}\left(f_K + \frac{1}{2} f_\pi\right)$
- Combine Wilson & Mixed Action (Wtm)
- SU(3) ChPT, $\mathcal{O}(a^2)$ effects, no data cuts
- Minimize:

$$\chi^2 = \sum_{i,j=1}^{N_{\text{dat}}} \left(y_i - f(x_i;\vec{p})\right) \mathcal{W}_{ij} \left(y_j - f(x_j;\vec{p})\right)$$

$$\mathcal{W}_{ij}^{-1} = C_{ij} \times \sqrt{1 + c_i^2 / C_{ii}} \sqrt{1 + c_j^2 / C_{jj}}$$

$$c_i^2 = c_\beta^2 \left(\frac{a^4}{t_0^2}\right)^2 + c_{\phi_2}^2 \left(\phi_2^2\right)^2$$



[M. Dalla Brida et al., 1607.06423], [M. Bruno et al., 1608.08900]

Scale Setting

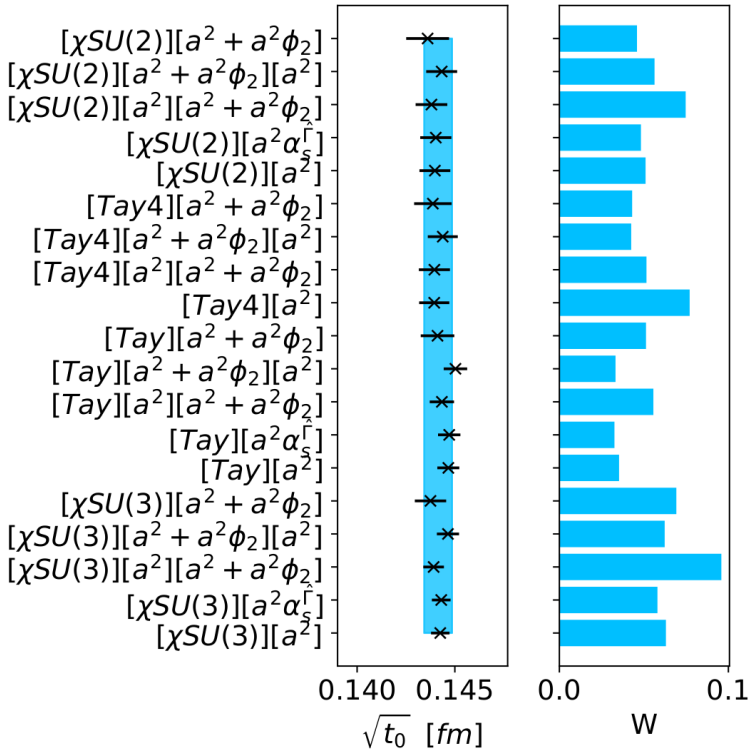
$$\left\langle \sqrt{t_0^{\rm ph}} \right\rangle = \sum_i \sqrt{t_0^{{\rm ph},(i)}} W^{(i)} \qquad \sigma^2_{\rm syst} = \left\langle \sqrt{t_0^{\rm ph^2}} \right\rangle - \left\langle \sqrt{t_0^{\rm ph}} \right\rangle^2$$

- **Model variation:**
  - SU(3), SU(2), Taylor,

    $\mathcal{O}(a^2)$, $\mathcal{O}(a^2\phi_2)$, $\mathcal{O}(a^2\alpha_s^{\Gamma_i})$

    $\Gamma_{i=1,2,3,4,5} = -0.111,\ 0.247,\ 0.519,\ 0.668,\ 0.760$
  - Cuts in data:
    - Lattice spacing
    - Pion mass
    - Volume

$$W \propto \exp\left(-\frac{1}{2}\left(\chi^2 - 2\left\langle\chi^2\right\rangle\right)\right)$$



[J. Frison, 2302.06550], [M. Bruno, R. Sommer, 2209.14188], [E. Neil et al., 2305.19417], [N. Husung, 2206.03536]

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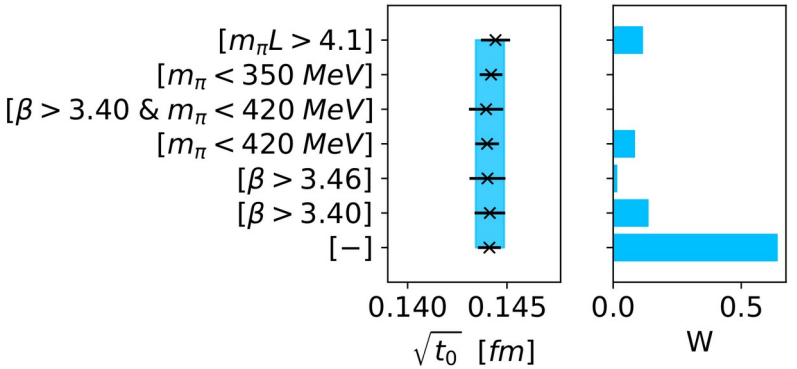
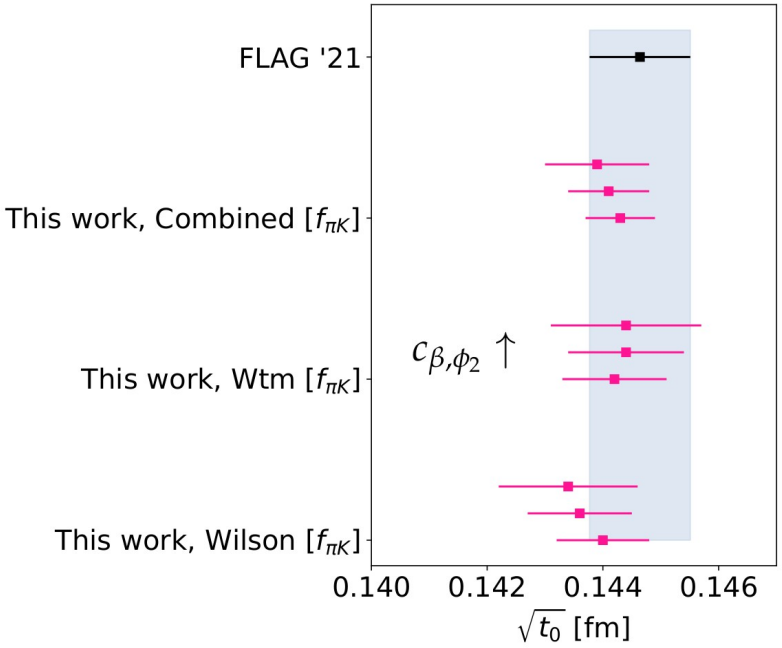
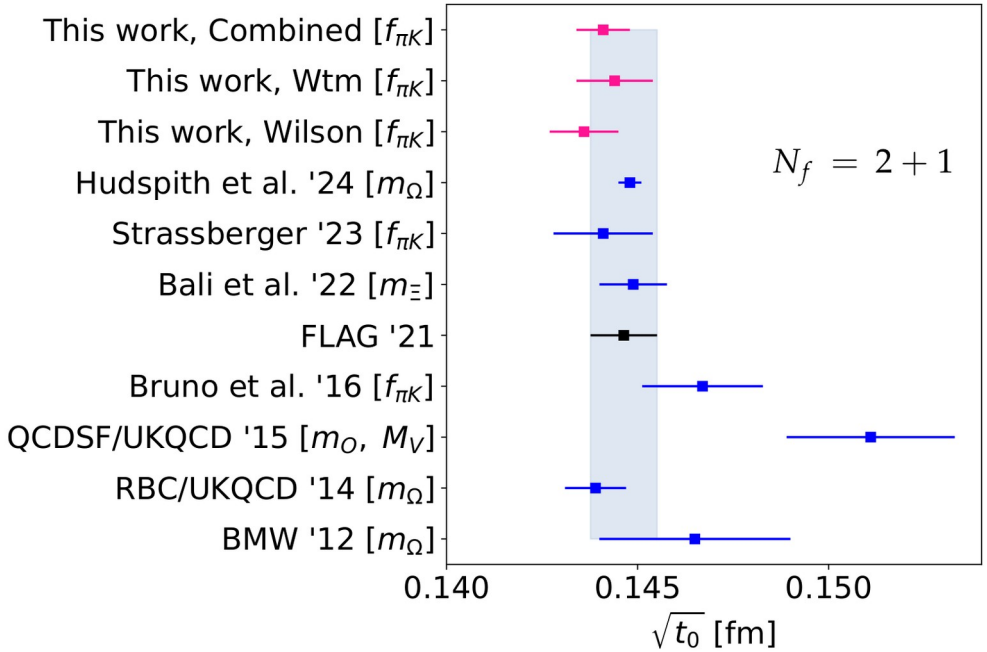


[J. Frison, 2302.06550], [M. Bruno, R. Sommer, 2209.14188], [E. Neil et al., 2305.19417], [N. Husung, 2206.03536]

Results





$$\mathcal{W}_{ij}^{-1} = C_{ij} \times \sqrt{1 + c_i^2/C_{ii}}\sqrt{1 + c_j^2/C_{jj}} \longrightarrow c_i^2 = c_\beta^2 \left(\frac{a^4}{t_0^2}\right)^2 + c_{\phi_2}^2 \left(\phi_2^2\right)^2$$

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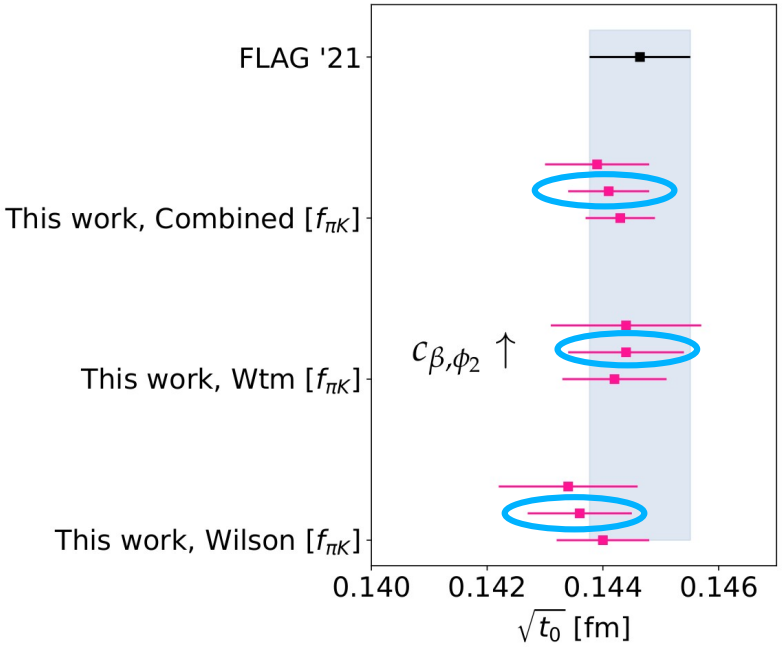

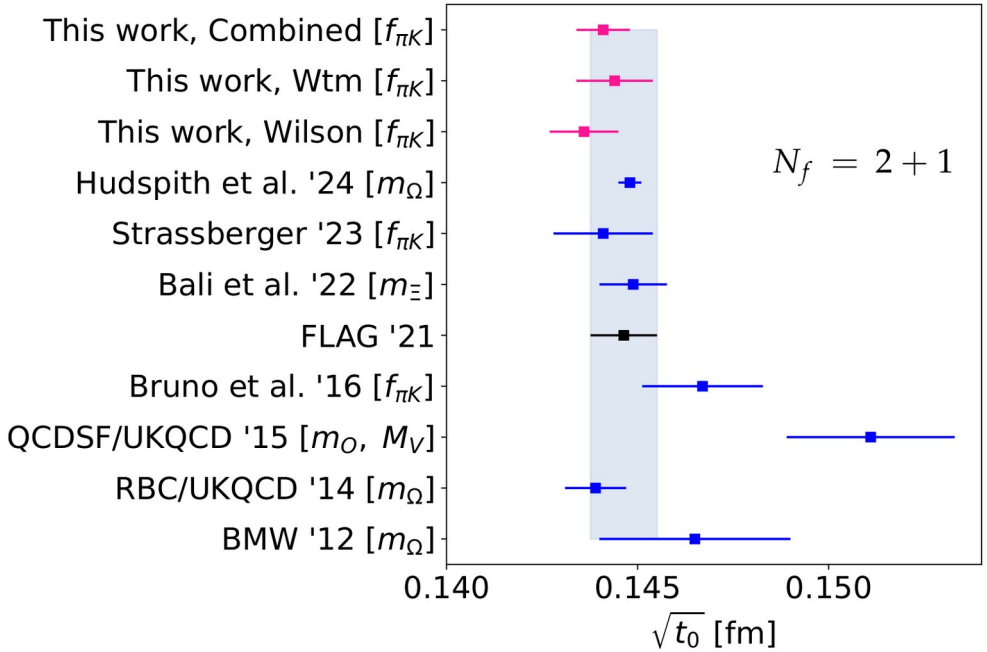

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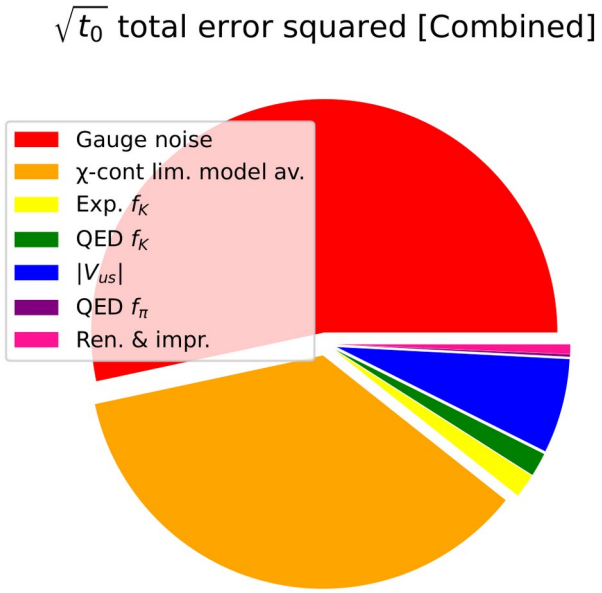
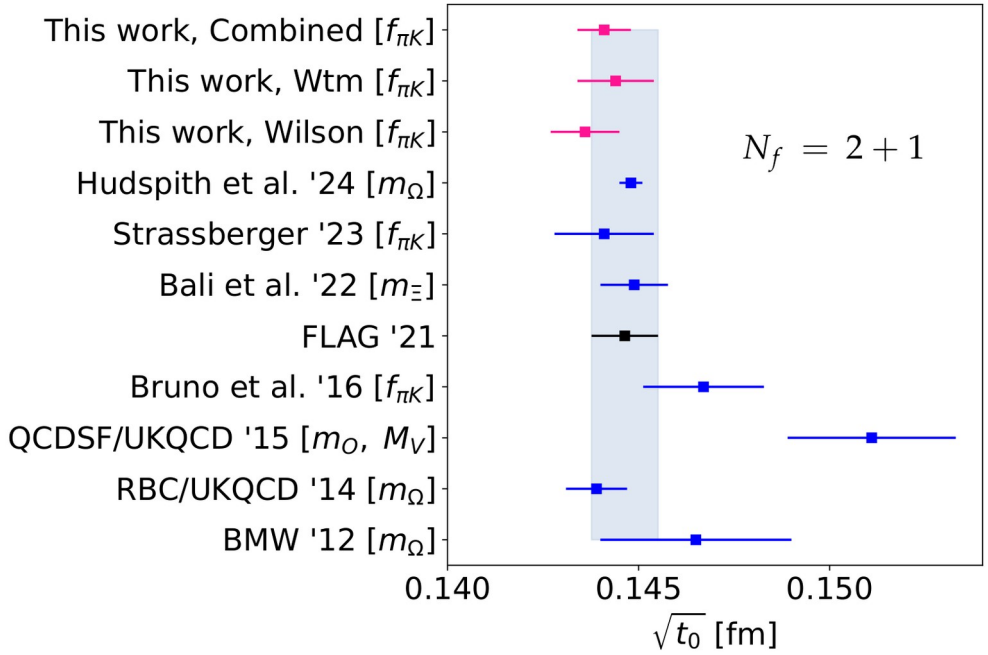

$\sqrt{t_0}$ total error squared [Combined]
Gauge noise
χ-cont lim. model av.
Exp. f_K
QED f_K
$|V_{us}|$
QED f_π
Ren. & impr.


This work, Combined [$f_{\pi K}$]
This work, Wtm [$f_{\pi K}$]
This work, Wilson [$f_{\pi K}$]
Hudspith et al. '24 [m_Ω]
Strassberger '23 [$f_{\pi K}$]
Bali et al. '22 [m_Ξ]
FLAG '21
Bruno et al. '16 [$f_{\pi K}$]
QCDSF/UKQCD '15 [m_O, M_V]
RBC/UKQCD '14 [m_Ω]
BMW '12 [m_Ω]
$N_f = 2+1$
0.140
0.145
0.150
$\sqrt{t_0}$ [fm]

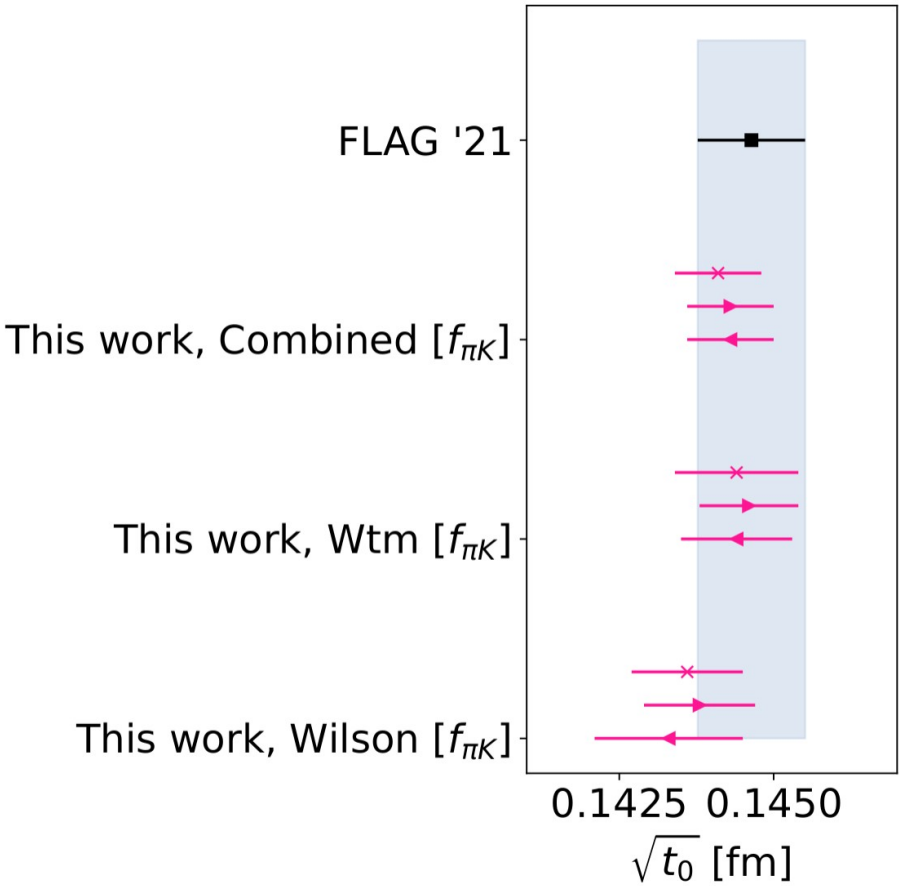
Information Criteria

$$\mathrm{TIC} = \chi^2 - 2\langle\chi^2\rangle$$

$$\mathrm{AIC}^{perf} = \chi^2 + 2n_{param} + 2n_{cut}$$

$$\mathrm{AIC}^{sub} = \chi^2 + 2n_{param} + n_{cut}$$

$$\mathrm{AIC}^{(0)} = \chi^2 + 2n_{param}$$



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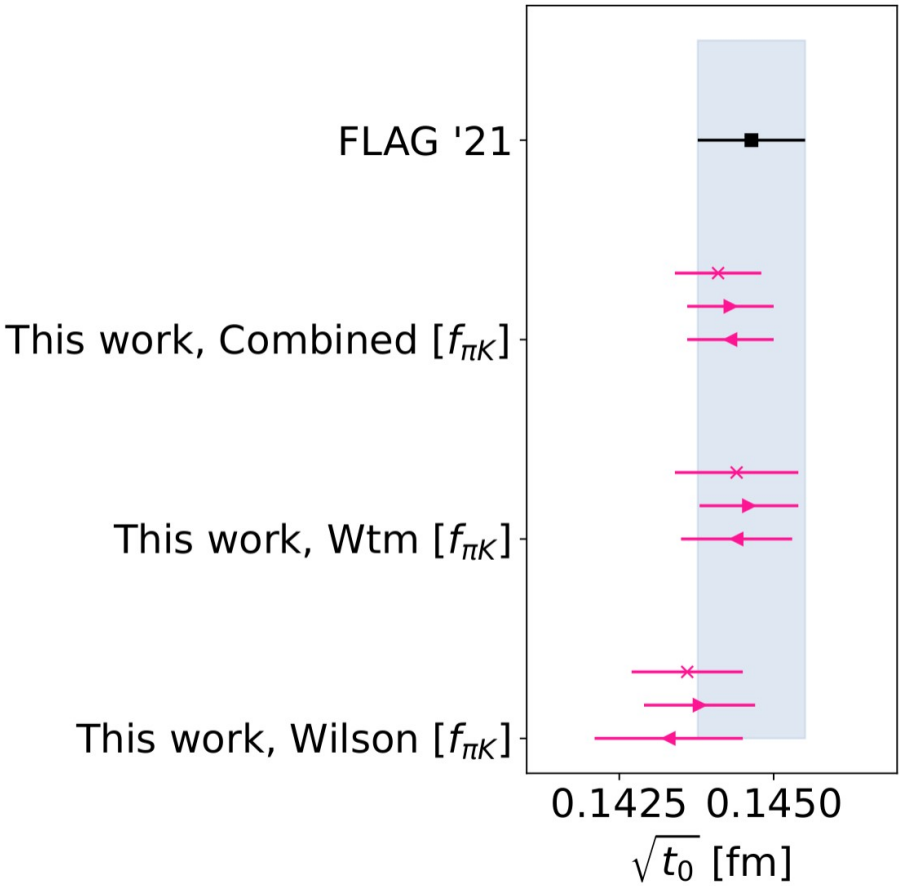

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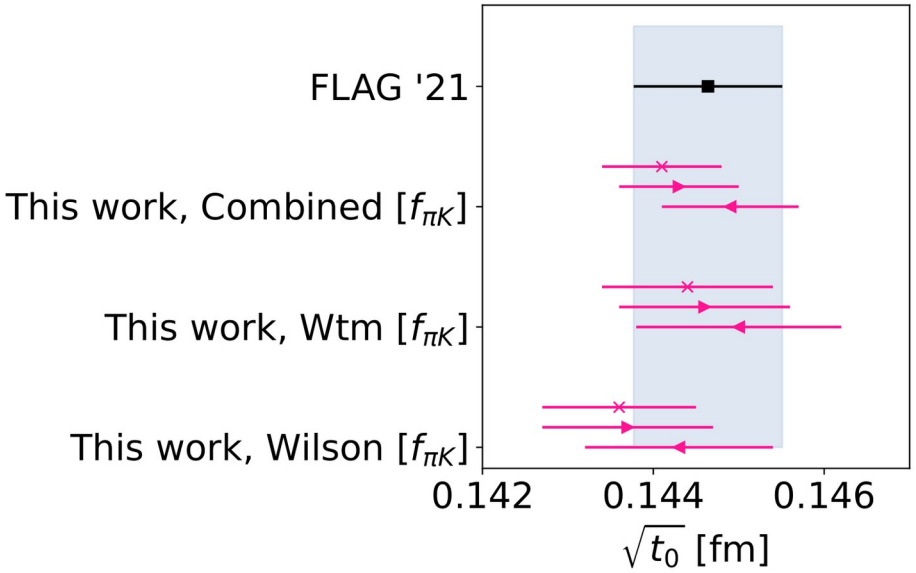
Physical Input



FLAG '21

$$m_\pi^{\text{isoQCD}} = 134.9768(5)\ \text{MeV},$$
$$m_K^{\text{isoQCD}} = 497.611(13)\ \text{MeV},$$
$$f_\pi^{\text{isoQCD}} = 130.56(2)_{\text{exp}}(13)_{\text{QED}}(2)_{|V_{ud}|}\ \text{MeV},$$
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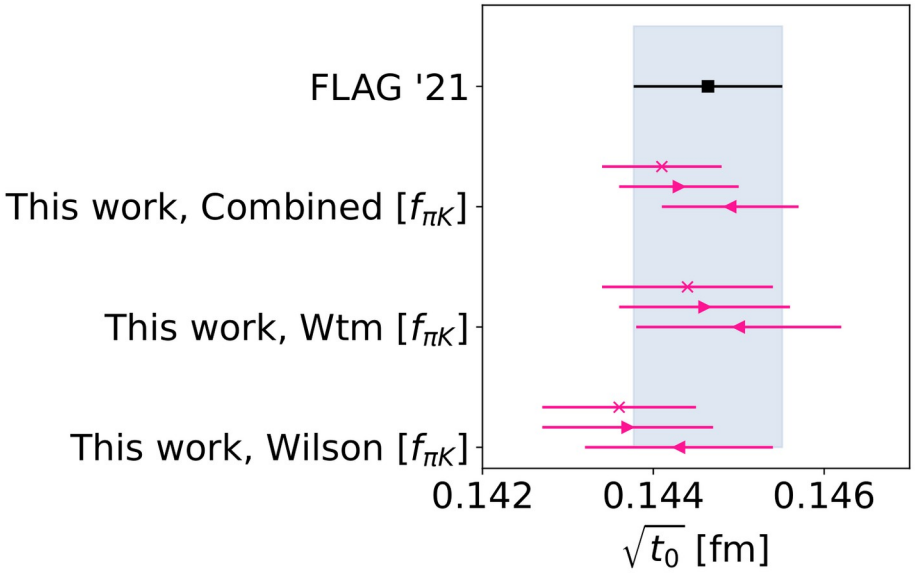
[QCD+QED Workshop, Edinburgh, 2023. **Link**]

$$m_\pi^{\text{isoQCD}} = 135.0\ \text{MeV},$$
$$m_K^{\text{isoQCD}} = 494.6\ \text{MeV},$$
$$f_\pi^{\text{isoQCD}} = 130.5\ \text{MeV}.$$

FLAG '16

$$m_\pi^{\text{isoQCD}} = 134.8(3)\ \text{MeV},$$
$$m_K^{\text{isoQCD}} = 494.2(3)\ \text{MeV},$$
$$f_\pi^{\text{isoQCD}} = 130.4(2)\ \text{MeV},$$
$$f_K^{\text{isoQCD}} = 156.2(7)\ \text{MeV},$$

Physical Input



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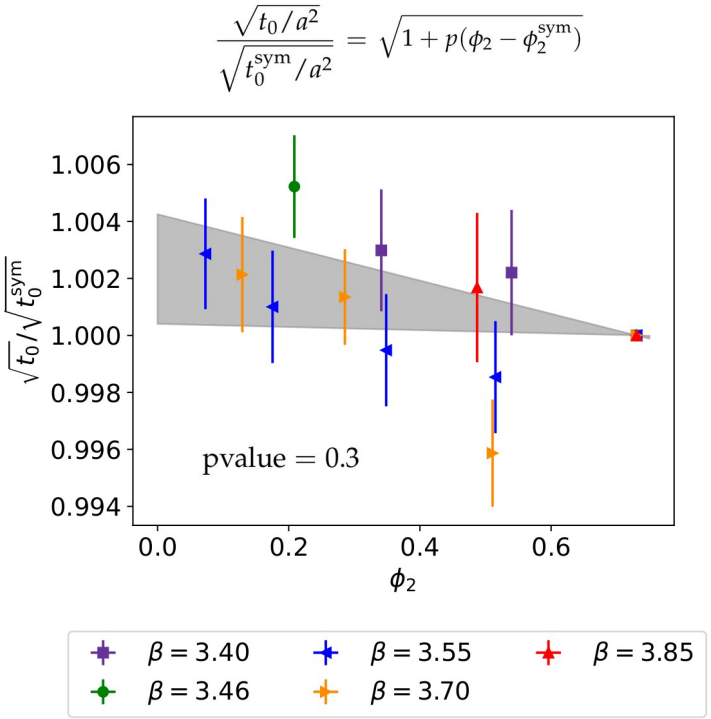
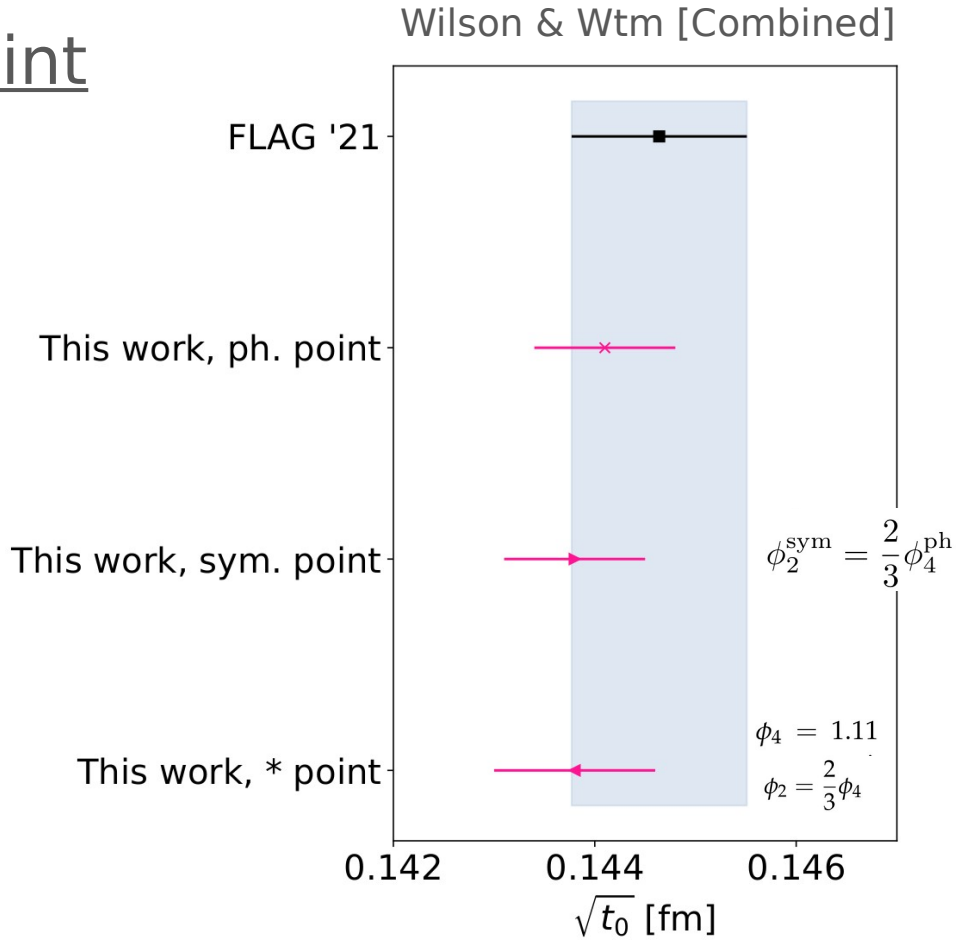
$\sqrt{t_0}$ at the symmetric point

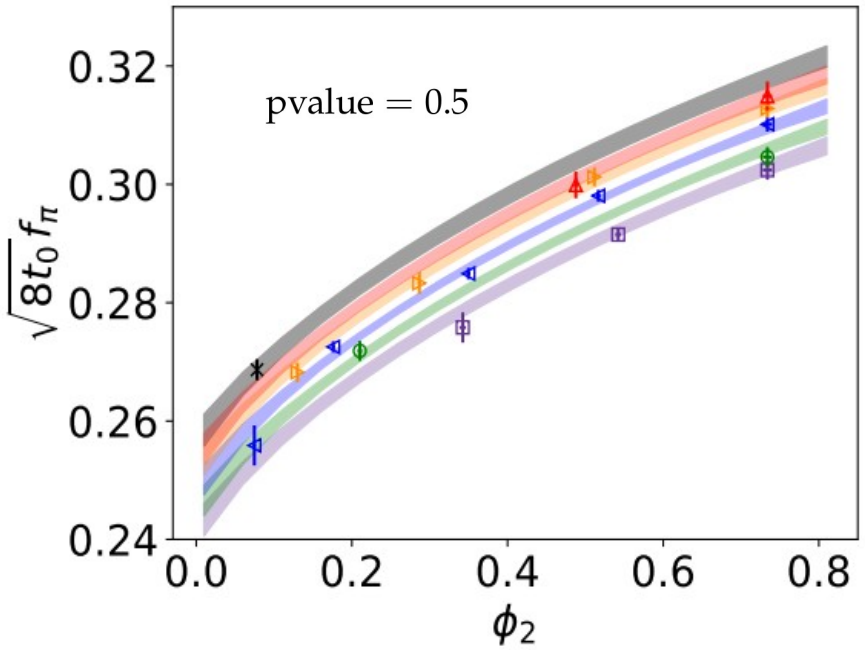


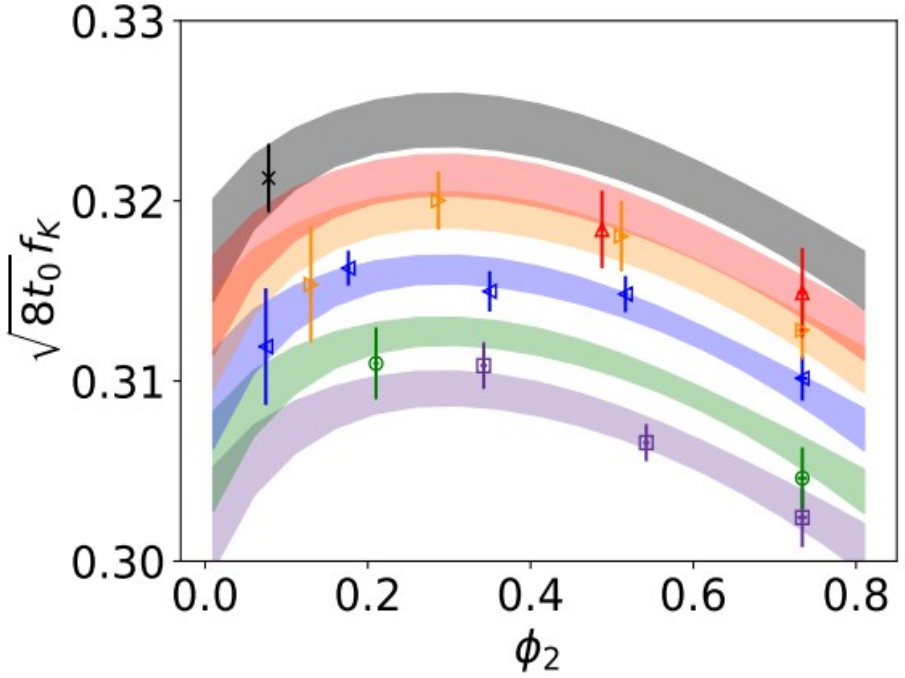


[Strassberger et al., 2112.06696]

Scale Setting with f_π

PRELIMINARY, Wilson





SU(3) ChPT, $\mathcal{O}(a^2)$ effects, no cuts in data

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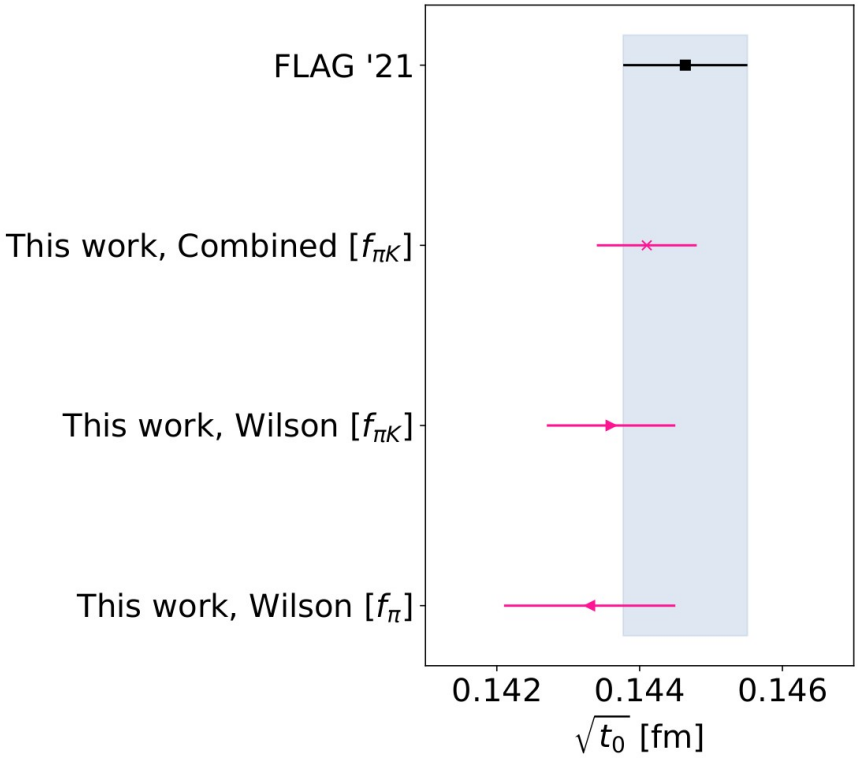
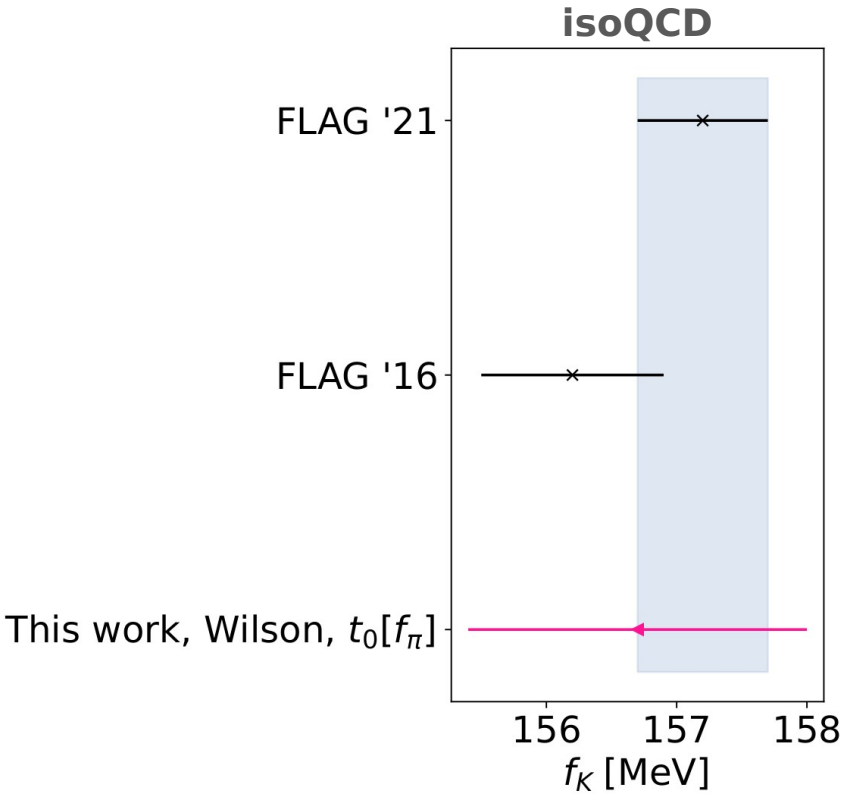


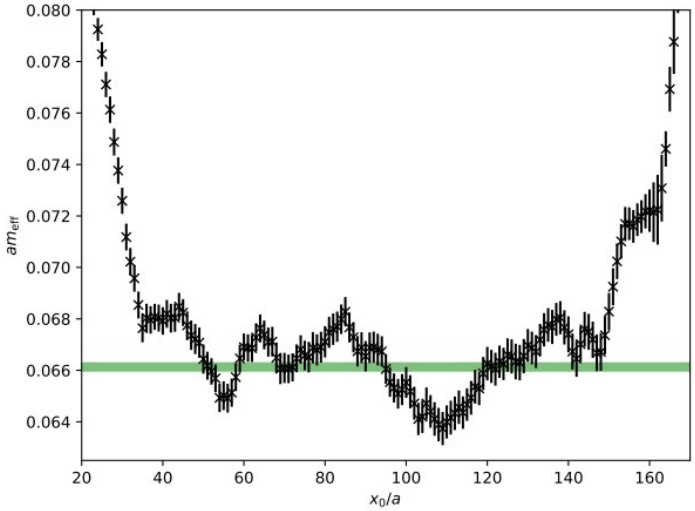


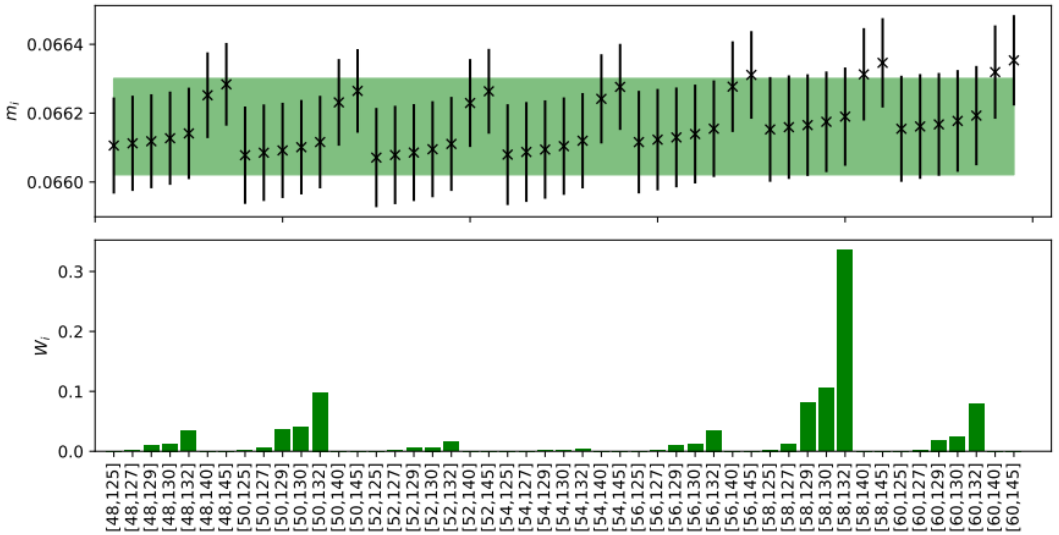
Thank you!

We acknowledge support of the project PID2021-127526NB-I00, funded by MCIN/AEI/10.13039/501100011033 and by FEDER, EU.

Ground state signals

$$W \propto \exp\left(-\frac{1}{2}\left(\chi^2 - 2\langle\chi^2\rangle\right)\right)$$





J501 pion effective mass, fit to a plateau, vary over plateau range

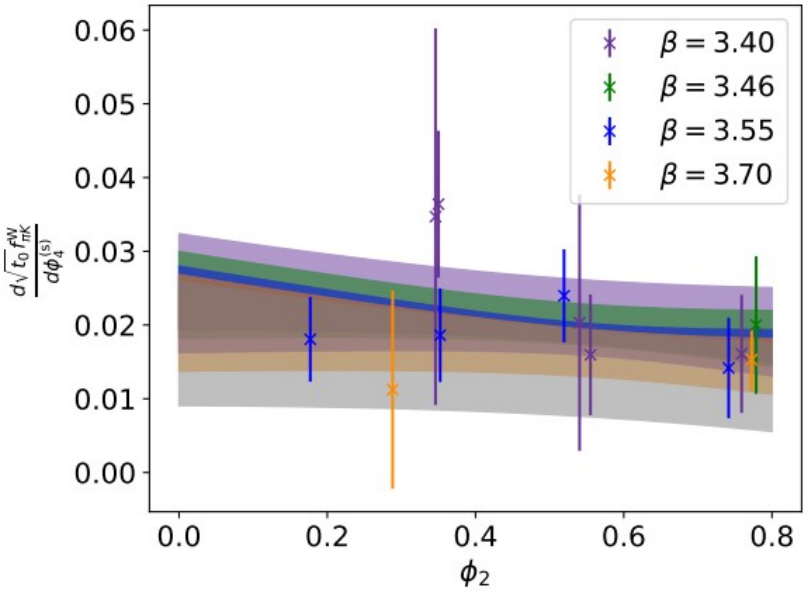
Mass shifting

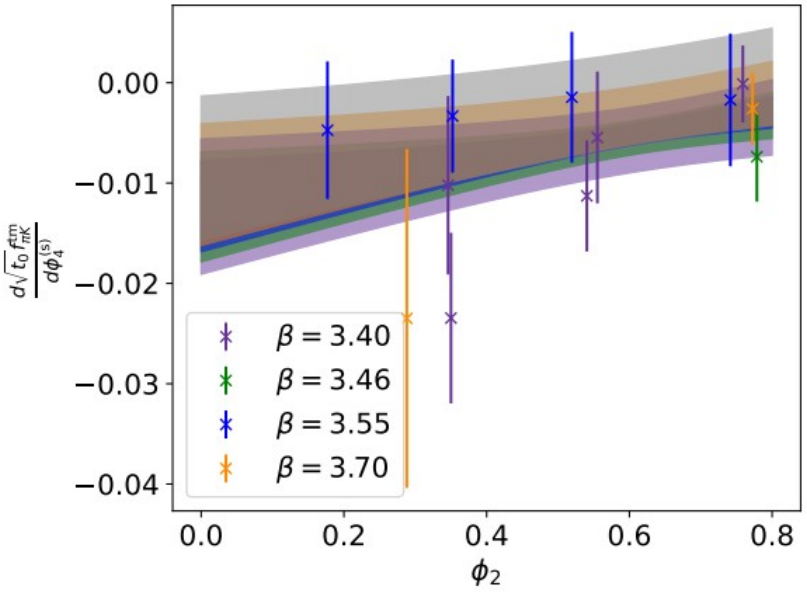
$$\phi_2 = 8t_0 m_\pi^2 \propto m_l^{\mathrm{R}},$$

$$\phi_4 = 8t_0 \left(m_K^2 + \frac{1}{2} m_\pi^2 \right) \propto 2m_l^{\mathrm{R}} + m_s^{\mathrm{R}} = \mathrm{tr}\left(M_q^{\mathrm{R}} \right)$$

$$O\left(\phi_4^{(\mathrm{s})\prime} = \phi_4^{\mathrm{ph}}\right) = O\left(\phi_4^{(\mathrm{s})}\right) + \left(\phi_4^{\mathrm{ph}} - \phi_4^{(\mathrm{s})}\right) \frac{dO}{d\phi_4^{(\mathrm{s})}}$$

$$\frac{dO}{dm_q^{(\mathrm{s})}} = \sum_i \frac{\partial O}{\partial \langle P_i \rangle} \left[\left\langle \frac{\partial P_i}{\partial m_q^{(\mathrm{s})}} \right\rangle - \left\langle P_i \frac{\partial S}{\partial m_q^{(\mathrm{s})}} \right\rangle + \langle P_i \rangle \left\langle \frac{\partial S}{\partial m_q^{(\mathrm{s})}} \right\rangle \right]$$





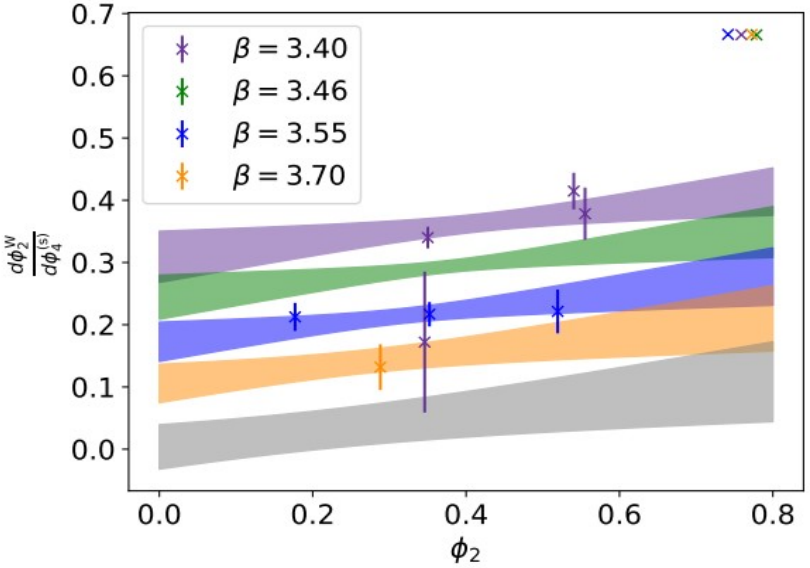
[Strassberger et al. 2112.06696]

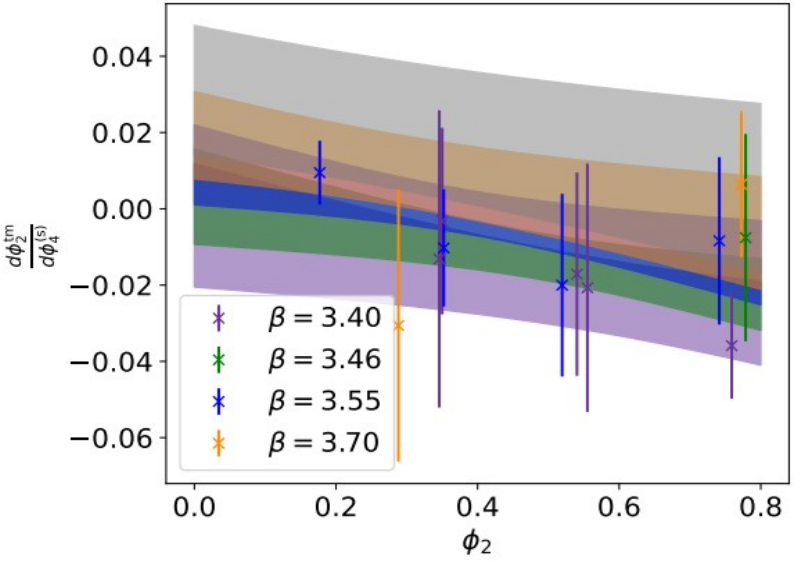
Mass shifting

$$O\left(\phi_4^{(s)\prime} = \phi_4^{ph}\right) = O\left(\phi_4^{(s)}\right) + \left(\phi_4^{ph} - \phi_4^{(s)}\right)\frac{dO}{d\phi_4^{(s)}}$$

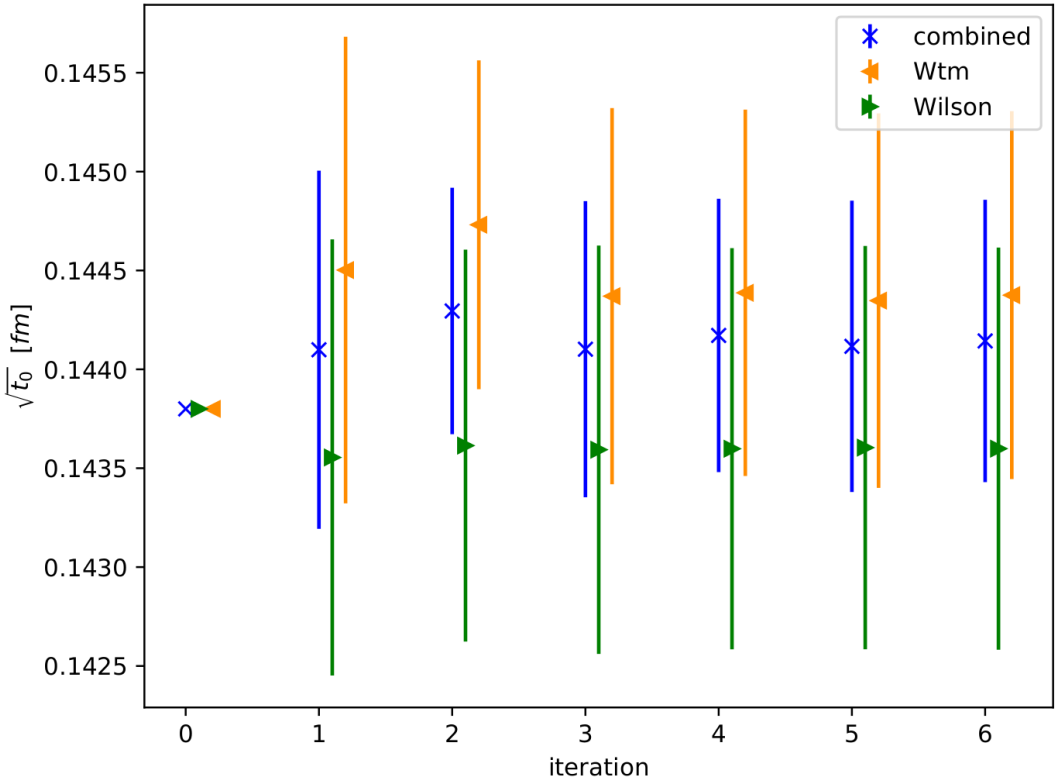
$$\phi_2 \propto m_l^R,$$
$$\phi_4 \propto 2m_l^R + m_s^R = \mathrm{tr}\left(M_q^R\right)$$

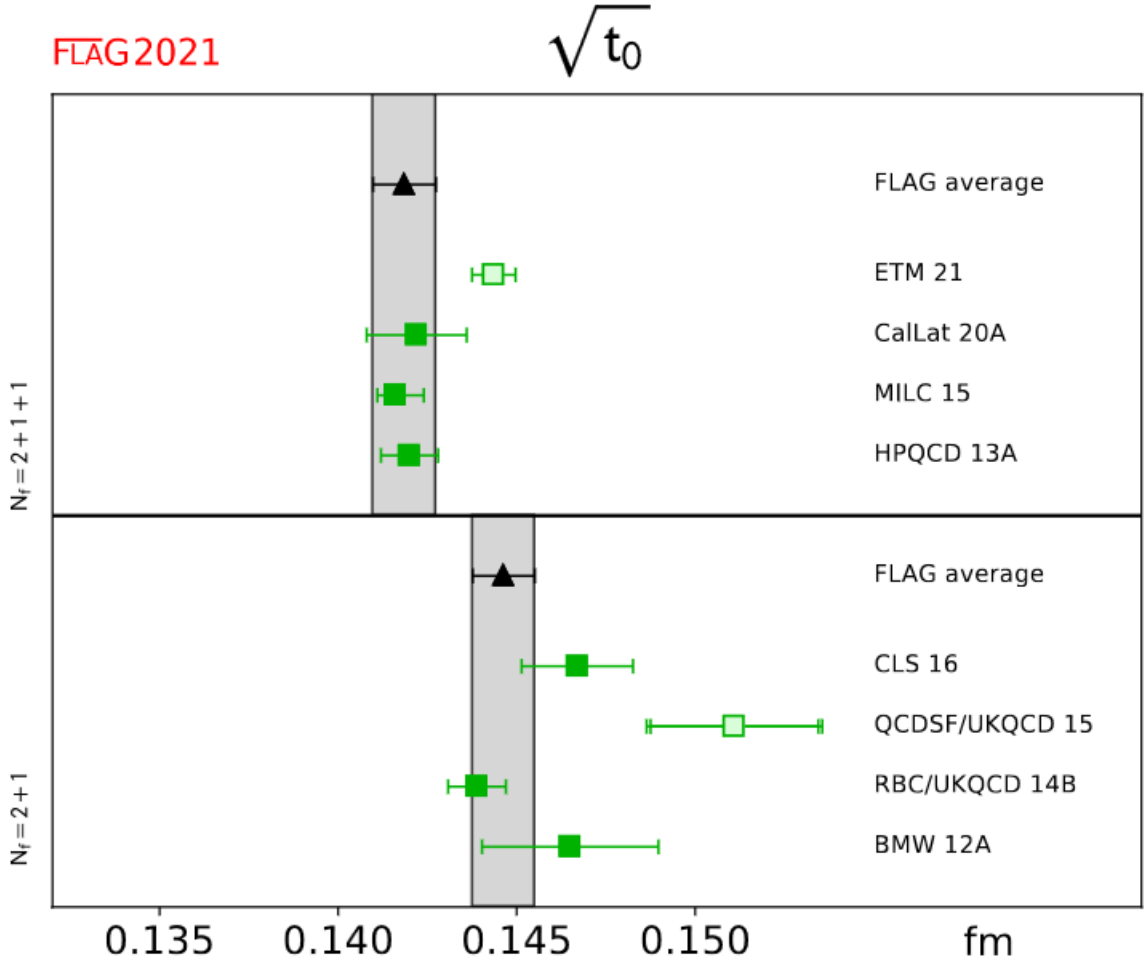
$$\frac{dO}{dm_q^{(s)}} = \sum_i \frac{\partial O}{\partial \langle P_i \rangle}\left[\left\langle \frac{\partial P_i}{\partial m_q^{(s)}} \right\rangle - \left\langle P_i \frac{\partial S}{\partial m_q^{(s)}} \right\rangle + \langle P_i \rangle \left\langle \frac{\partial S}{\partial m_q^{(s)}} \right\rangle\right]$$





Iterative determination of $\sqrt{t_0}$




FLAG2021
$\sqrt{t_0}$
$N_f = 2+1+1$
FLAG average
ETM 21
CalLat 20A
MILC 15
HPQCD 13A
$N_f = 2+1$
FLAG average
CLS 16
QCDSF/UKQCD 15
RBC/UKQCD 14B
BMW 12A
0.135
0.140
0.145
0.150
fm