PRIMORDIAL BLACK HOLE FORMATION FROM THE COLLAPSE OF A MASSLESS SCALAR FIELD

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A BRIEF INTRODUCTION

- **Primordial Black Holes** are formed in the early Universe after inflation, spanning a wide range of order of magnitudes.
- Cosmological perturbations with an amplitude larger than threshold δ_c collapse into PBHs after re-entering the cosmological horizon.
- The aim of this work is to study the **threshold for PBH formation** from the collapse of adiabatic perturbations of a massless scalar field.
- This scenario was investigated by *T. Harada & B. Carr:* Growth of primordial black holes in a universe containing a massless scalar field, PRD (2005)

MATHEMATICAL FORMALISM

- We formulate our project working with the 3+1 conformal decomposition with the metric line element:

$$ds^2 = -\alpha^2 dt^2 + a^2(t) e^{2\zeta} \tilde{\gamma}_{ij} \left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

- Stress-energy tensor for a massless scalar field:

$$T_{\mu\nu} = -\frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi + \partial_\mu \phi \partial_\nu \phi$$

- The **conjugate momentum**: $\Pi = \frac{1}{\alpha}\left(\partial_t - \beta^i \partial_i\right)\phi$

- In this formalism the Klein-Gordon equation could be decomposed:

$$\begin{cases} \left(\partial_t - \beta^i \partial_i\right)\Pi = \alpha \bigtriangleup \phi + \alpha K \Pi + D_i \phi D^i \phi \\ \left(\partial_t - \beta^i \partial_i\right)\phi = \alpha \Pi \end{cases}$$

SCALAR FIELD-PERFECT FLUID EQUIVALENCE

- In spherical symmetry the metric in **cosmic time slicing** is:

$$ds^2 = -A^2(t,r)dt^2 + B^2(t,r)dr^2 + R^2(t,r)d\Omega^2$$

- Misner-Sharp differential operators:

$$\begin{cases} D_t = \frac{1}{A}\partial_t \\ D_r = \frac{1}{B}\partial_r \end{cases} \Rightarrow \begin{cases} \eta = D_t\phi \\ \psi = D_r\phi \end{cases}$$

- The stress-energy tensor becomes: $T_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}\left(\psi^2 - \eta^2\right) + \partial_\mu\phi\partial_\nu\phi$

In the comoving gauge $\psi = 0$ and $E := T^{00} = \frac{\eta^2}{2} = P := T^{11}$

$$\Downarrow$$

The scalar field behaves as a perfect fluid with an equation of state $p = \rho$.

GRADIENT EXPANSION

- We applied the **gradient expansion approach**, expanding each variable in spatial gradient of the curvature perturbation on super horizon scales.

- According to *Shibata-Sasaki* (1999)

$$\varepsilon(t) \propto \frac{k}{H(t)a(t)} \ll 1$$

is measuring the order of the gradient expansion.

- Adiabatic cosmological perturbations of the scalar field ϕ and its conjugate momentum Π

$$\phi(t,r) = \frac{1}{3}\sqrt{\frac{3}{4\pi}}\ln(t) + \lambda(t,r) + O\left(\varepsilon^3\right)$$

$$\Pi(t,r) = \frac{1}{3t}\sqrt{\frac{3}{4\pi}} + \Omega(t,r) + O\left(\varepsilon^3\right)$$

CURVATURE PERTURBATION PROFILE

- At **super-horizon scales** the perturbed Universe is described by the asymptotic form of the spatial metric:

$$dl^2 = a^2(t)e^{2\zeta(r)}\left[dr^2 + r^2 d\Omega^2\right]$$

- The **perturbation amplitude** is defined as the **peak of the compaction function** in the comoving gauge:

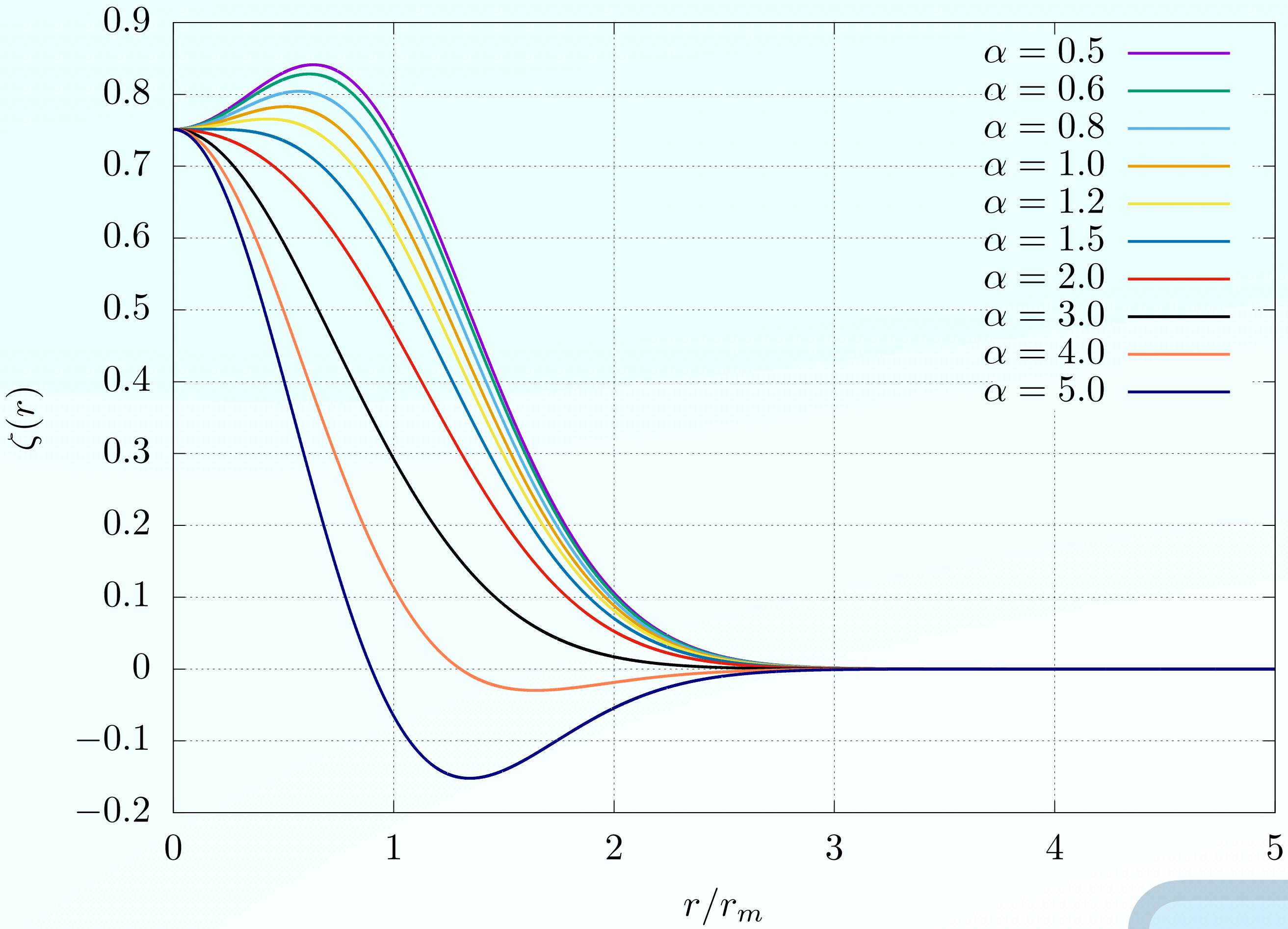
$$C(r) = -\frac{3}{4} r\zeta'(r)\left[2 + r\zeta'(r)\right] \qquad \Rightarrow \qquad \delta := C(r_m) \quad \text{where} \quad C'(r_m) = 0$$

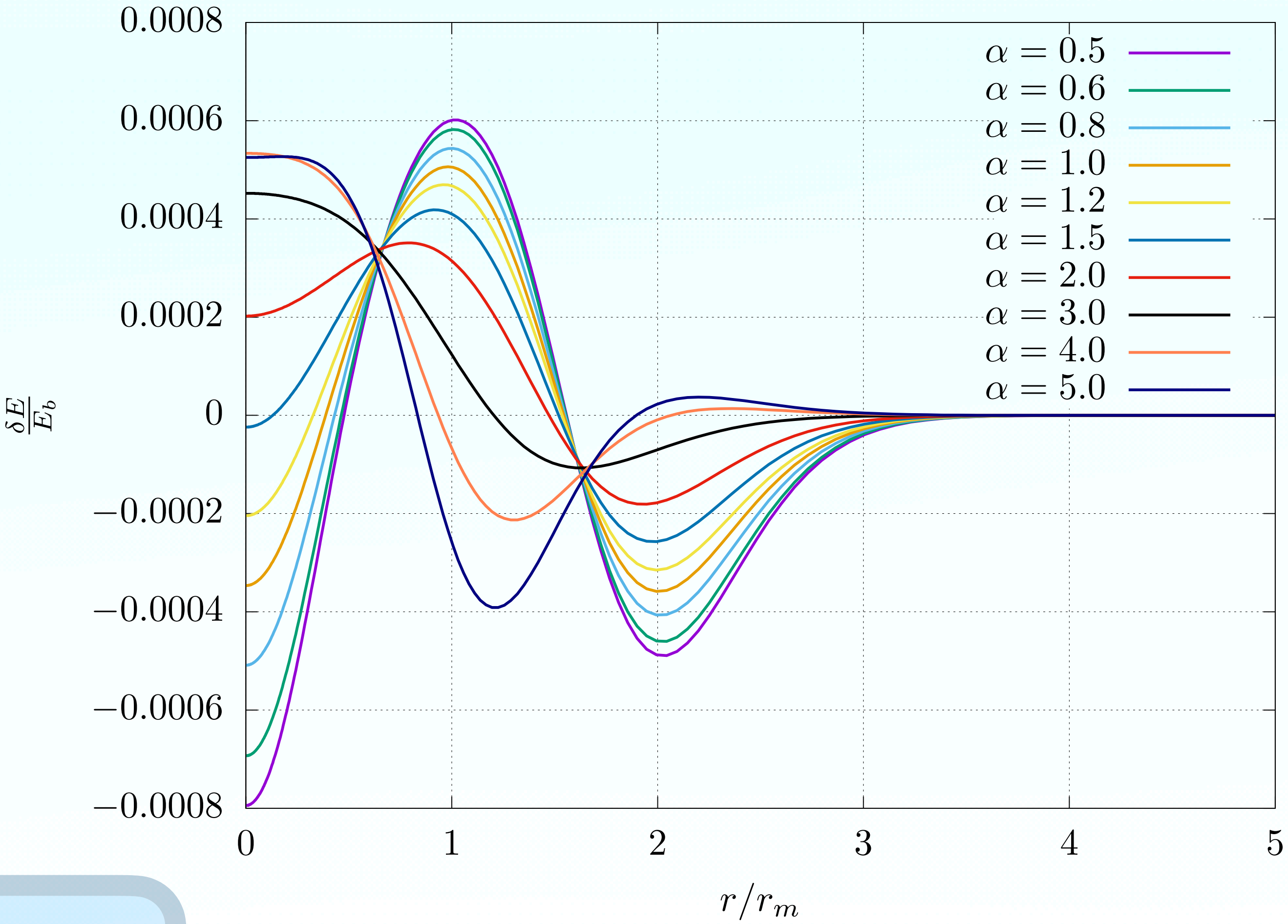
- The threshold δ_c depends on the **shape parameter** α:

$$\tilde{r} = re^{\zeta(r)}, \quad \alpha = -\frac{C''\left(\tilde{r}_m\right)\tilde{r}_m^2}{4C\left(\tilde{r}_m\right)} \qquad \Rightarrow \qquad \alpha = -\frac{C''\left(r_m\right) r_m^2}{4C\left(r_m\right)\left[1 - \frac{3}{2}C\left(r_m\right)\right]}$$

CURVATURE PERTURBATION PROFILE

$$\zeta(r) = A\left[1 + 2(1-\beta)\frac{r^2}{r_m^2}\right]e^{-\frac{r^2}{r_m^2}}; \qquad A = \frac{e}{2}\left(1 - \sqrt{1 - \frac{4}{3}\delta}\right), \qquad \beta = \alpha\left(1 - \frac{A}{e}\right)\left(1 - \frac{2A}{e}\right)$$



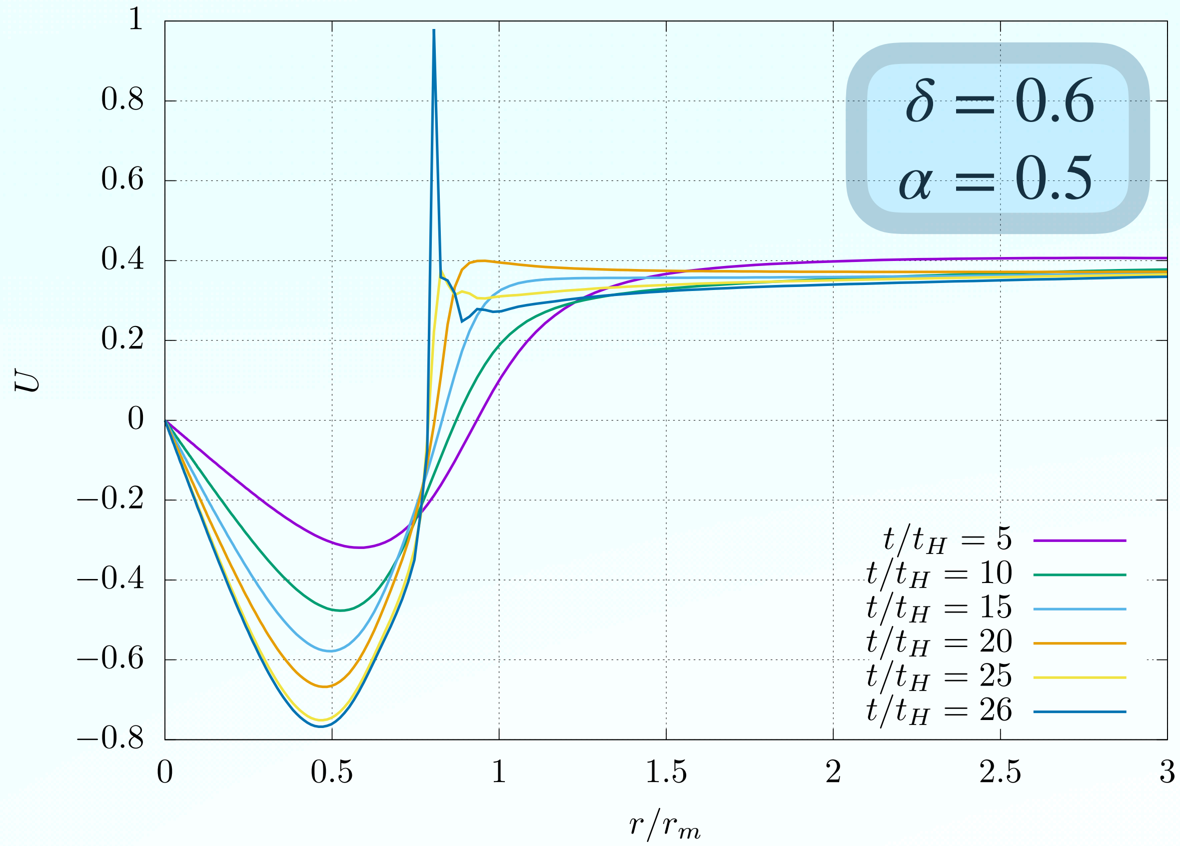


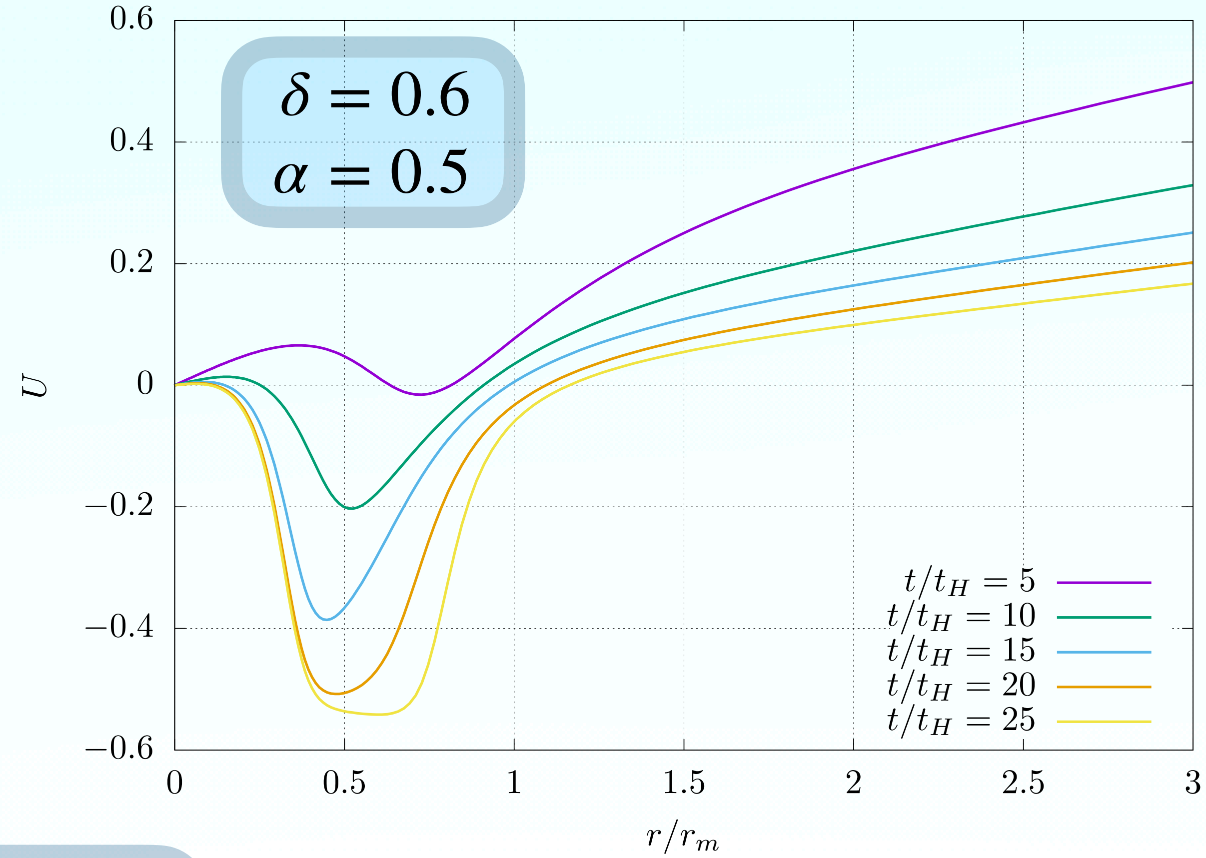
$\delta = 0.6$

NUMERICAL SIMULATIONS

- **Spherically symmetric relativistic hydrodynamical code** based on the BSSN formalism using a non-uniform (exponential growth) grid, extended up to 10 perturbation length-scales.

- **Comoving against CMC gauge:** the comoving gauge is failing because of a coordinate singularity forming during the collapse and before the formation of the apparent horizon.





Comoving gauge

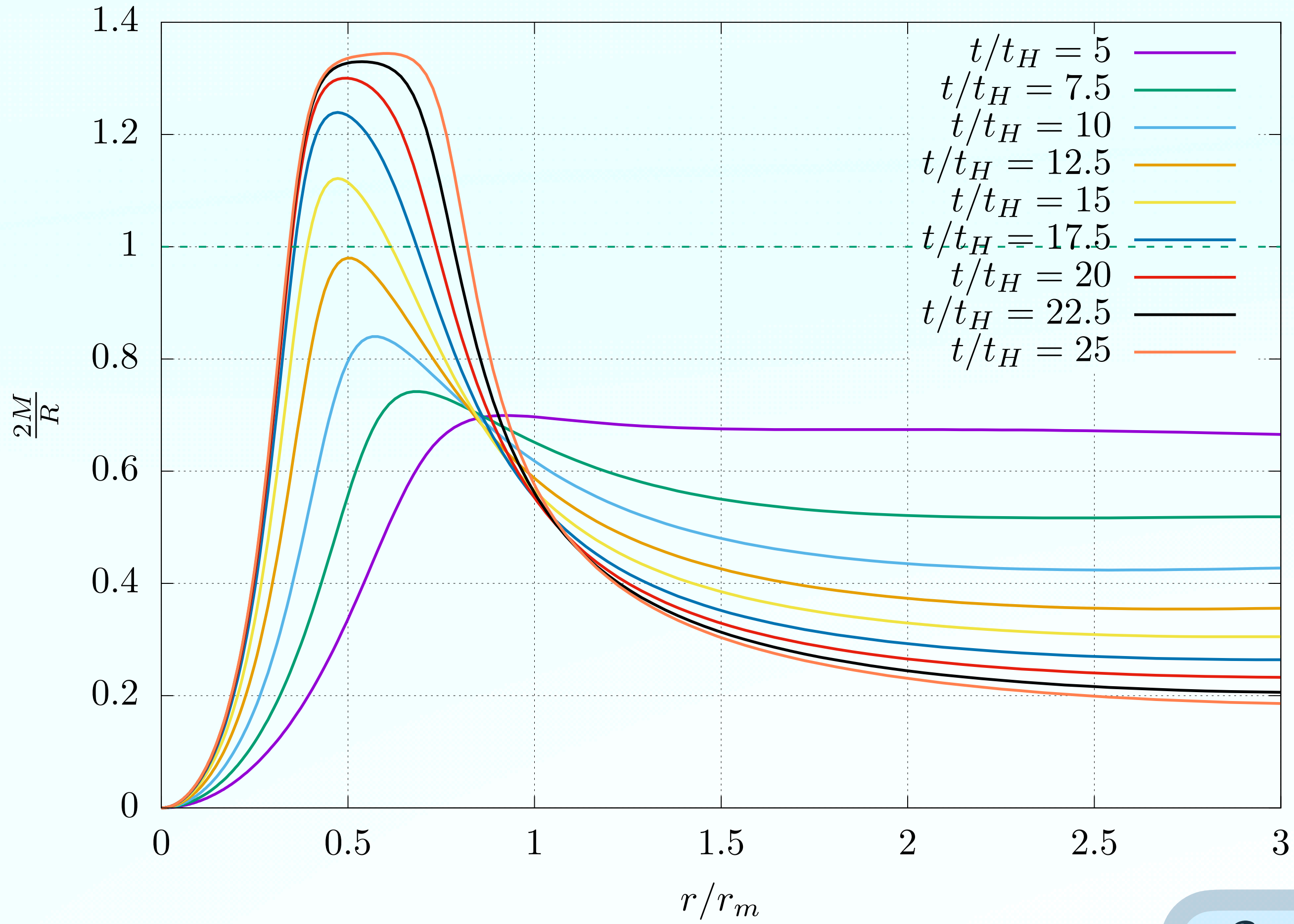
$U = D_t R$

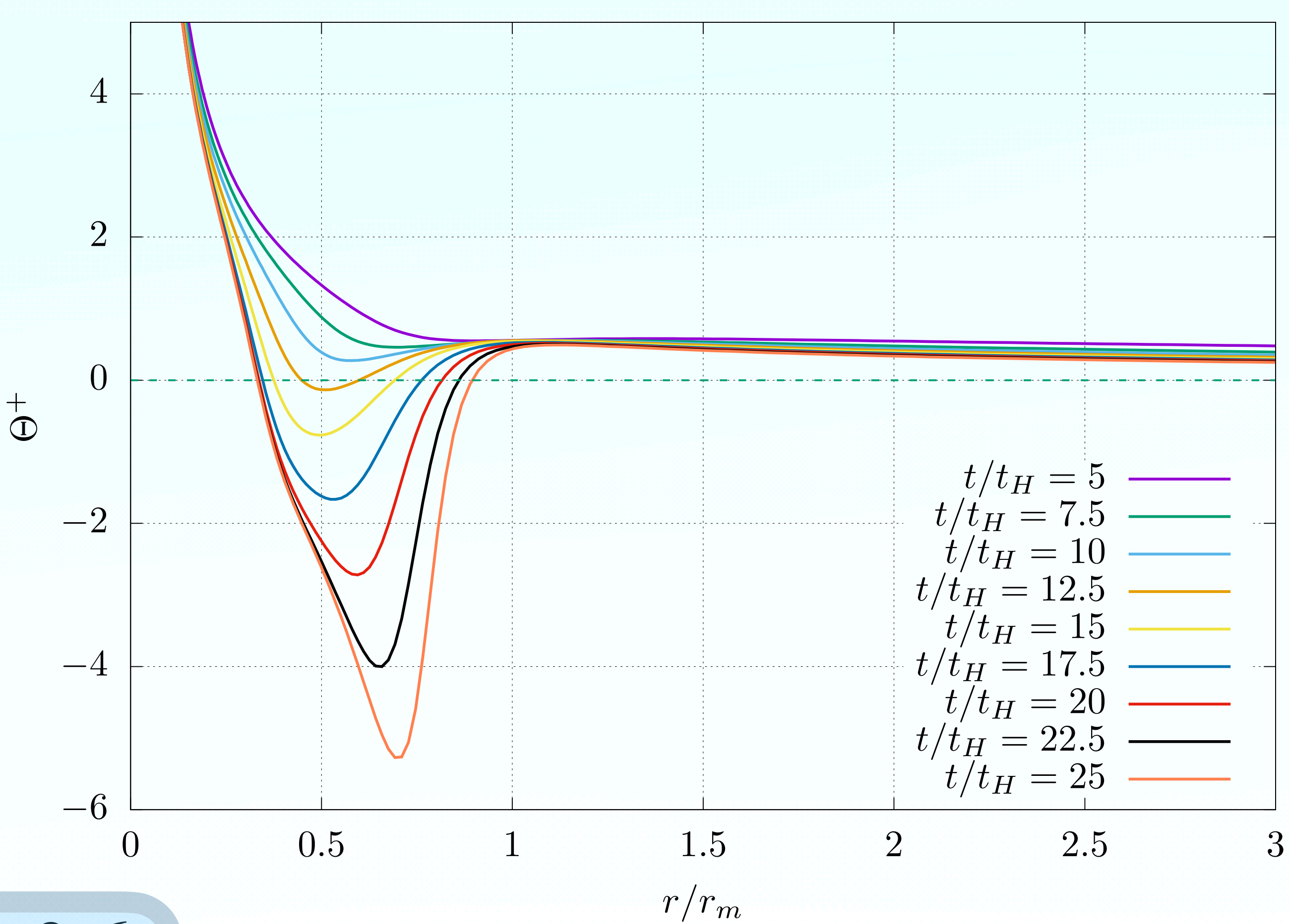
CMC gauge

APPARENT HORIZON FORMATION

$R = 2M$

$\theta^+ = 0$

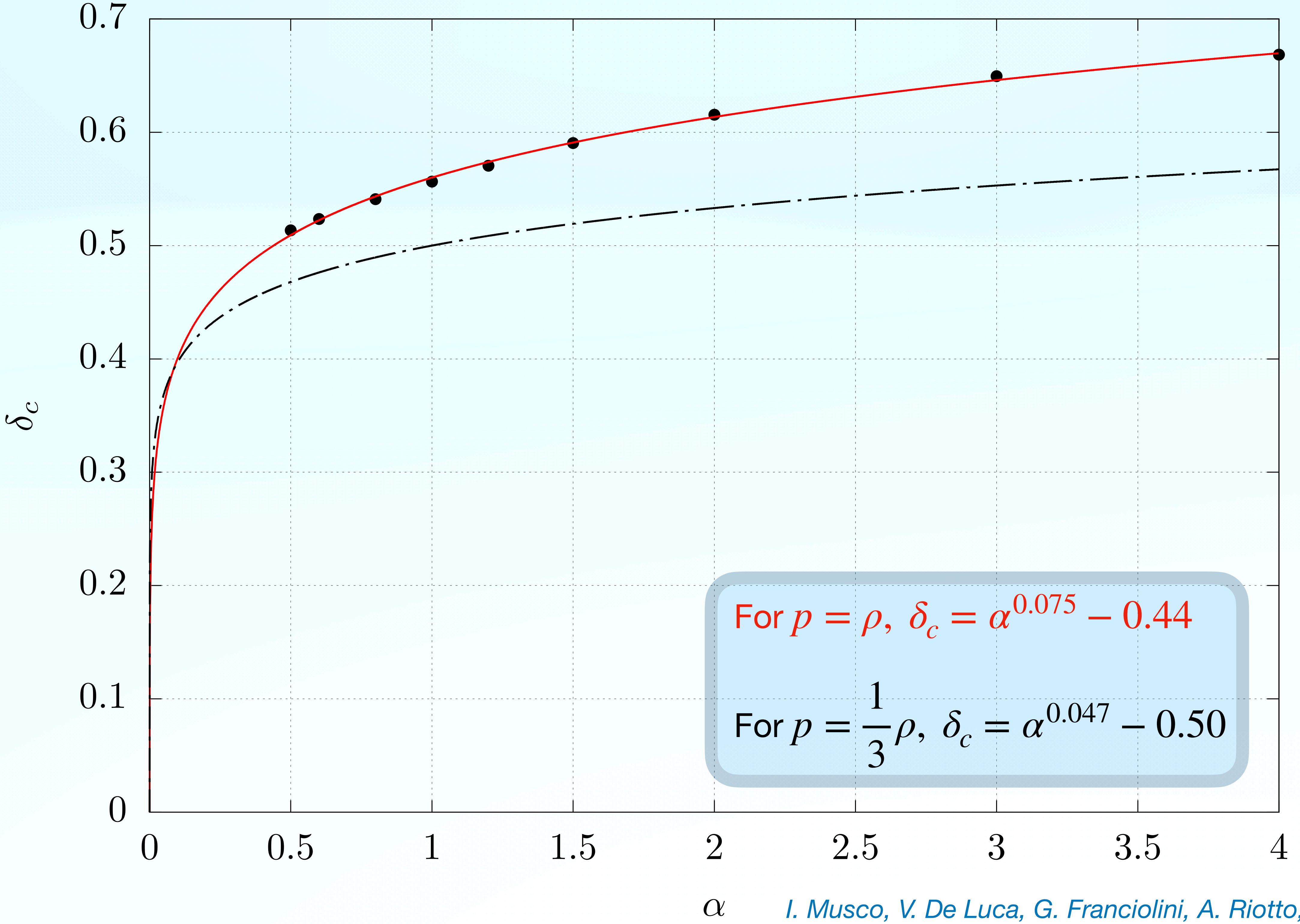




$\delta = 0.6$

$\alpha = 0.5$

THRESHOLD FOR PBH FORMATION

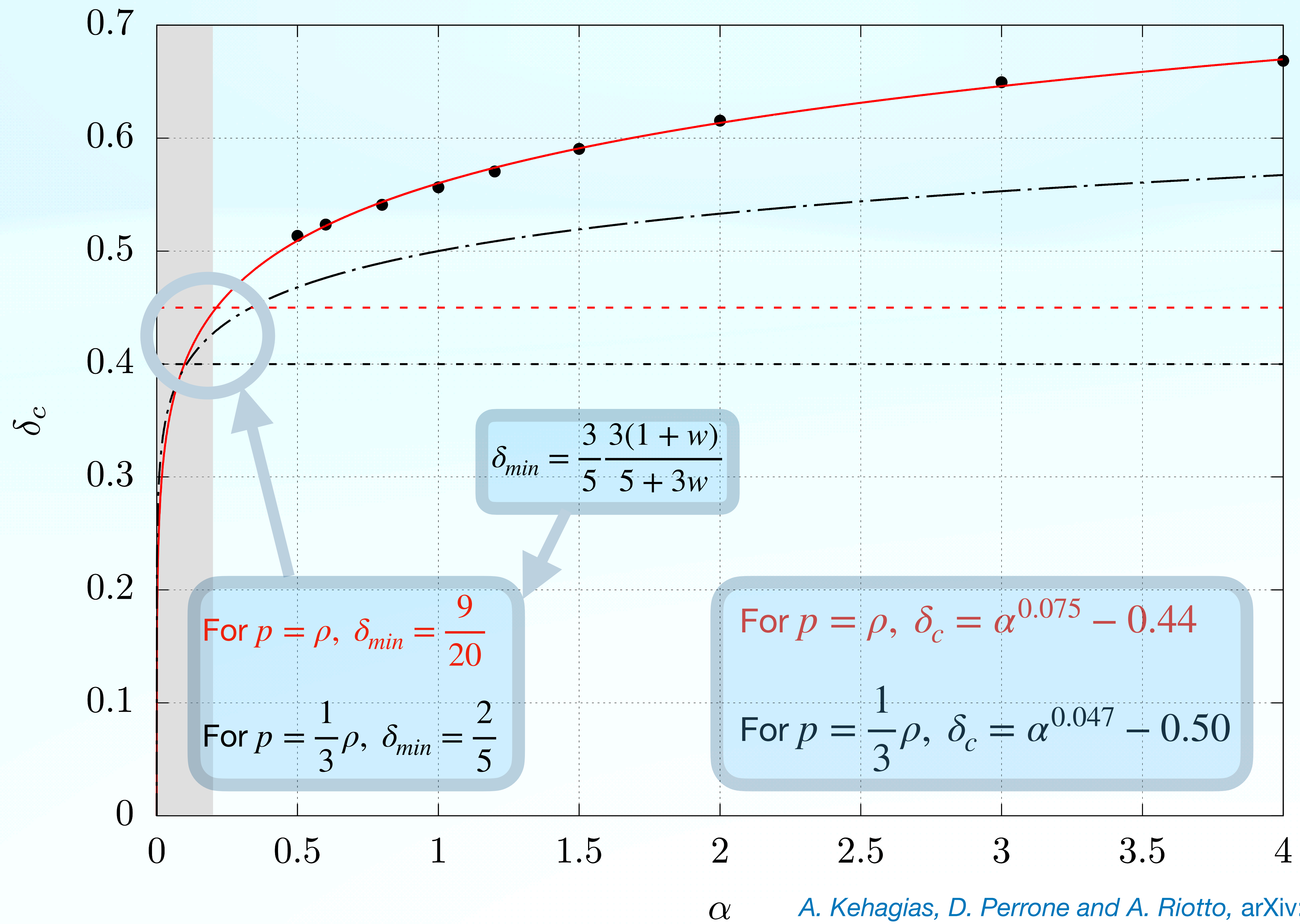


For $p = \rho,\ \delta_c = \alpha^{0.075} - 0.44$

For $p = \frac{1}{3}\rho,\ \delta_c = \alpha^{0.047} - 0.50$

I. Musco, V. De Luca, G. Franciolini, A. Riotto, PRD 2021

THRESHOLD FOR PBH FORMATION



A. Kehagias, D. Perrone and A. Riotto, arXiv:2405.05208

SUMMARY

This is a work in progress…

1. More simulations for the **threshold for PBH** are coming.

2. We will investigate numerically the **equivalence between the fluid and the scalar field**, recovering the same results using an hydrodynamical numerical code (*Musco*) for a perfect fluid with equation of state $p = \rho$.

3. These results will be used in a follow up work to study the critical collapse, computing the **mass function** and the **abundance** of the population of PBHs formed from the collapse of a massless scalar field.

Thanks for your attention!

3+1 conformal decomposition

$$\tilde{\triangle}\psi = \frac{\tilde{R}^k_k}{8}\psi - 2\pi\psi^5 a^2 E - \frac{\psi^5 a^2}{8}\left(\tilde{A}_{ij}\tilde{A}^{ij} - \frac{2}{3}K^2\right)$$

$$\tilde{D}^j\left(\psi^6\tilde{A}_{ij}\right) - \frac{2}{3}\psi^6\tilde{D}_i K = 8\pi\psi^6 J_i$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)\psi = -\frac{\dot{a}}{2a}\psi + \frac{\psi}{6}\left(\overline{\mathcal{D}}_i\beta^i - \alpha K\right)$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)K = \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) - \mathcal{D}^i\mathcal{D}_i\alpha + 4\pi\alpha\left(E + S\right)$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)\tilde{A}_{ij} = \frac{1}{a^2\psi^4}\left[\alpha\left(R_{ij} - \frac{1}{3}\gamma_{ij}R_{(sc)}\right) - \left(\mathcal{D}_i\mathcal{D}_j\alpha - \frac{\gamma_{ij}}{3}\mathcal{D}_k\mathcal{D}^k\alpha\right)\right] + \alpha\left(K\tilde{A}_{ij} - 2A_{ik}A^k_j\right) + \\ -\frac{2}{3}\tilde{A}_{ij}\overline{\mathcal{D}}_k\beta^k - \frac{8\pi\alpha}{a^2\psi^4}\left(S_{ij} - \frac{\gamma_{ij}}{3}S^k_k\right)$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} - \frac{2}{3}\tilde{\gamma}_{ij}\overline{\mathcal{D}}_k\beta^k$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)\Pi = \alpha\triangle\phi + \alpha K\Pi + D_i\phi D^i\phi$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)\phi = \alpha\Pi$$

Perfect fluid - scalar field equivalence

$$\frac{D_t B}{B} = \frac{D_r U}{\Gamma}$$

$$D_t \eta + \frac{2U}{R}\eta + \frac{\eta}{\Gamma} D_r U = 0$$

$$D_r M = 4\pi \Gamma E R^2$$

$$D_t \Gamma = \frac{U D_r A}{A}$$

$$D_t M = -4\pi R^2 P U$$

$\rightarrow$

$$D_t U = -\left[\frac{\Gamma}{e+p} D_r p + \frac{M}{R^2} + 4\pi R p\right]$$

$$\frac{D_t \rho}{\rho} = -\frac{1}{\Gamma R^2} D_r \left(R^2 U\right)$$

$$D_t e = \frac{e+p}{\rho} D_t \rho$$

$$D_r A = -\frac{A}{e+p} D_r p$$

$$D_r M = 4\pi \Gamma e R^2$$

$$D_t \Gamma = \frac{U D_r A}{A}$$

$$D_t M = -4\pi R^2 p U$$

Lapse equation

- Starting from the definition of $D_t\phi = \frac{1}{A}\partial_t\phi$, and differentiating with respect D_r, we obtain:

$$\frac{D_r A}{A} = -\frac{D_r \eta}{\eta}$$

- From the definition of $E = P = \frac{\eta^2}{2}$, we finally obtain:

$$\frac{D_r A}{A} = -\frac{D_r P}{E + P}$$

10-Einstein equation

- We can mix the 10-Einstein equation with the Klein-Gordon equation:

$$\frac{D_t \eta}{\eta} = -\frac{1}{\Gamma R^2} D_r(R^2 U)$$

Continuity equation

- Starting from the definition of E and differentiating with respect to D_t, we get:

$$\frac{D_t \eta}{\eta} = \frac{D_t E}{2E}$$

- In particular we can rewrite for the fluid:

$$D_t e = \frac{e+p}{\rho} D_t \rho \quad \Rightarrow \quad \frac{D_t \rho}{\rho} = \frac{D_t e}{2e}$$

11-Einstein equation

- We can rewrite the 11-Einstein equation for the scalar field in the following way:

$$D_t U = -\left[\frac{\Gamma}{E+P} D_r P + \frac{M}{R^2} + 4\pi R P\right]$$

- While for the fluid, assuming the definition of the Misner-Sharp mass we obtain:

$$D_t U = -\left[\frac{\Gamma}{e+p} D_r p + \frac{M}{R^2} + 4\pi R p\right]$$

Gradient expansion equations

$$
\begin{cases}
\zeta(t,r) = \zeta_b(r) + \xi(t,r) + O\left(\varepsilon^3\right) \\
\tilde{\gamma}_{ij(t,r)} = \eta_{ij} + h_{ij}(t,r) + O\left(\varepsilon^3\right) \\
K(t,r) = K_b(t)\big(1+\kappa(t,r)\big) + O\left(\varepsilon^3\right) \\
\alpha(t,r) = 1 + \chi(t,r) + O\left(\varepsilon^3\right) \\
\beta^i(t,r) = O\left(\varepsilon\right) \\
\phi(t,r) = \Phi(t) + \lambda(t,r) + O\left(\varepsilon^3\right) \\
\Pi(t,r) = \Pi_b(t) + \Omega(t,r) + O\left(\varepsilon^3\right)
\end{cases}
$$

$$
\begin{cases}
\partial_t\lambda = \Omega + \chi\Pi_b + \mathcal{O}\left(\epsilon^3\right) \\
\partial_t\Omega = -3H_b\Pi_b\left(\chi + \kappa + \frac{\Omega}{\Pi_b}\right) + \mathcal{O}\left(\epsilon^3\right) \\
\partial_t h_{ij} = -2\tilde{A}_{ij} - \frac{2}{3}\eta_{ij}\overline{D}_k\beta^k + \overline{D}_j\beta_i + \overline{D}_i\beta_j + \beta^k\eta_{ij,k} + \mathcal{O}\left(\epsilon^3\right) \\
3\dot{\xi} - 3\beta^k\partial_k\zeta_b - 3H_b\chi - 3H_b\kappa - \overline{D}_i\beta^i = \mathcal{O}\left(\epsilon^3\right) \\
\kappa\left(2 + \frac{\partial_t H_b}{H_b^2}\right) + \frac{1}{H_b}\dot{\kappa} + 3\chi + \frac{8\Omega}{H_b}\sqrt{\frac{\pi}{3}} = \mathcal{O}\left(\epsilon^3\right) \\
\partial_t\tilde{A}_{ij} + 3\frac{\dot{a}}{a}\tilde{A}_{ij} = a^{-2}e^{-2\zeta_b}\mathfrak{R}_{ij}\left(\zeta_b\right) + \mathcal{O}\left(\epsilon^4\right) \\
\overline{\triangle}\zeta_b + \frac{1}{2}\overline{D}^k\zeta_b\overline{D}_k\zeta_b = 4\pi a^2 e^{2\zeta_b}\Pi_b^2\left(\kappa - \frac{\Omega}{\Pi_b}\right) + \mathcal{O}\left(\epsilon^4\right) \\
\overline{D}_j\tilde{A}_i^j + 3\tilde{A}_i^j\overline{D}_j\zeta_b + \Pi_b\left(2\sqrt{\frac{4\pi}{3}}\overline{D}_i\kappa + 8\pi\partial_i\lambda\right) = \mathcal{O}\left(\epsilon^5\right)
\end{cases}
$$

$$
\mathfrak{R}_{ij}\left(\zeta_b\right) = \overline{D}_i\zeta_b\overline{D}_j\zeta_b + \frac{1}{3}\eta_{ij}\overline{\triangle}\zeta_b - \overline{D}_i\overline{D}_j\zeta_b - \frac{1}{3}\eta_{ij}\overline{D}^k\zeta_b\overline{D}_k\zeta_b
$$

$$
\overline{\triangle}\zeta_b = \frac{e^{\frac{\zeta_b}{2}}}{4}\left(\overline{D}^k\zeta_b\overline{D}_k\zeta_b + 2\overline{\triangle}\zeta_b\right)
$$

Gradient expansion solution

$$\Omega = -\frac{3}{16a_0^2}\sqrt{\frac{3}{\pi}}e^{-2\zeta_b}\left(\overline{D}^k\zeta_b\overline{D}_k\zeta_b + 2\overline{\triangle}\zeta_b\right)t^{\frac{1}{3}}$$

$$\kappa = \frac{3}{8a_0^2}e^{-2\zeta_b}\left(\overline{D}^k\zeta_b\overline{D}_k\zeta_b + 2\overline{\triangle}\zeta_b\right)t^{\frac{4}{3}}$$

$$\chi = \frac{9}{8a_0^2}e^{-2\zeta_b}\left(\overline{D}^k\zeta_b\overline{D}_k\zeta_b + 2\overline{\triangle}\zeta_b\right)t^{\frac{4}{3}}$$

$$\lambda = 0$$

$$\hat{\xi} = \frac{3e^{-2\zeta_b}}{8a_0^2}\left(\overline{D}^k\zeta_b\overline{D}_k\zeta_b + 2\overline{\triangle}\zeta_b\right)t^{\frac{4}{3}}$$

$$\tilde{A}_{ij} = \frac{3}{4a_0^2}\zeta_{ij}\left(\zeta_b\right)e^{-2\zeta_b}t^{\frac{1}{3}}$$

$$h_{ij} = -\frac{9}{8a_0^2}\mathfrak{R}_{ij}\left(\zeta_b\right)e^{-2\zeta_b}t^{\frac{4}{3}}$$

$$\Omega = -\frac{1}{4a_0^2}\sqrt{\frac{3}{\pi}}e^{-2\zeta_b}\left(\overline{D}^k\zeta_b\overline{D}_k\zeta_b + 2\overline{\triangle}\zeta_b\right)t^{\frac{1}{3}}$$

$$\kappa = 0$$

$$\chi = \frac{2e^{-2\zeta_b}}{a_0^2}\left(\overline{D}^k\zeta_b\overline{D}_k\zeta_b + 2\overline{\triangle}\zeta_b\right)t^{\frac{4}{3}}$$

$$\lambda = \frac{1}{16a_0^2}\sqrt{\frac{3}{\pi}}e^{-2\zeta_b}\left(\overline{D}^k\zeta_b\overline{D}_k\zeta_b + 2\overline{\triangle}\zeta_b\right)t^{\frac{4}{3}}$$

$$\hat{\xi} = \frac{e^{-2\zeta_b}}{2a_0^2}\left(\overline{D}^k\zeta_b\overline{D}_k\zeta_b + 2\overline{\triangle}\zeta_b\right)t^{\frac{4}{3}}$$

$$\tilde{A}_{ij} = \frac{3}{4a_0^2}\mathfrak{R}_{ij}\left(\zeta_b\right)e^{-2\zeta_b}t^{\frac{1}{3}}$$

$$h_{ij} = -\frac{9}{8a_0^2}\mathfrak{R}_{ij}\left(\zeta_b\right)e^{-2\zeta_b}t^{\frac{4}{3}}$$