How does Your Black Hole Grow?

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Motivation

Black holes are fascinating test grounds for GR.

They represent regions of very strong gravity, and constrain any candidate to challenge GR.

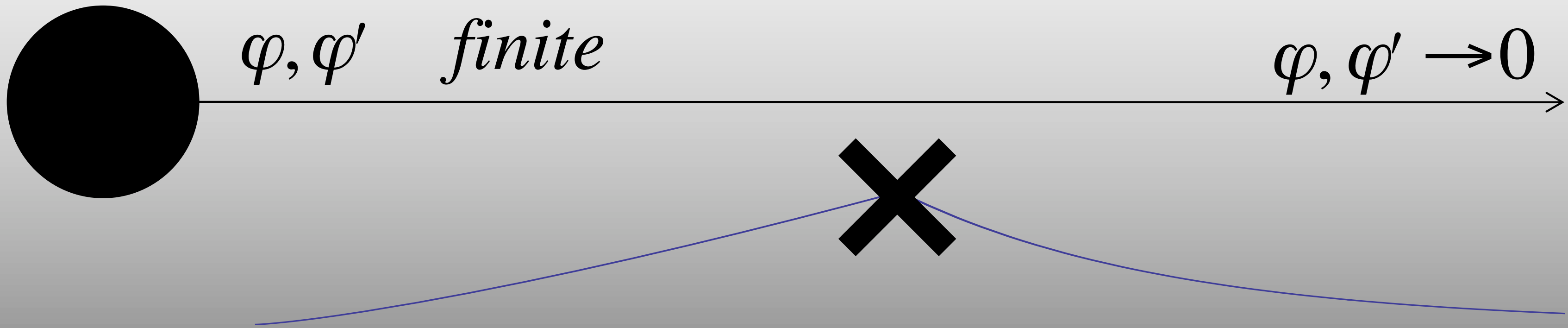
Scalar fields are ubiquitous in cosmology as a mechanism for inflation, dark energy, or even dark matter.

Understanding how scalars (particularly rolling scalars) interact with black holes from an analytic perspective is important – and revealing.

No Scalar Hair

Black holes in 4D obey a set of theorems: They are spherical, obey laws of thermodynamics, and are characterized by relatively few "numbers" – or "Black Holes Have No Hair".

The essence of "no hair" is that the scalar field must have finite energy and fall off at infinity. Integrating the equation of motion gives a simple relation, only satisfied for $\varphi = \varphi' \equiv 0$



No-No Hair!

But this is highly idealised:

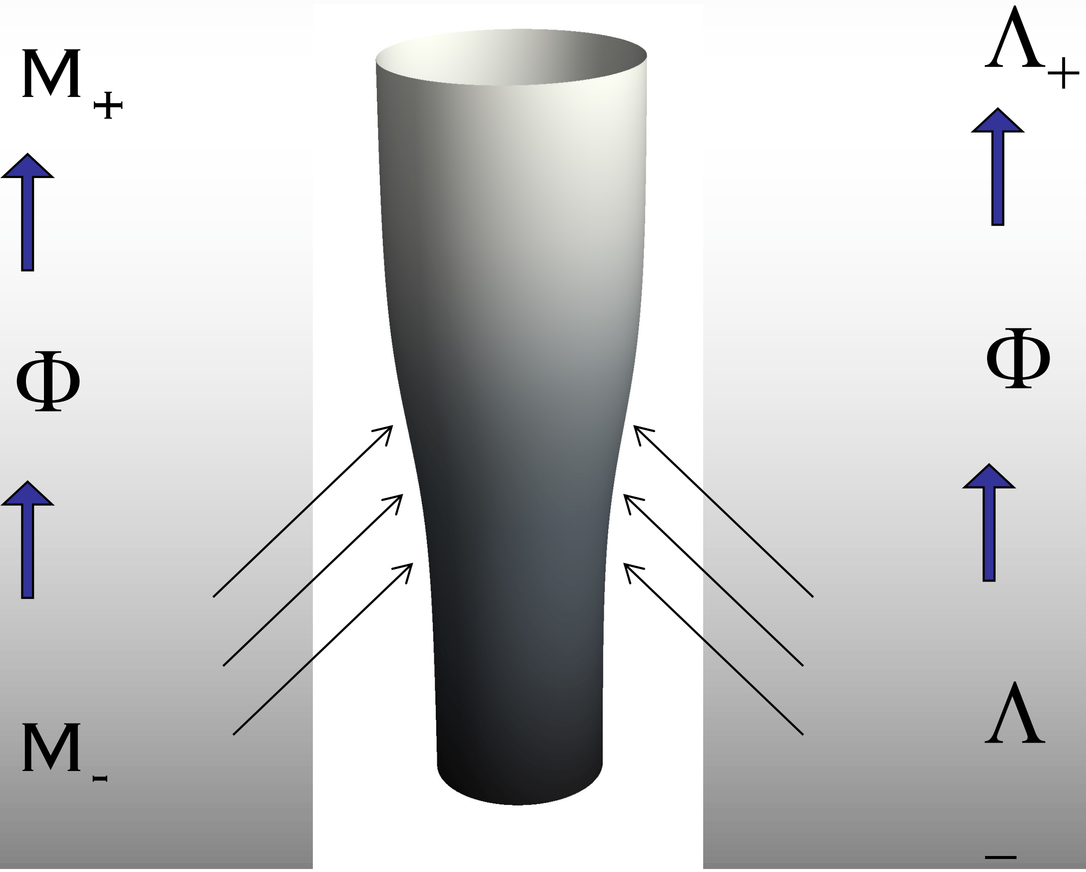
- Static
- Vacuum
- Convex potential
- Λ not negative
- Einstein (or similar) gravity

“No hair” came to mean the much stronger “no field profiles” which was rapidly shown not to be valid by using one of these fixes.

Varying Scalar

One of the clearest ways of evading the no-hair axioms is to have time-dependence.

A rolling scalar corresponds to many ideas of dark energy.



The black hole should absorb scalar energy and grow, while the cosmological constant drops.

Controlling Lambda

The idea of a varying Lambda is very familiar – in slow roll inflation, lambda varies gradually, while our universe is quasi-de Sitter.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_p^2}\left[\frac{\dot{\phi}^2}{2} + W(\phi)\right]$$

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial W}{\partial \phi}$$

Small slow-roll parameters ensure that inflation in maintained:

$$\varepsilon = \frac{M_p^2}{2}\frac{W'^2}{W^2} \quad , \quad \eta = M_p^2 \frac{W''}{W}$$

Black Hole in de Sitter

A black hole with cosmological constant is given by the Schwarzschild de Sitter solution:

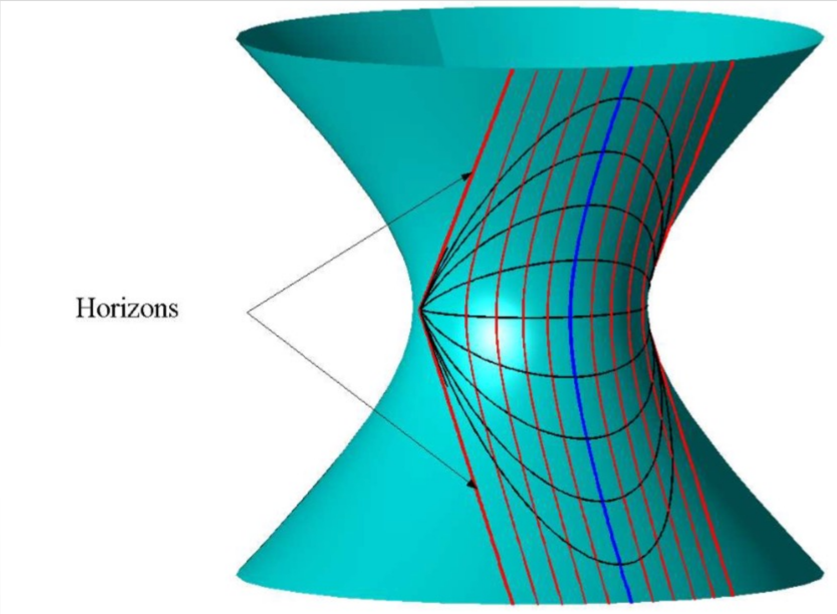
$$ds^2 = f\,dt^2 - \frac{dr^2}{f} - r^2 d\Omega_{II}^2$$

Where

$$f = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2$$

But this is not explicitly time dependent – what does “slow roll” mean?

STATIC PATCHES:



Without the black hole, have a "static patch" de Sitter potential:

$$f = 1 - H^2 r^2$$

– not the familiar (flat) cosmological coordinates

$$d\tau^2 - e^{2H\tau} d\mathbf{x}^2$$

The transformation to cosmological time is nontrivial:

$$\tau_{cos} = t_s + \frac{1}{2H} \log(1 - H^2 r_s^2) \qquad \rho_{cos} = \frac{r_s e^{-Ht_s}}{\sqrt{1 - H^2 r_s^2}}$$

And with black hole, even this relative simplicity is lost.

Black Hole-Slow Roll Approximation

With a black hole, intuition is that the geometry is approximately SDS, the scalar still slow-rolls, but that this produces a sub-leading effect on the background black hole geometry. The spacetime slides from one Λ to a lower one, and the black hole accretes a little mass.

$$\phi = \phi_0 + \delta\phi_{SR}$$

$$g_{\mu\nu} = g_{0\,\mu\nu} + \delta g_{SR\,\mu\nu}$$

SDS $\quad \mathcal{O}\left(\delta\phi_{SR}\right)^2$

Scalar Field Eqn

Idea is to turn e.o.m for ϕ

$$\frac{\phi_{,tt}}{f} - \frac{1}{r^2}\left(r^2 f \phi_{,r}\right)_{,r} = -\frac{\partial W}{\partial \phi}$$

into something like a slow roll equation by taking $\phi = \phi(T)$, then writing

$$T = t + \xi(r)$$

T is constructed so that ϕ is regular at both horizons.

Substitute in: $\phi = \phi(t + \xi(r))$

$$\frac{1}{r^2}\left(r^2 f \xi'\right)' \dot{\phi} - \frac{\ddot{\phi}}{f}\left(1 - f^2 \xi'^2\right) = \frac{\partial W}{\partial \phi}$$

Substitute in: $\phi = \phi(t + \xi(r))$

$$\frac{1}{r^2}\left(r^2 f \xi'\right)' \dot{\phi} - \frac{\ddot{\phi}}{f}\left(1 - f^2 \xi'^2\right) = \frac{\partial W}{\partial \phi}$$

Dropping second term, and remember ϕ = ϕ(T), we must have

$$\frac{1}{r^2}\left(r^2 f \xi'\right)' = -3\gamma$$

γ constant, and hence

$$\xi' = \frac{1}{f}\left(-\gamma r + \frac{\beta}{r^2}\right)$$

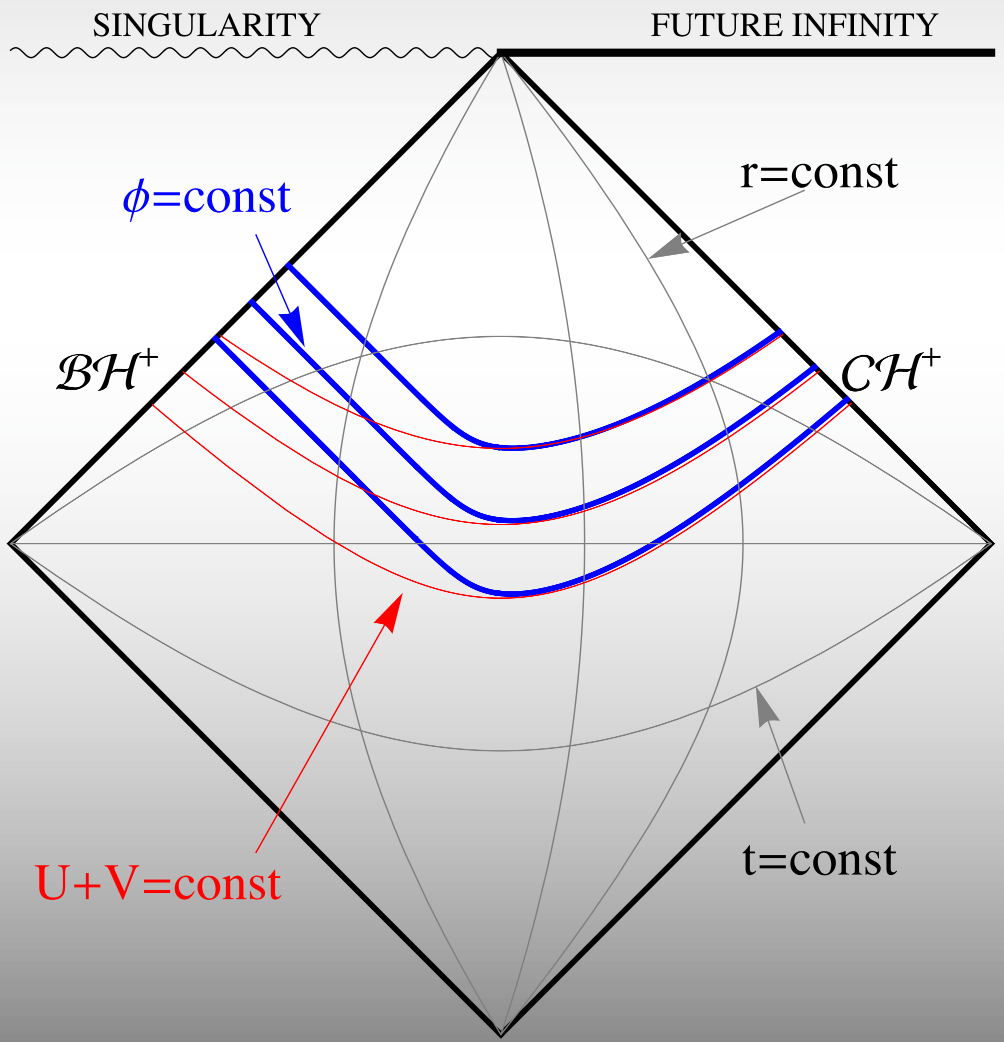
Find γ and β by regularity: $\phi(T)$ must be ingoing on event horizon and outgoing on cosmological horizon.
Final answer gives T:

$$T = t - \frac{1}{2\kappa_c} \log \left| \frac{r - r_c}{r_c} \right| + \frac{1}{2\kappa_b} \log \left| \frac{r - r_b}{r_b} \right| + \frac{r_b r_c}{r_c - r_b} \log \frac{r}{r_0} + \left(\frac{r_c}{4\kappa_b r_b} - \frac{r_b}{4\kappa_c r_c} \right) \log \left| \frac{r - r_n}{r_n} \right|$$

T looks like Kruskal V at the black hole horizon (r_b) and Kruskal U at the cosmological horizon (r_c)

The T Coordinate

The T coordinate becomes the local Kruskal at each horizon



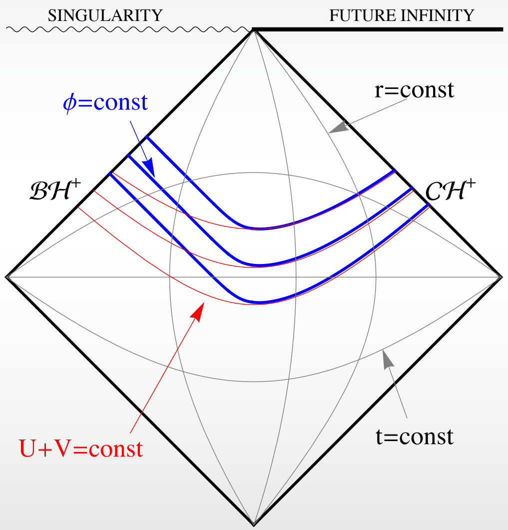
Phi Equation

The phi equation is now a standard slow-roll type

$$3\gamma\dot{\phi}(T) = -\frac{\partial W}{\partial \phi}$$

but with friction parameter modified from H:

$$\gamma = \frac{r_c^2 + r_h^2}{r_c^3 - r_h^3} = \frac{A_{TOT}}{3V}$$



Physical effect of black hole is (apparently) to add friction to roll, or to slow down the scalar.

Horizon Evolution

Can follow evolution of event horizons, using null coordinates:

$$ds^2 = 4e^{2\nu}\sqrt{\frac{B_0}{B}}dU\,dV - Bd\Omega_{\rm II}^2$$

See how leading order is ϕ rolling on black hole background

$$\phi_{,UV} = -W_{,\phi}(\phi)\sqrt{\frac{B_0}{B}}e^{2\nu} - \frac{1}{2B}\left(B_{,U}\phi_{,V} + B_{,V}\phi_{,U}\right)$$

$$B_{,UV} = 2\left(\frac{W(\phi)}{M_p^2}B^{1/2} - B^{-1/2}\right)e^{2\nu}B_0^{1/2}$$

$$\nu_{,UV} = \frac{1}{2}\left(\frac{W(\phi)}{M_p^2}B^{-1/2} + B^{-3/2}\right)e^{2\nu}B_0^{1/2} - \frac{\phi_{,U}\phi_{,V}}{2M_p^2}$$

$$B_{,VV} = 2\nu_{,V}B_{,V} - B\phi_{,V}^2/M_p^2$$

$$B_{,UU} = 2\nu_{,U}B_{,U} - B\phi_{,U}^2/M_p^2$$

Background

Starting point is constant ϕ, and equations integrate up to give SDS in different gauge

$$B = B\left[F(V) + G(U)\right] \qquad\qquad e^{2\nu} = F'G'B'$$

$$B' = \frac{4W_0}{3M_p^2}B^{3/2} - 4B^{1/2} + \mu$$

$\propto GM$

Physically, easiest to set $B=r^2$. U,V can be Kruskals at each horizon.

$$F, G \leftrightarrow \mp\frac{(t \pm r^\star)}{4}$$

Horizon Growth

The solution at each horizon can be integrated up e.g. black hole horizon:

$$\delta B = -\frac{B_{0h}}{3\gamma\kappa_h M_p^2} v \int_v^\infty \left(W[\phi(v')] - W[\phi(0)]\right) \frac{dv'}{v'^2}$$

Shows teleological behaviour of horizon, total area shift of

$$\delta A_h = \frac{A_h}{3\gamma\kappa_h M_p^2}(W_i - W_f)$$

Note

$$\frac{\delta A_h / A_h}{\delta A_c / A_c} = \frac{|\kappa_c|}{\kappa_h} < 1$$

(fractional change in cosmological horizon area greater than that of black hole).

Bulk Back-Reaction

Given this Eddington-Finkelstein behaviour, look at SDS metric in (T,r) coords:

$$ds^2 = f(r,T)\,dT^2 - 2h(r,T)\,dT\,dr - \frac{dr^2}{f}\left(1-h^2\right) - r^2 d\Omega^2$$

The energy momentum of the scalar has 2 independent cpts:

$$T_{TT} = \left(W(\phi) + \frac{1+h^2}{2f}\dot{\phi}^2\right)|g_{TT}|\ ,$$

$$T_{ab} = \left(-W(\phi) + \frac{1-h^2}{2f}\dot{\phi}^2\right)g_{ab}$$

$$h = f\xi'$$

Which we relate to the Einstein tensor:

$$G_{TT} = \left[\frac{1}{r^2}(1 - f - rf') - \frac{h\dot{f}}{rf}\right]|g_{TT}|$$

$$G_{rr} = \left[-\frac{1}{r^2}(1 - f - rf') + \frac{h\dot{f}}{rf} + \frac{2\dot{h}}{r(1-h^2)}\right]g_{rr}$$

$$G_{rT} = \left[-\frac{1}{r^2}(1 - f - rf') - \frac{(1-h^2)\dot{f}}{rhf}\right]g_{rT}$$

$$G_{\theta\theta} = \left[\frac{f''}{2} + \frac{f'}{r} - \frac{h'\dot{f}}{2f} + \frac{\dot{h}f'}{2f} + \frac{\dot{h}}{r} + \dot{h}' + \frac{1}{2}\left(\frac{(h^2-1)}{f}\right)^{\cdot\cdot}\right]g_{\theta\theta} = \frac{G_{\phi\phi}}{\sin^2\theta}$$

Slow Roll with a Black Hole

Need to have control of the slow-roll approximation to identify the key dependences in these equations:

- Scalar:

$$\frac{1-h^2}{f}\ddot{\phi} - \frac{(r^2 h)'}{r^2}\dot{\phi} = -W'(\phi)$$

- Einstein:

Recall: $(r^2 h)' = -3\gamma r^2$

$$\left[\frac{1}{r^2}(1-f-rf') + \frac{(1-h^2)\dot{f}}{rhf}\right] = \frac{1}{M_p^2}\left(W(\phi) - \frac{1-h^2}{2f}\dot{\phi}^2\right)$$

Slow roll conditions:

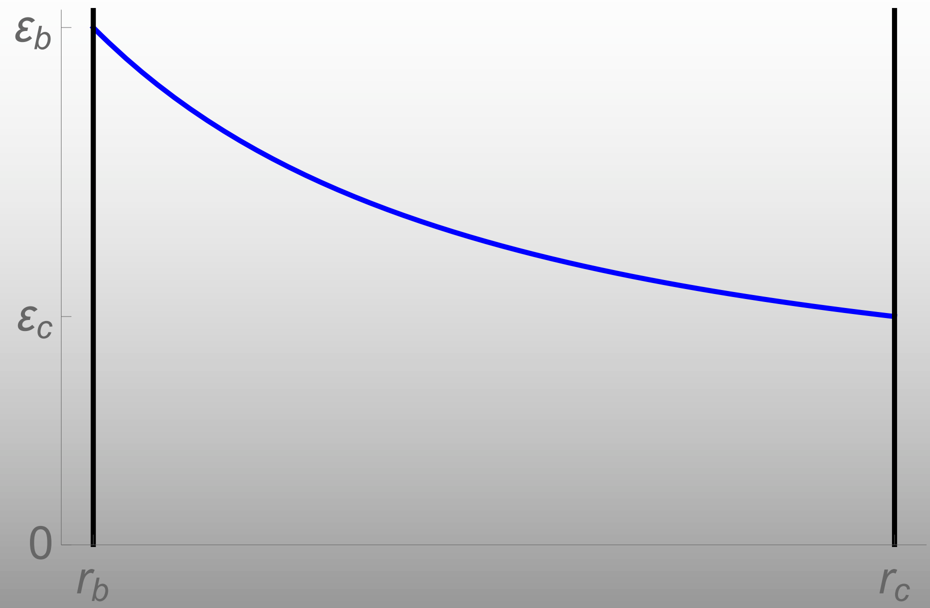
Geometry should be dominated by the potential energy:

$$\frac{1-h^2}{f}\dot{\phi}^2 \ll W \;\Rightarrow\; \frac{1-(rf)'}{r^2} = 3H^2 = \frac{W}{M_p^2}$$

Requires

$$M_p^2 \frac{W'^2}{W^2} \frac{(1-h^2)}{f} \frac{H^2}{3\gamma^2} \ll 1$$

$$\varepsilon = M_p^2 \frac{W'^2}{W^2} \ll 1$$



Slow roll conditions:

Scalar motion dominated by friction:

$$\frac{1-h^2}{f}\ddot{\phi} \ll 3\gamma\dot{\phi} \quad \Rightarrow \quad -3\gamma\dot{\phi} = W'(\phi)$$

..the same spatial dependence as before, giving a bound

$$\eta = M_p^2 \frac{W''}{W} \ll 1$$

The eta parameter.

This allows us to solve the Einstein equations to leading order in the slow-roll parameters.

➢ TT + Tr:

$$\dot{f} = -rh\,\frac{\dot{\phi}^2}{M_p^2}$$

implies

$$f(r,T) = f_0(r) + \delta f(r,T) - rh_0 \int \frac{\dot{\phi}^2}{M_p^2}$$

Where δf is order εη, but slowly varying. We can then integrate the ϕ kinetic energy:

$$\int_{T_0}^{T} \dot{\phi}^2 dT' = -\frac{1}{3\gamma}\left\{W(\phi(T)) - W(\phi(T_0)\right\} = -\frac{\delta W}{3\gamma}$$

And end up with a familiar expression:

$$f(r,T) = 1 - \frac{\Lambda(T)}{3M_p^2}r^2 - \frac{2GM(T)}{r} + \delta f$$

With

$$\Lambda(T) = W[\phi(T)] \qquad\qquad M(T) = M_0 - 4\pi\beta\frac{\delta W}{3\gamma}$$

In fact, even before slow roll, we can determine features of the spacetime from the simplicity of the Einstein equations!

Equality of the T-r and r-r Einstein implies $$\frac{f}{(1-h^2)} = g(r)$$

With the time dependence of the two functions being given by including T-T eqn:

$$\frac{\dot{f}}{rh} = -\frac{2g(r)\dot{h}}{r} = -\frac{\dot{\phi}^2}{M_p^2}$$

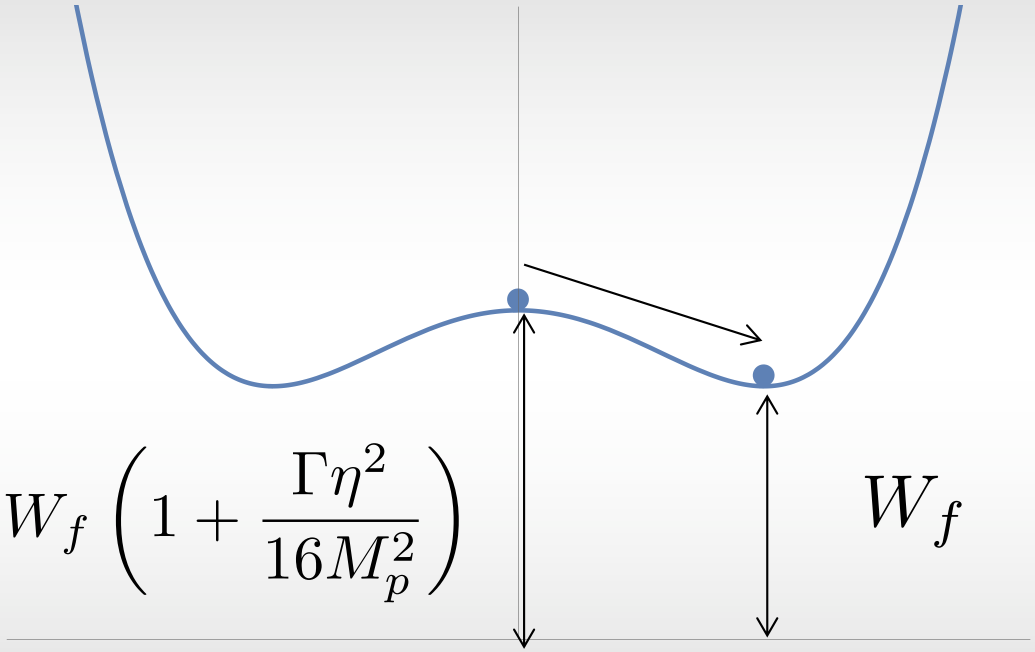
Which in turn allows us to integrate:

$$f = 1 - \frac{r^2 W(\phi)}{3M_p^2} - \frac{2GM(T)}{r} - \underbrace{\frac{\dot{\phi}^2}{2rM_p^2}\int \frac{r^2}{g(r)}}_{\text{Subdominant, transient}}$$

Subdominant, transient

Explicit Example

Double well potential

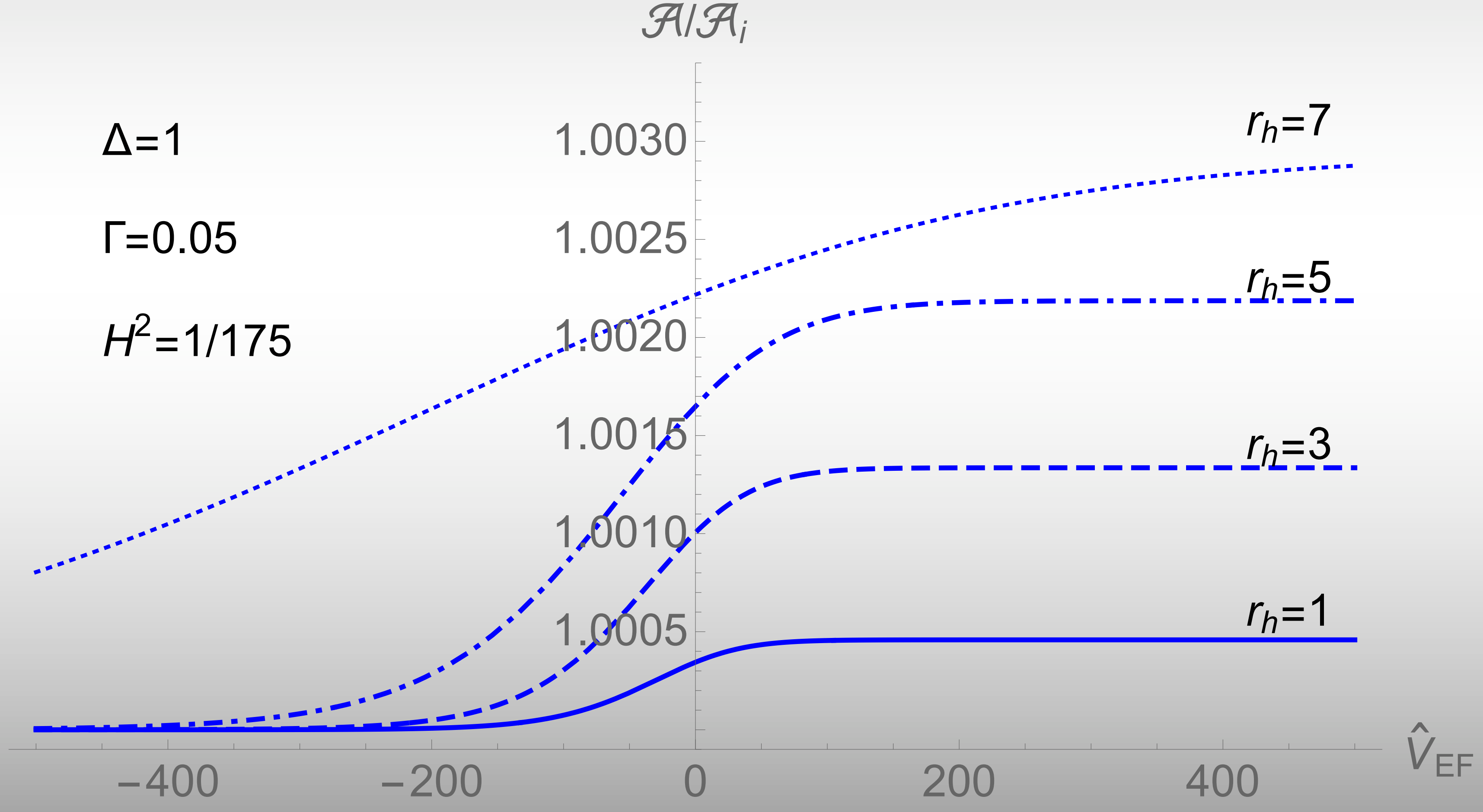


"Cosmological" soln:

$$\phi^2 = \frac{\eta^2}{2}\left[1+\tanh\left(\frac{H_f\Gamma}{4}\tau\right)\right]$$

Horizon Growth

Horizon growth depends primarily on Δ but the rate of growth determined by the slow roll friction parameter



Kerr & Scalars

Want a rotating black hole with nontrivial scalar

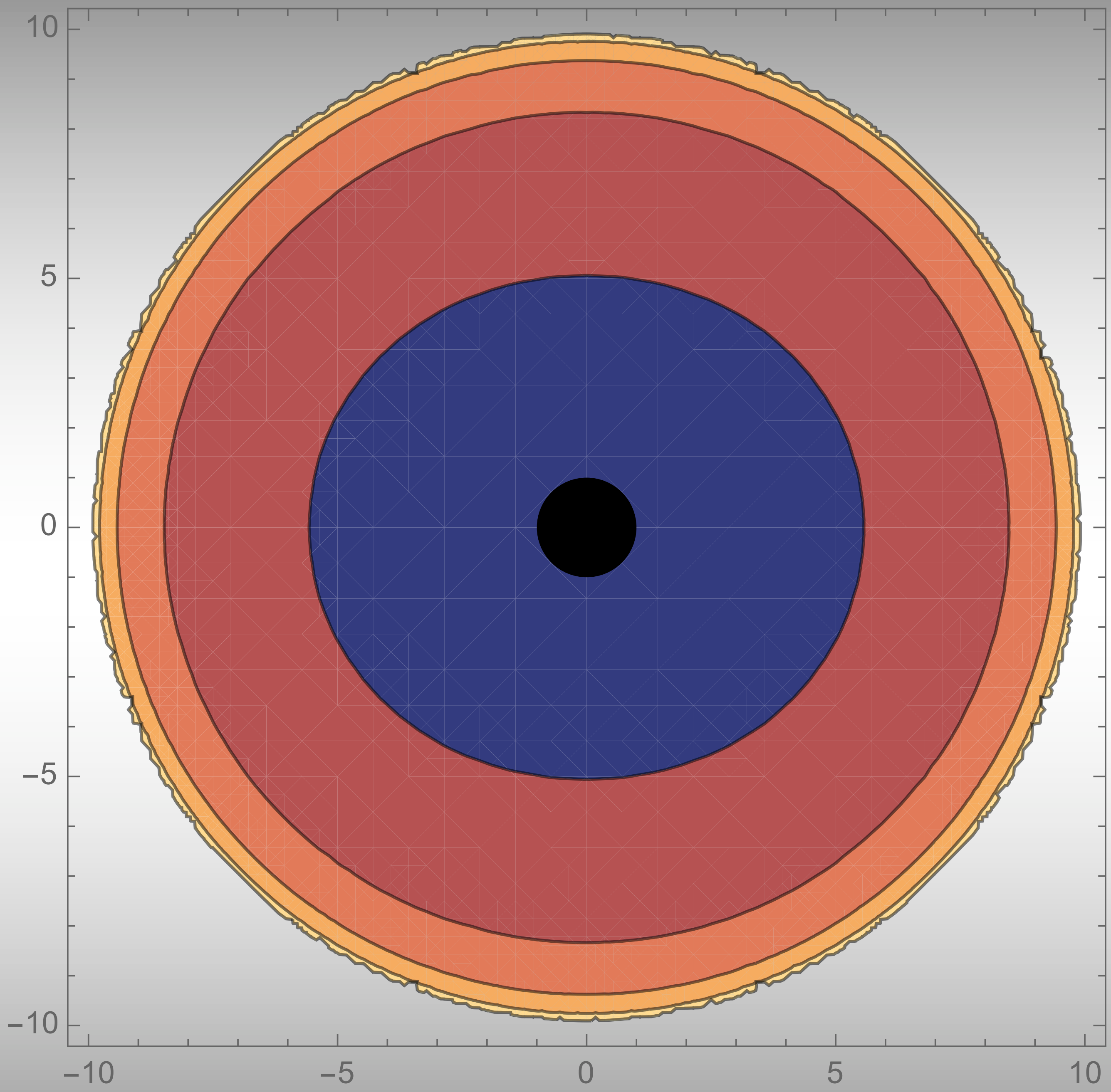
$$ds^2 = -\frac{\Delta_r}{\Xi^2\rho^2}\left[dt - a\,\sin^2\theta d\varphi\right]^2 + \rho^2\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) + \frac{\Delta_\theta \sin^2\theta}{\Xi^2\rho^2}\left[a\,dt - \left(r^2 + a^2\right) d\varphi\right]^2$$

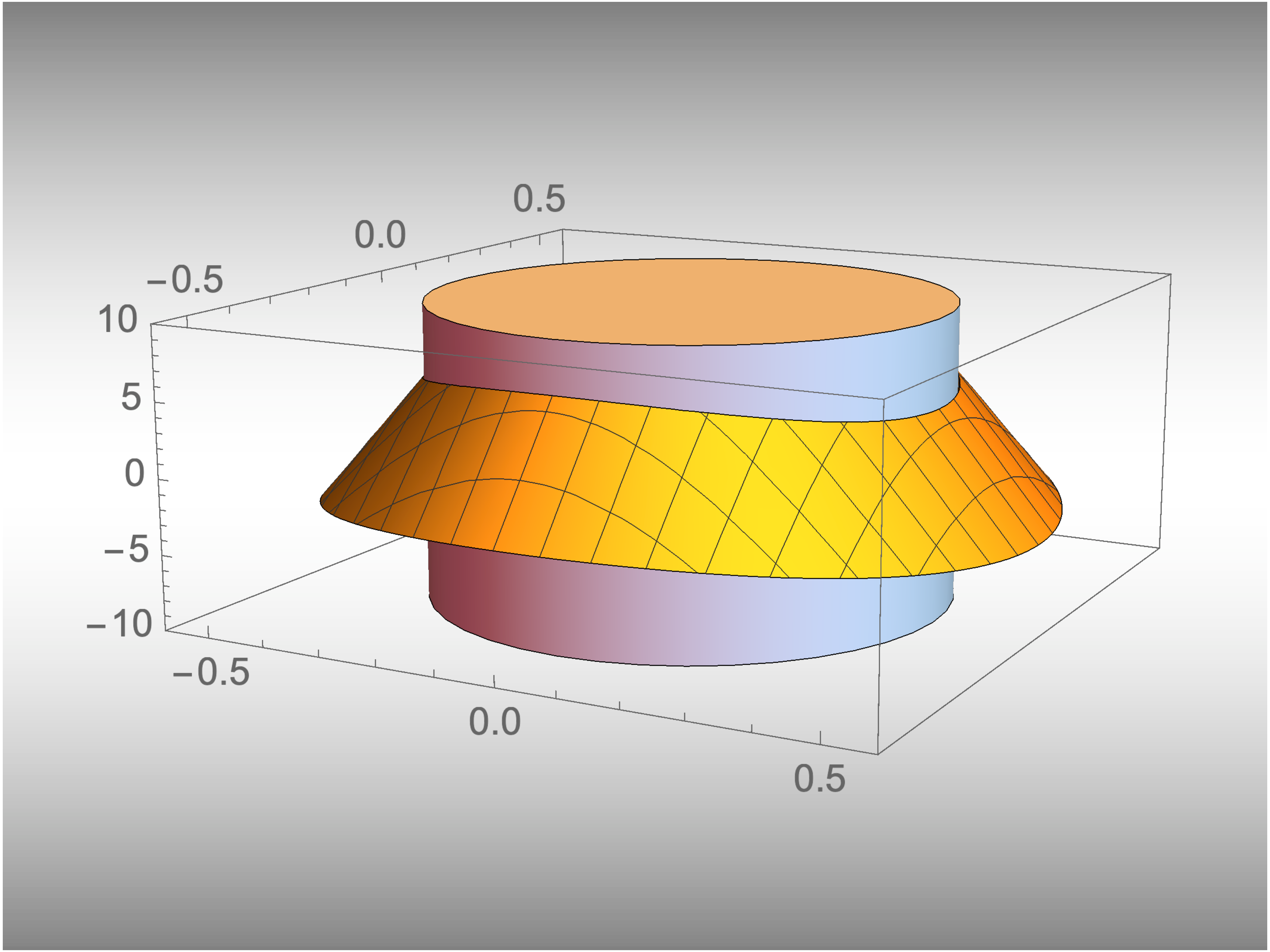
$$\Delta_r = \left(1 - \frac{r^2}{\ell^2}\right)\left(r^2 + a^2\right) - 2Mr \;\;,\;\; \Xi = 1 + \frac{a^2}{\ell^2} \;\;,\;\; \Delta_\theta = 1 + \frac{a^2}{\ell^2}\cos^2\theta \;\;,\;\; \rho^2 = r^2 + a^2\cos^2\theta$$

The method can be applied to the phi equation to find an expression for the rolling scalar with rotation. This complicates the expression for T, but it can be found and is now angularly dependent

$$T = t + \int dr \frac{r^2 + a^2}{\Delta_r} h(r) - \frac{\gamma\ell^2}{2}\log\Delta_\theta \qquad -\gamma r + \frac{\beta}{r^2 + a^2}$$

Kerr Slow Roll




0.5
0.0
−0.5
10
5
0
−5
−10
−0.5
0.0
0.5

Summary

- Can generalise slow-roll description to non-homogeneous black hole background.
- The slow roll conditions come from spatial, not temporal, derivatives.
- The black hole geometry is to a very good approximation quasi-Schwarzschild de Sitter.
- Have a similar picture for Kerr, but scalar breaks Kerr symmetries.