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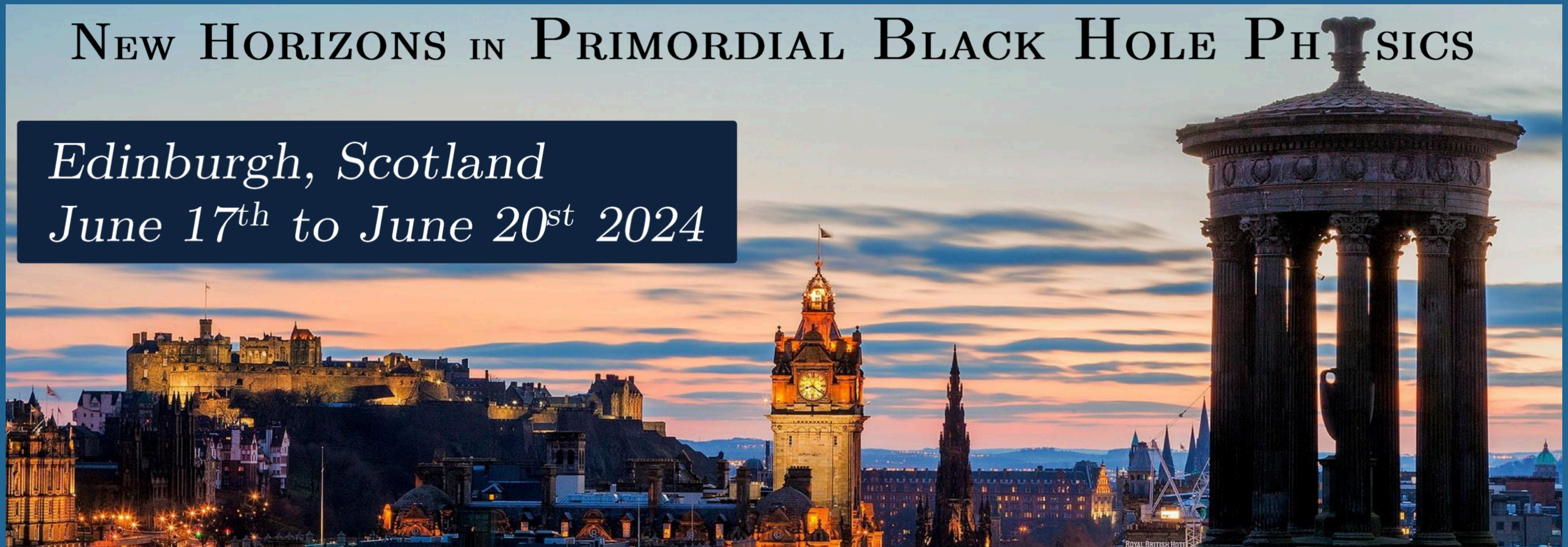
Clustering of primordial black holes from quantum diffusion during inflation

Chiara Animali

Tuesday 18 June 2024

work with Vincent Vennin
arXiv: 2402.08642



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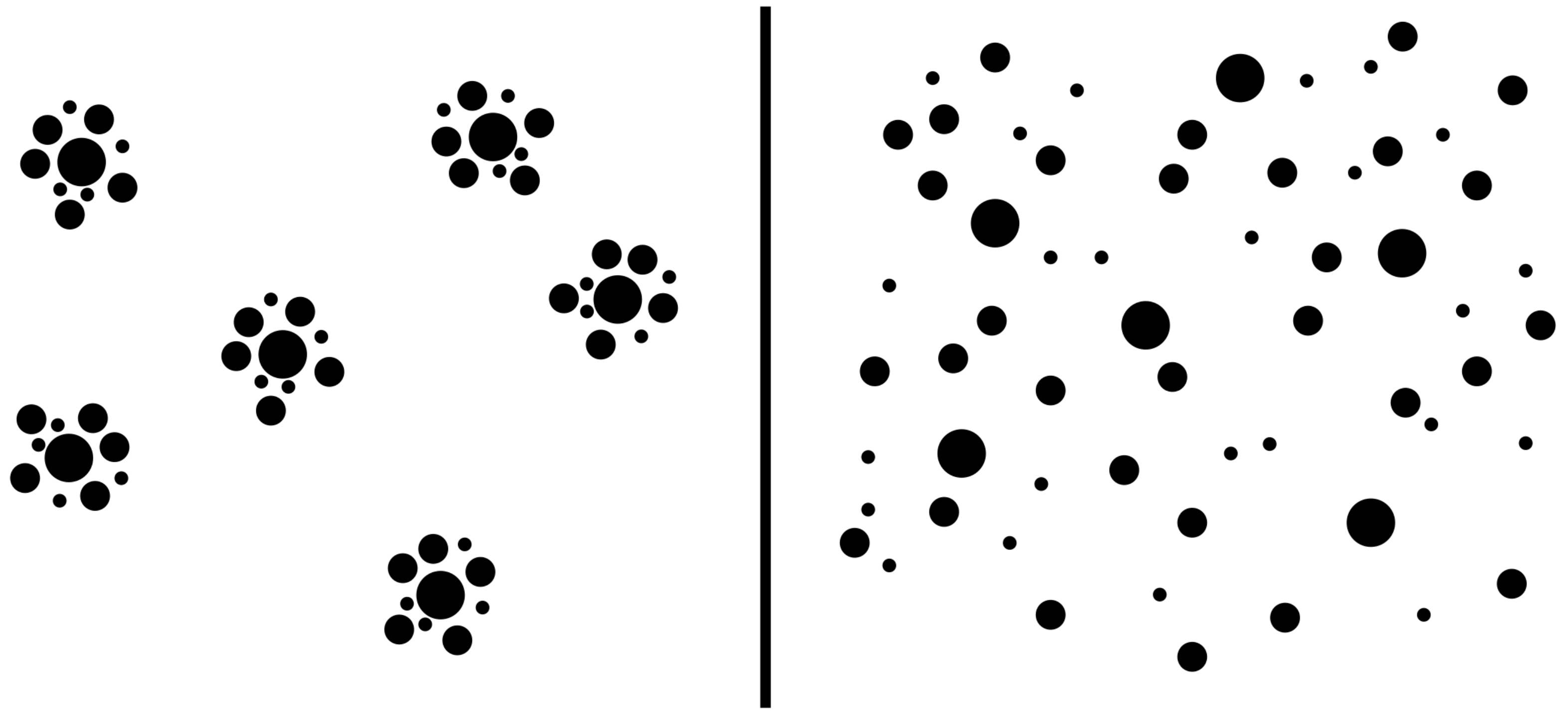

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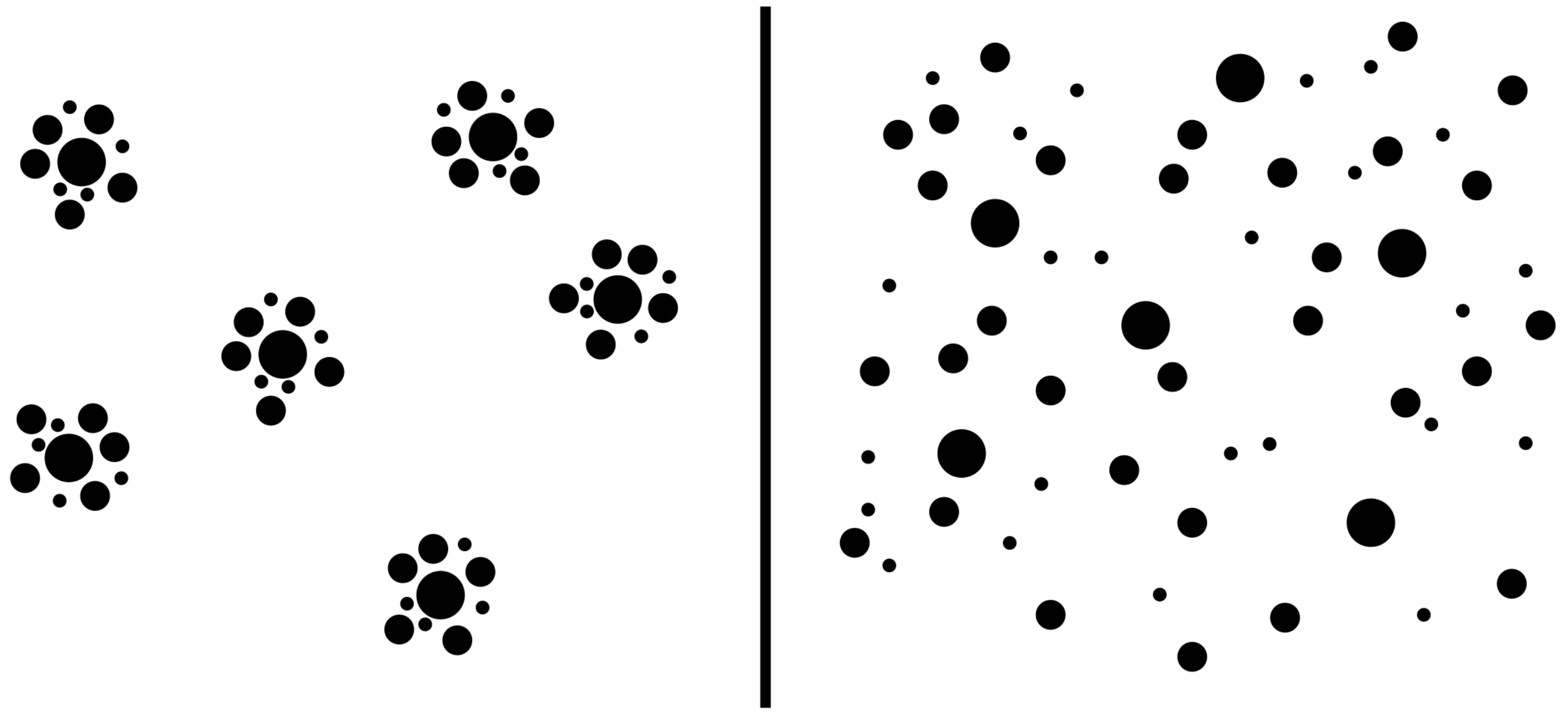

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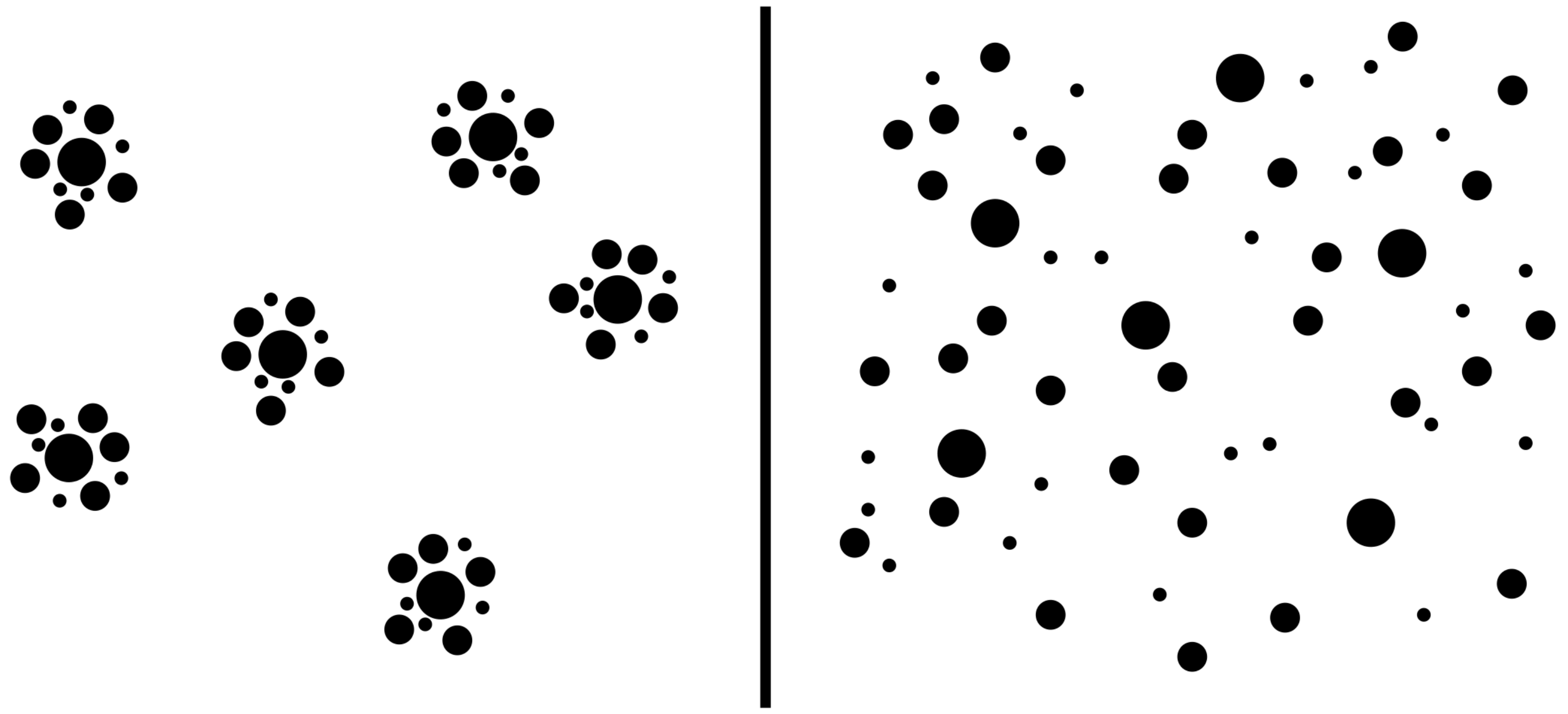

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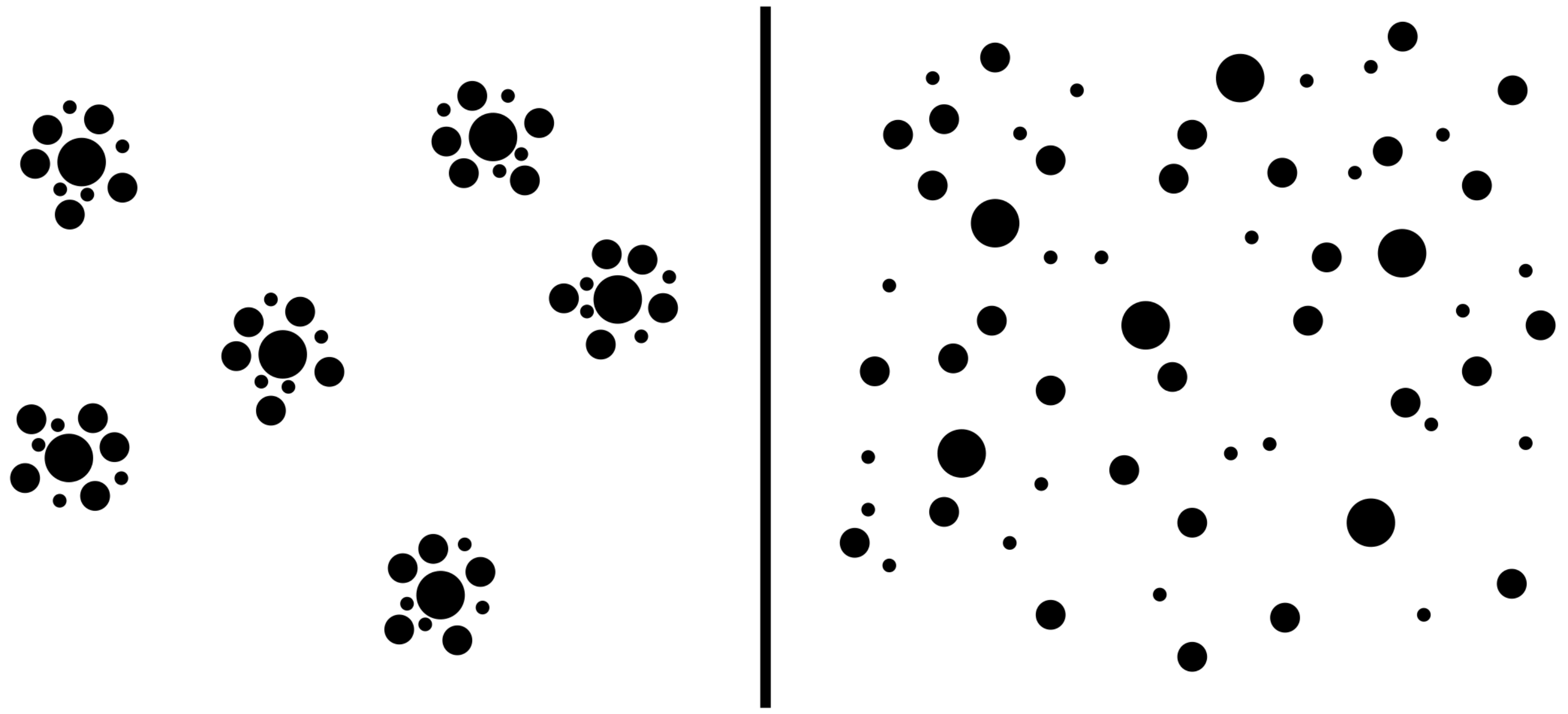

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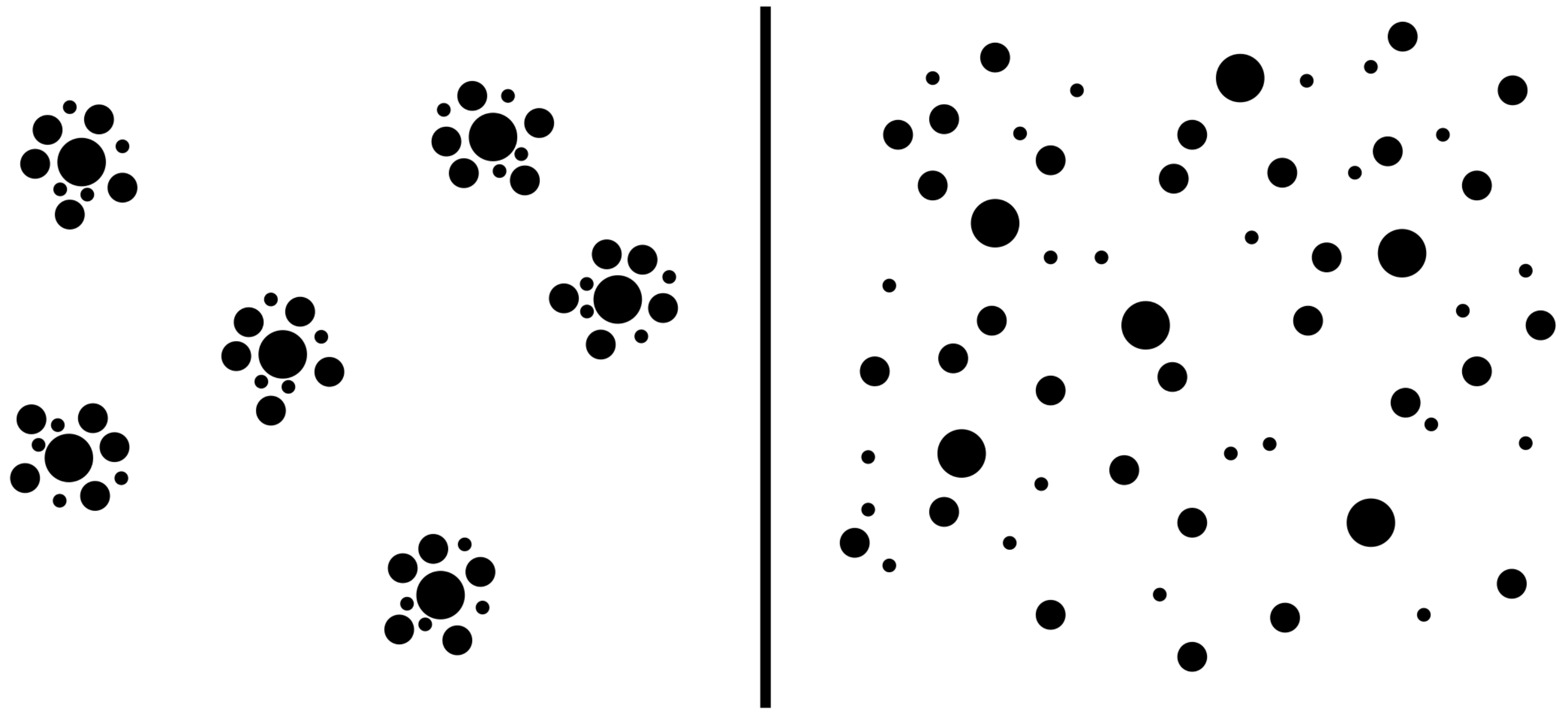

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→ **Central question**: characterise the **initial clustering** which then determines the clustering evolution throughout cosmic history

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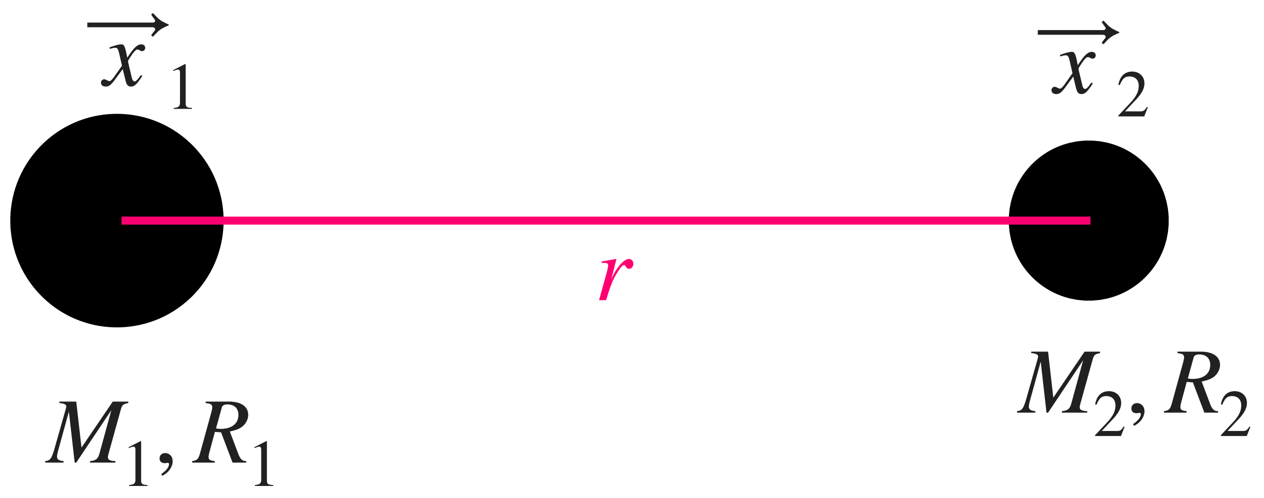

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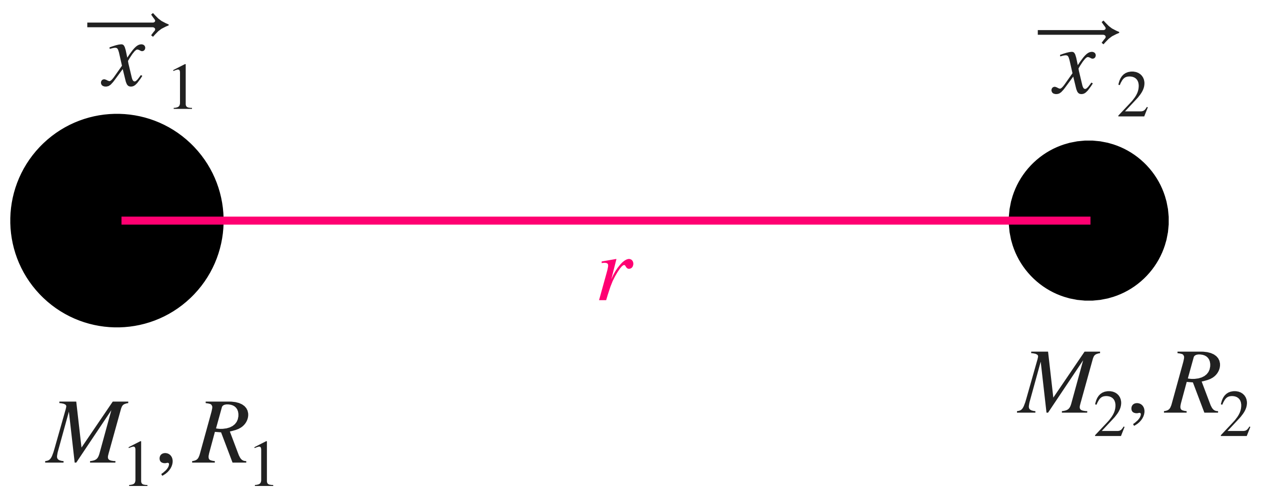

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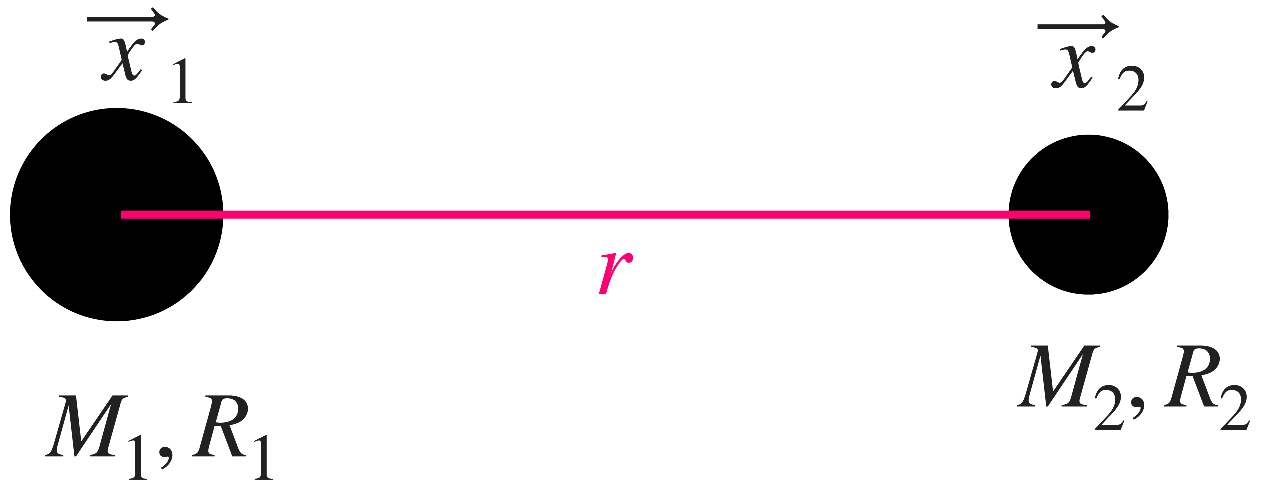

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Deviations from Poisson: $$\xi_{M_1,M_2}(r) = \frac{p(M_1, \vec{x}\,; M_2, \vec{x} + \vec{r})}{p_{M_1} p_{M_2}} - 1$$ **reduced correlation** N. Kaiser [1984]

$\xi > 0$: positive clustering ; $\xi < 0$: negative clustering

Primordial black holes: clustering & non Gaussianities

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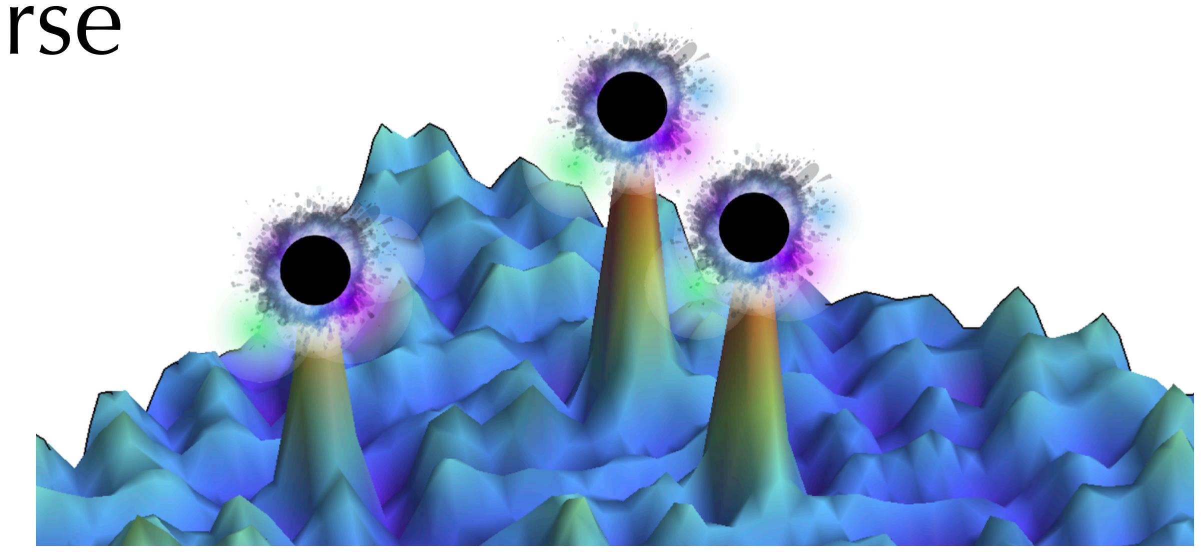
PBHs may originate from peaks of the density perturbations generated in the early universe



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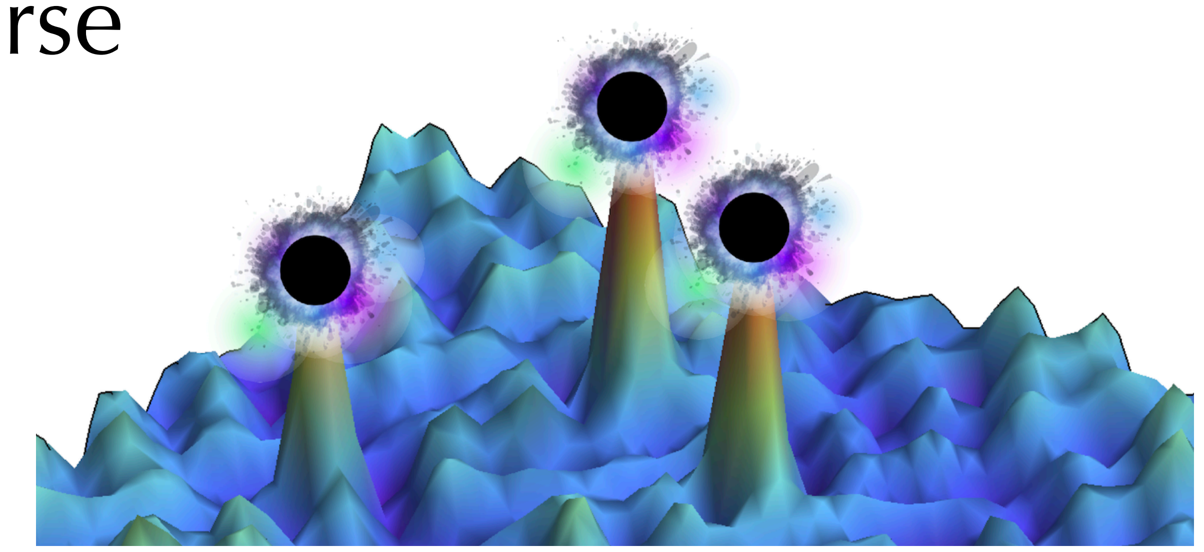

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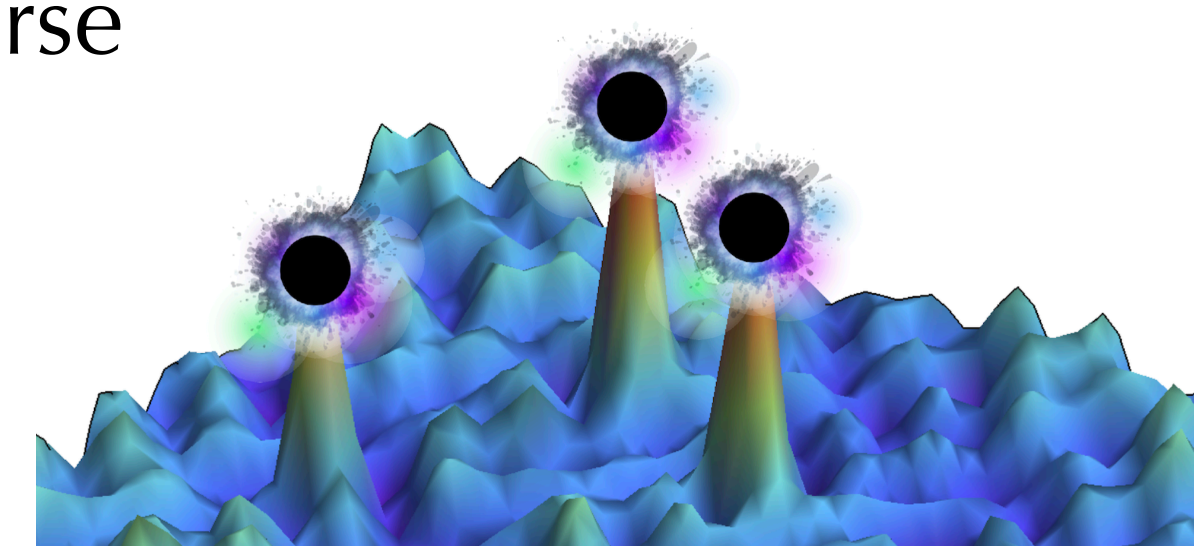

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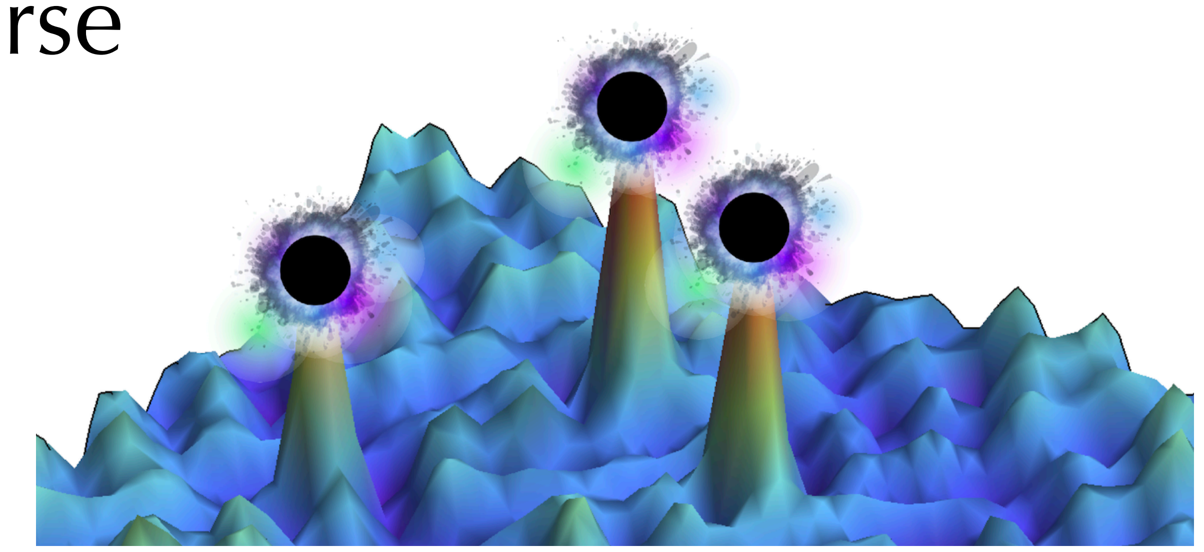

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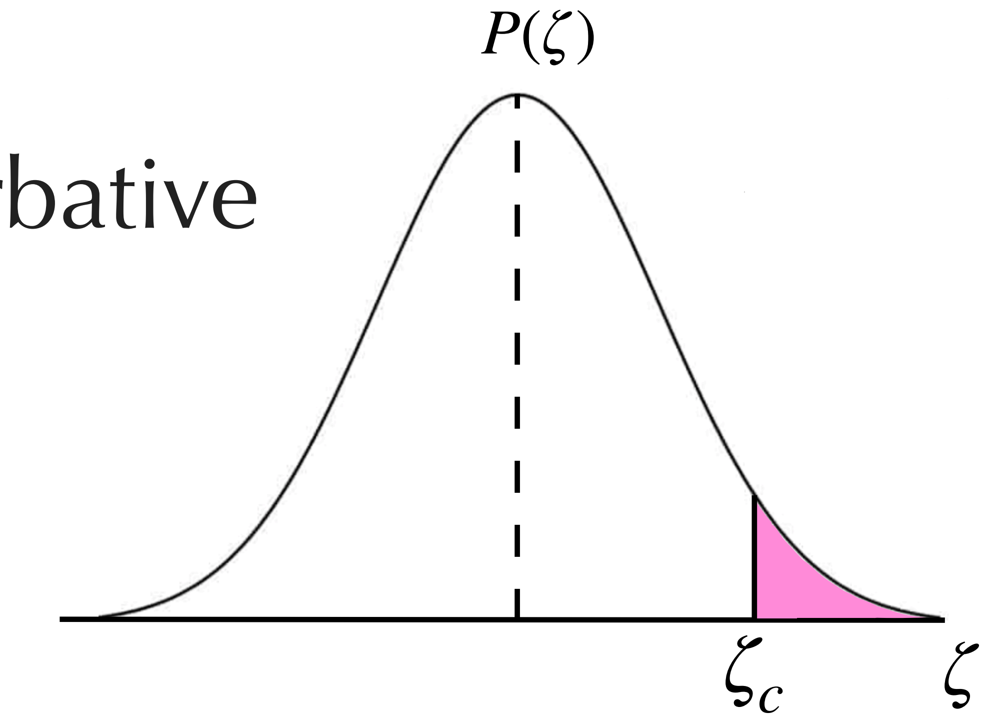

PBHs form in the heavy **tails** of distribution functions: non Gaussianities are not under perturbative control C. Pattison, V. Vennin, H. Assadullahi and D. Wands [2017]

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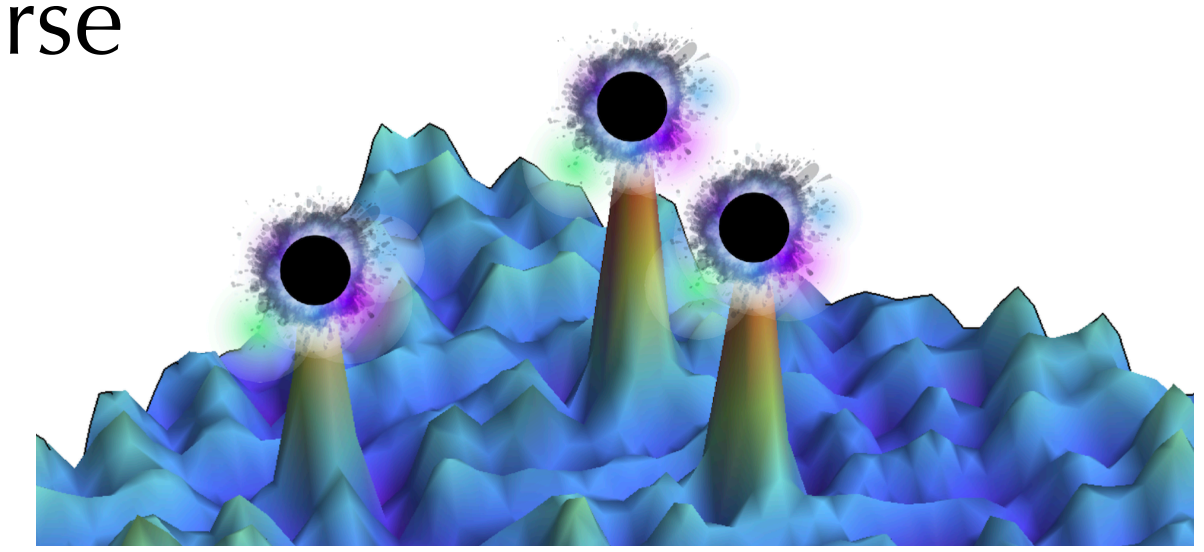

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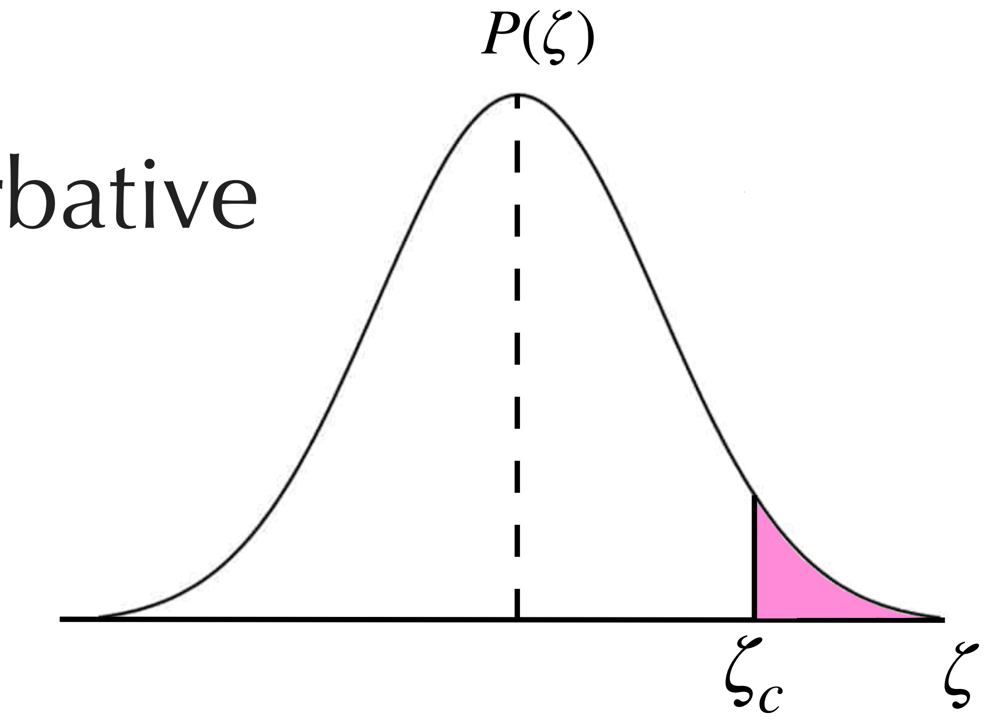

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→ **Goal: clustering in the stochastic-δN formalism**

Stochastic inflation

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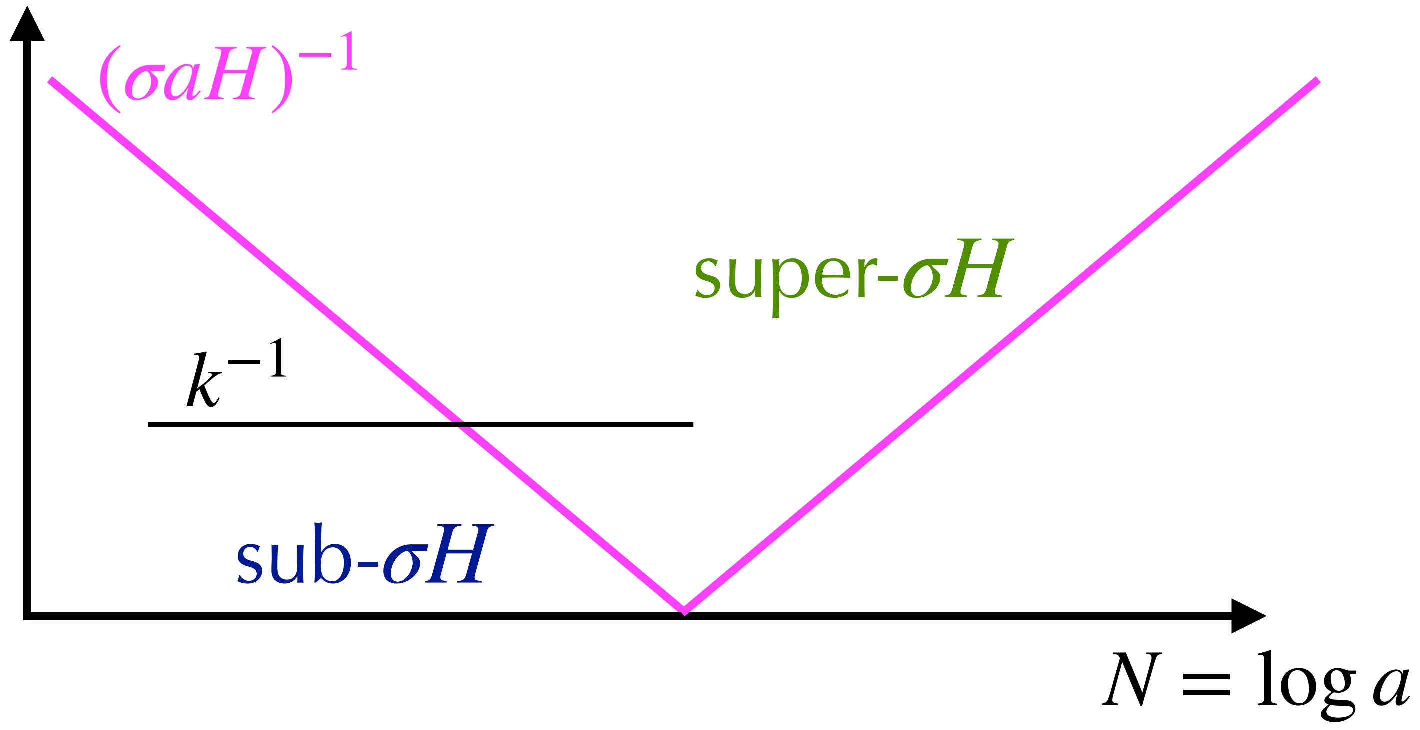
Effective theory for the long-wavelength part of quantum fields during inflation, which are coarse grained above the Hubble radius

$$\Phi = (\phi_1, \pi_1, \ldots \phi_n, \pi_n) \qquad \pi_i = \mathrm{d}\phi_i/\mathrm{d}N$$

$$\Phi(x)_{\mathrm{cg}}(N, \vec{x}) = \int \frac{d\vec{k}}{(2\pi)^{3/2}} \widetilde{W}\left(\frac{k}{\sigma a H}\right)\left[\Phi_k(N)\, e^{-i\vec{k}\cdot\vec{x}}\, \hat{a}_{\vec{k}} + \mathrm{h.c.}\right]$$



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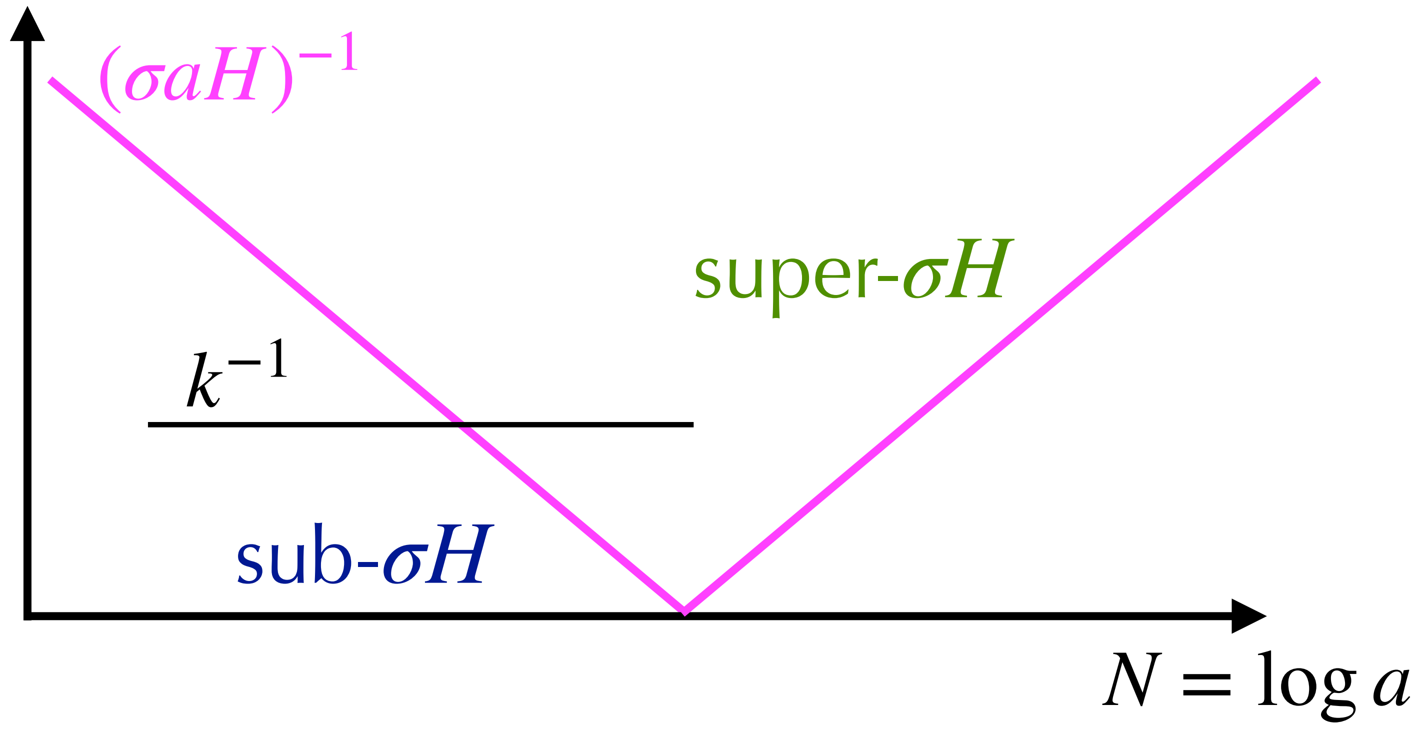

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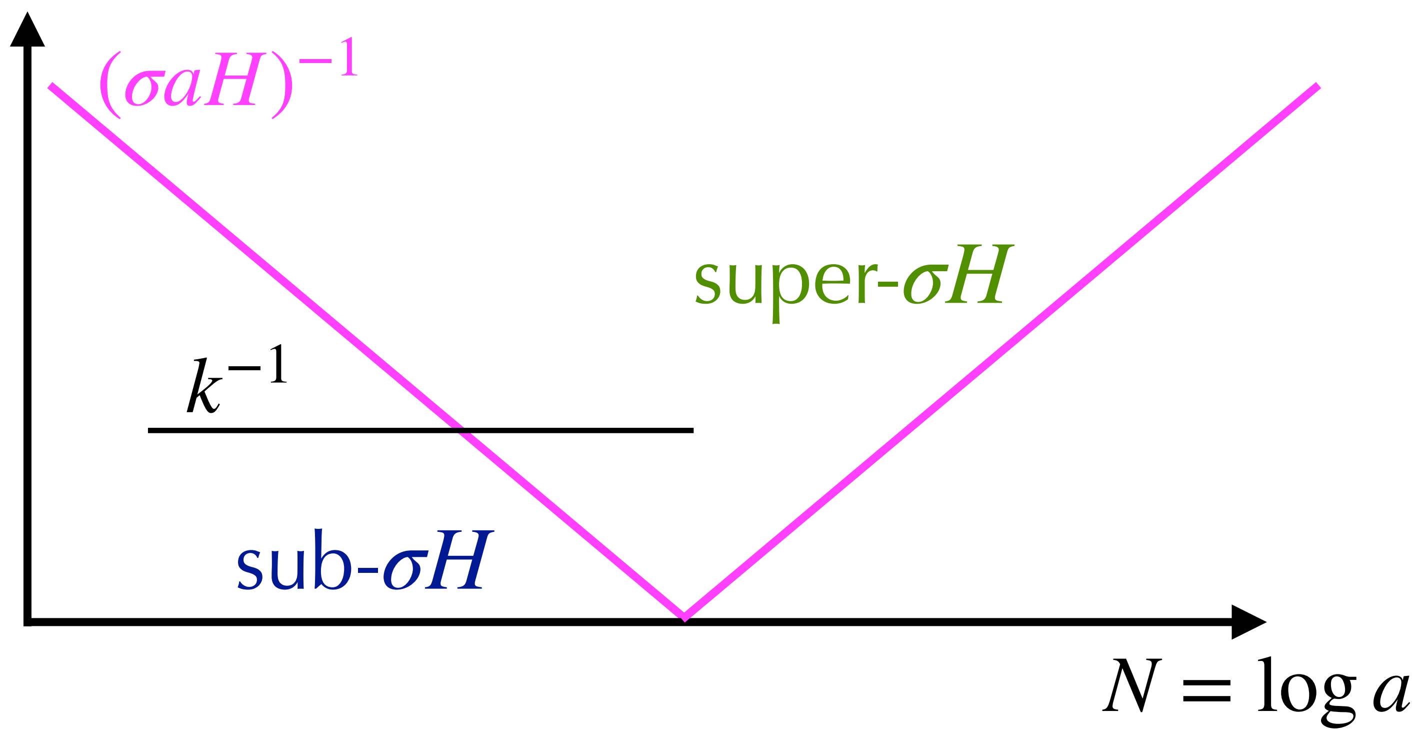

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Stochastic classical theory for Φ_{cg}: $\dfrac{\mathrm{d}\Phi_{\mathrm{cg}}}{\mathrm{d}N} = F_{\mathrm{cl}}(\Phi_{\mathrm{cg}}) + \xi$

- $F_{\mathrm{cl}}(\Phi_{\mathrm{cg}})$: classical eom
- ξ : white Gaussian noise
- $\left\langle \xi_i(\vec{x}, N_i)\, \xi_j(\vec{x}, N_j) \right\rangle = \dfrac{\mathrm{d}\ln(\sigma a H)}{\mathrm{d}N} \mathscr{P}_{\Phi_i, \Phi_j}\left[\sigma a H(N_i), N_i\right] \delta\left(N_i - N_j\right)$

Stochastic-δN formalism

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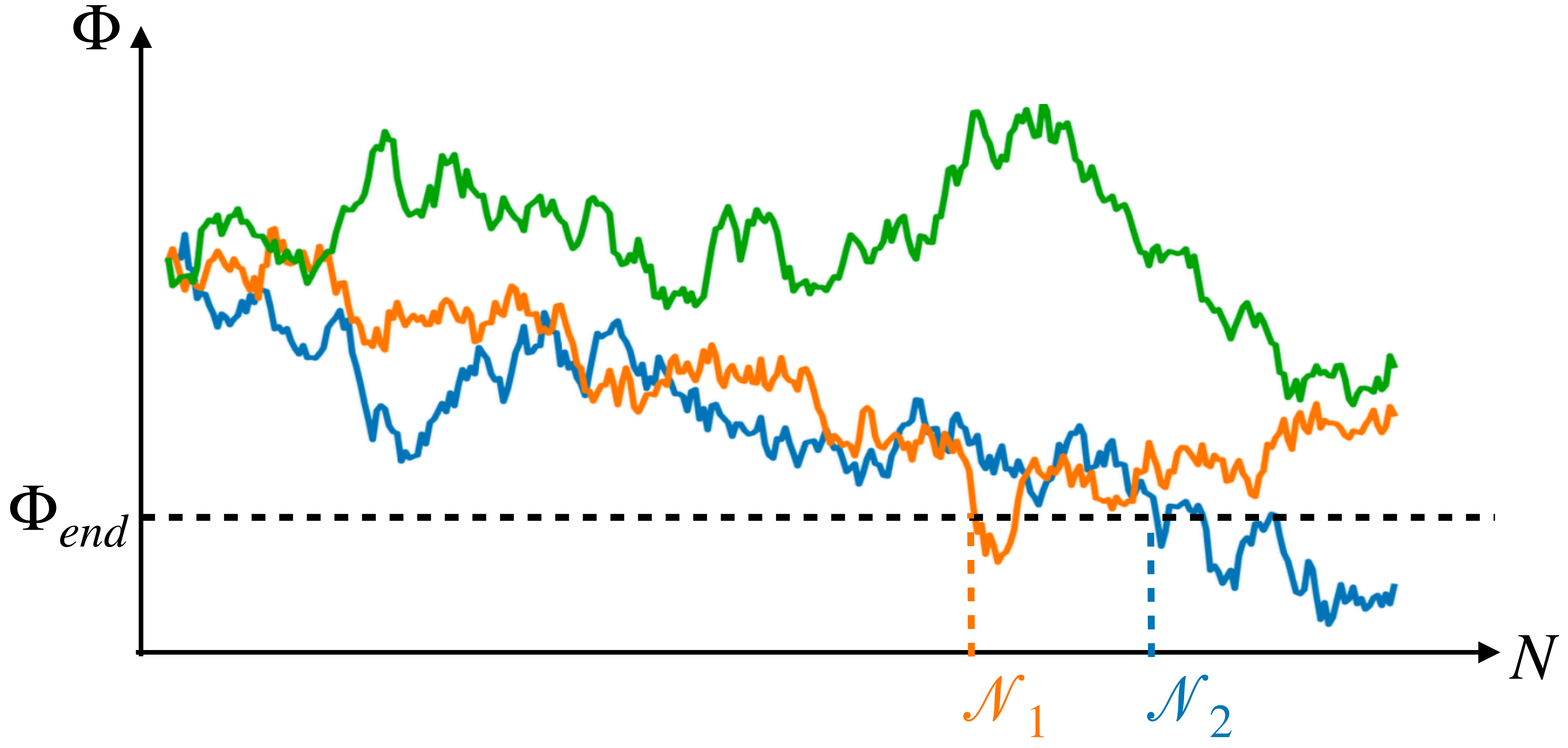
Duration of inflation becomes a stochastic variable: $\mathcal{N}$

First-passage time problem:

$$\frac{\mathrm{d}P_{\mathrm{FPT},\Phi}(\mathcal{N})}{\mathrm{d}\mathcal{N}} = \mathcal{L}^{\dagger}_{\mathrm{FP}}(\Phi) \cdot P_{\mathrm{FPT},\Phi}(\mathcal{N}) \qquad P_{\mathrm{FPT},\Phi=\Phi_{\mathrm{end}}}(\mathcal{N}) = \delta(\mathcal{N})$$



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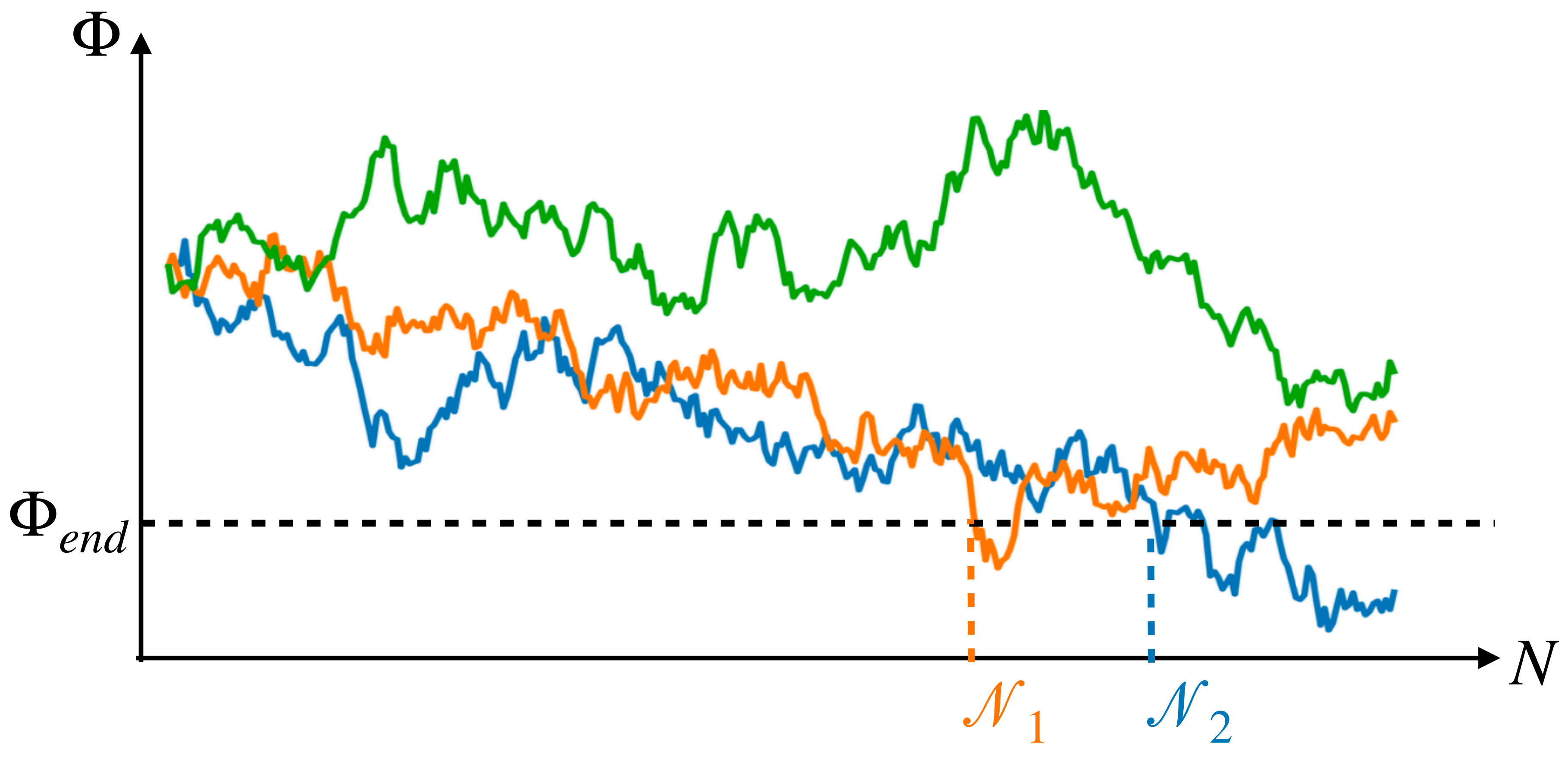

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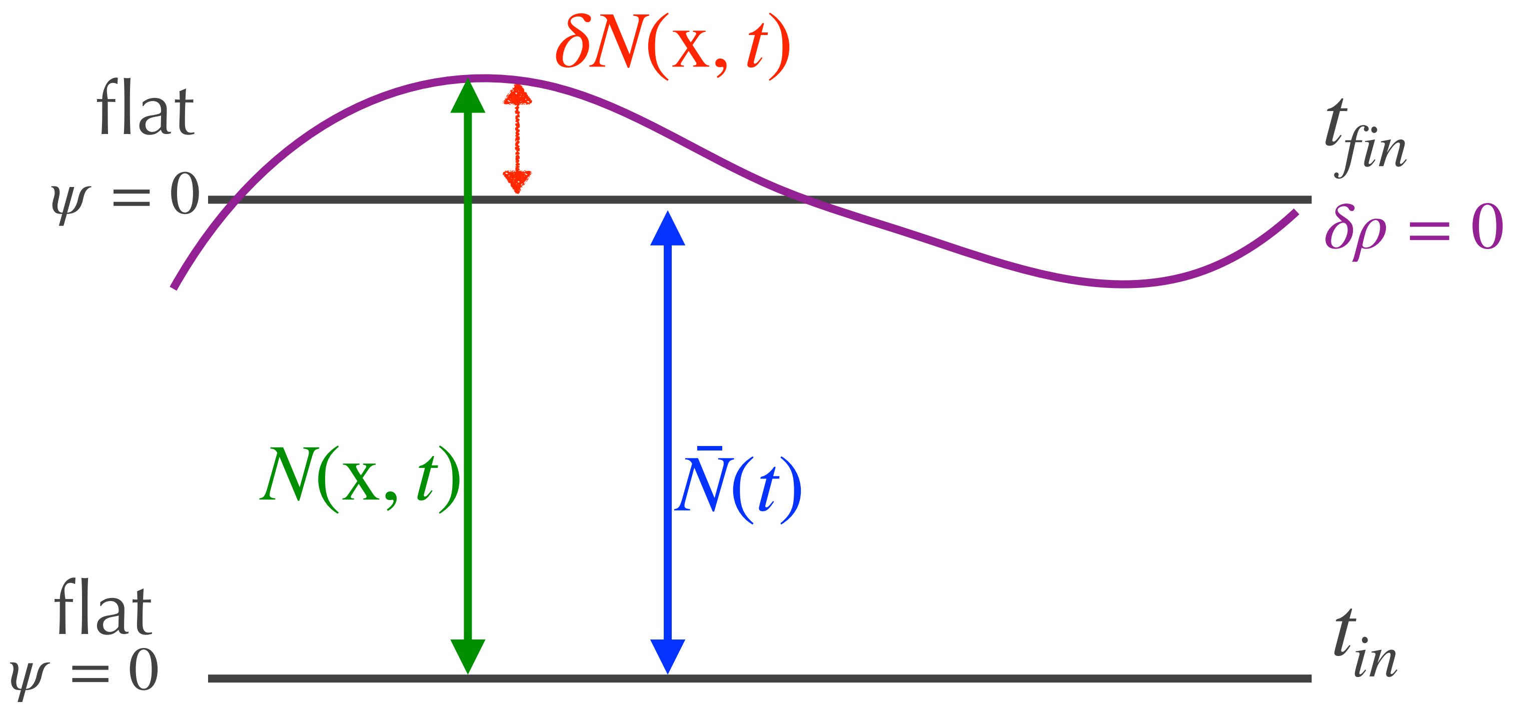

$$\zeta(t,\mathrm{x}) = N(t,\vec{x}) - \overline{N}(t) \equiv \delta N$$

δN formalism

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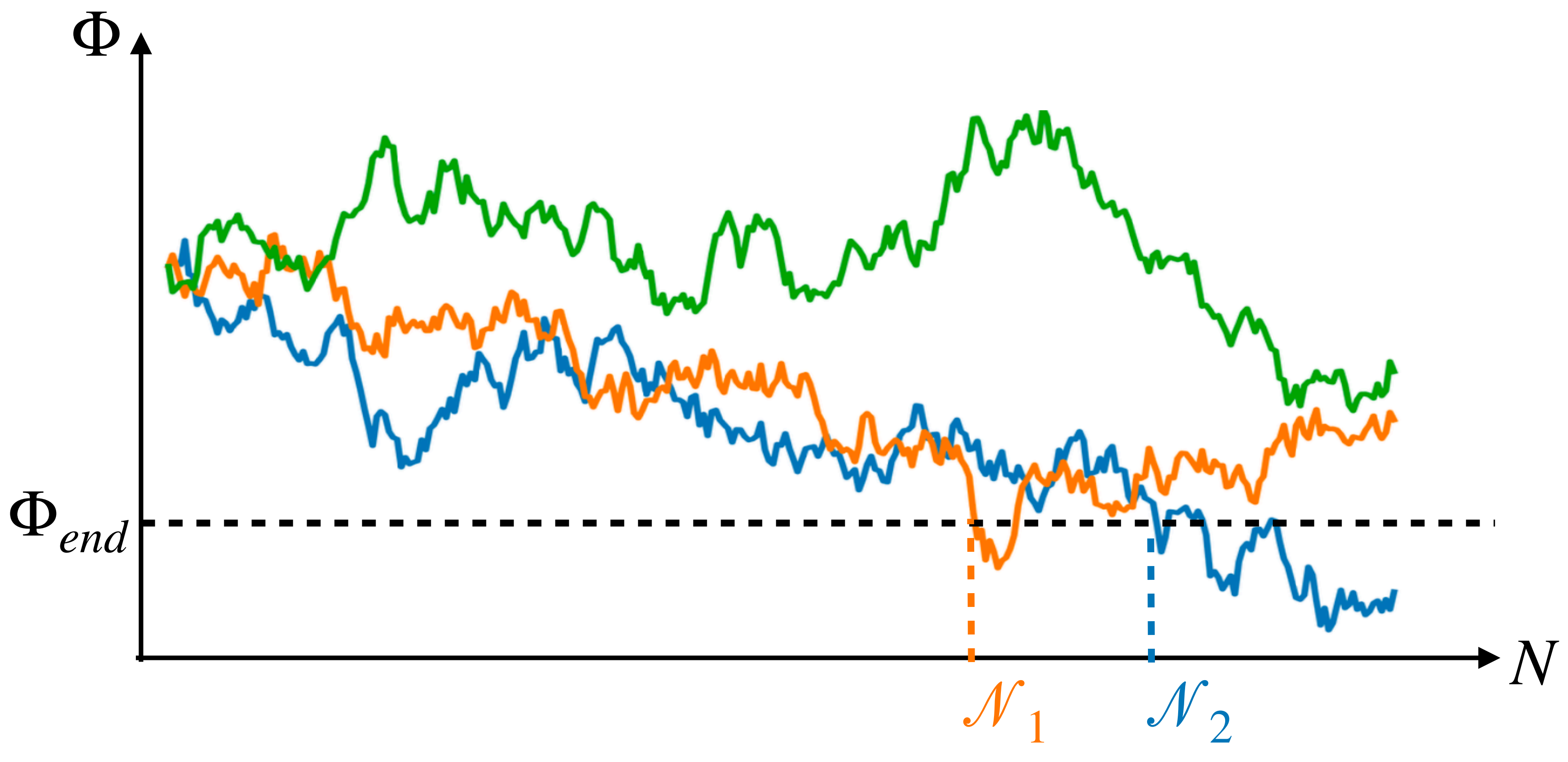

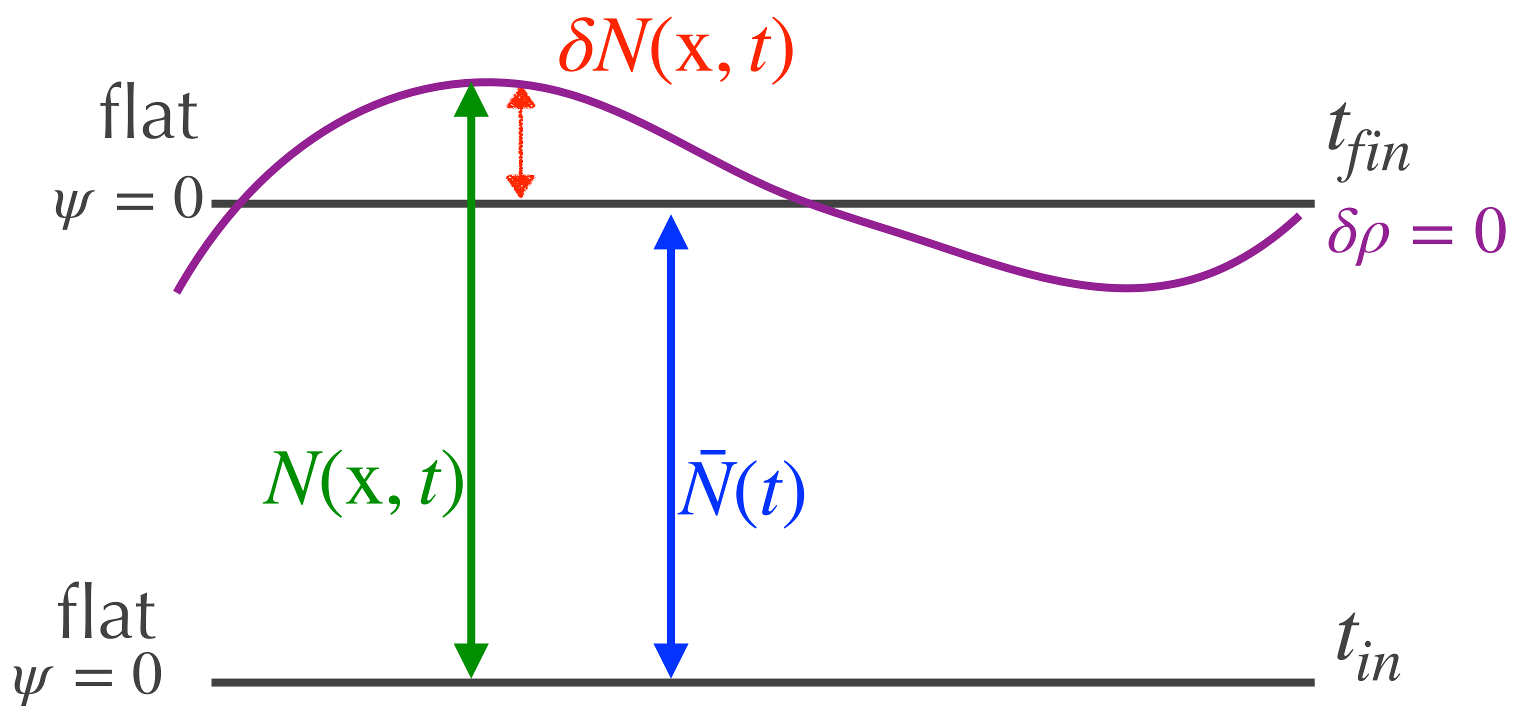

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Statistics of the duration of inflation (*first passage time problem*) gives the statistics of the coarse-grained curvature perturbation in a non-perturbative way:

[Enqvist, Nurmi, Podolsky, Rigopoulos [2008]

Vennin, Starobinsky [2015]

$$\zeta_{cg}(\mathrm{x}) = \mathcal{N} - \overline{N}$$

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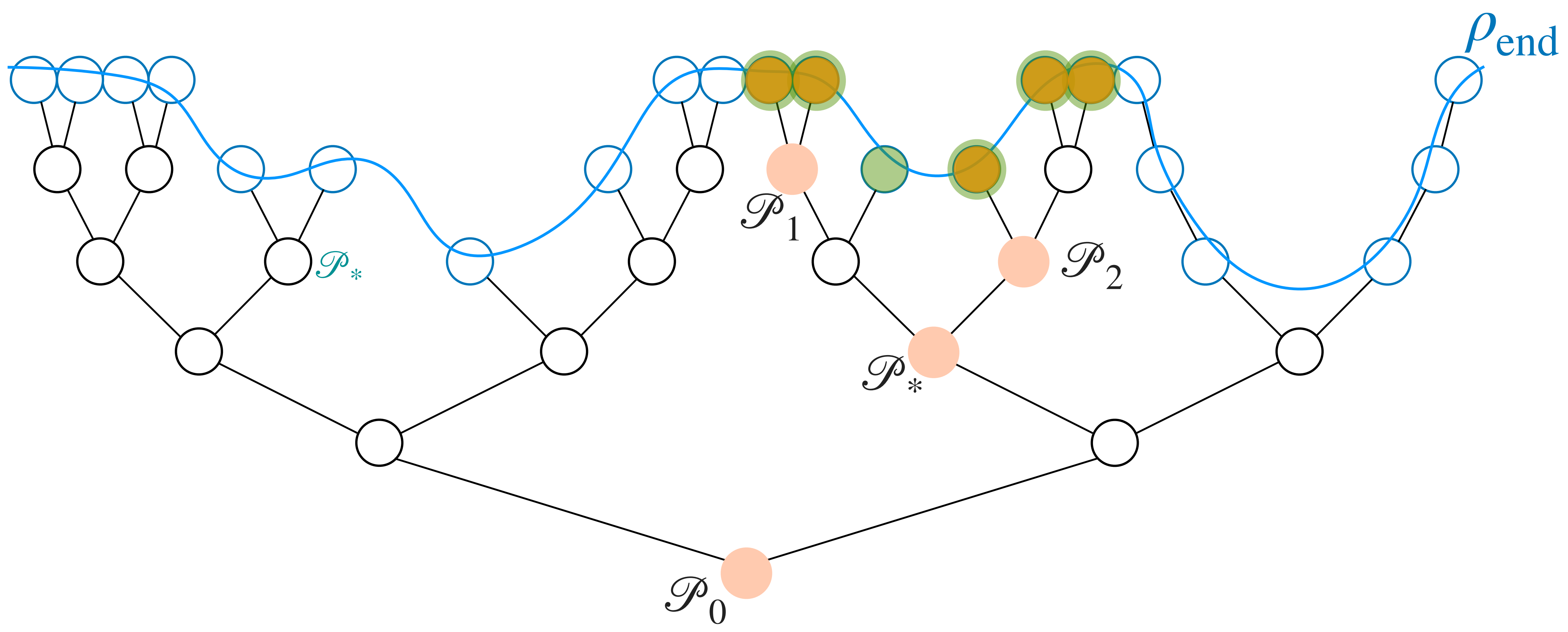

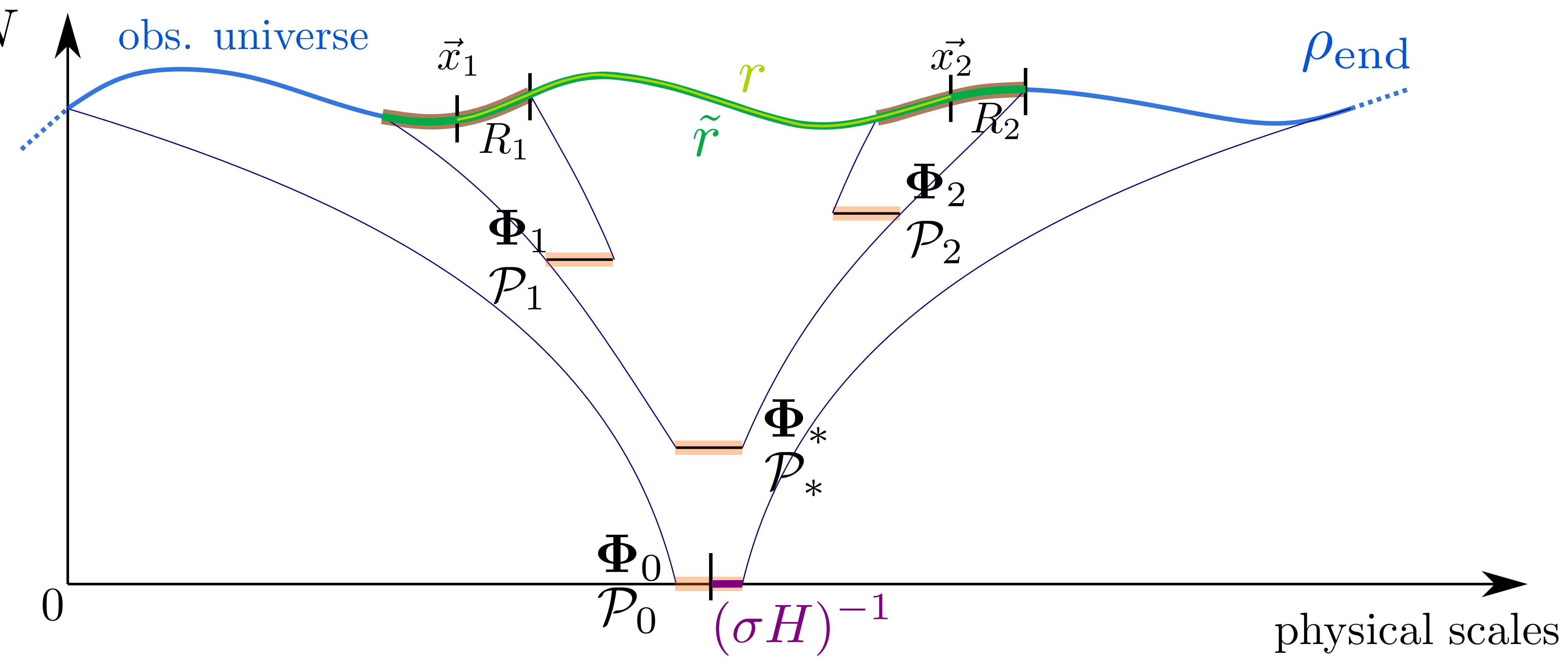

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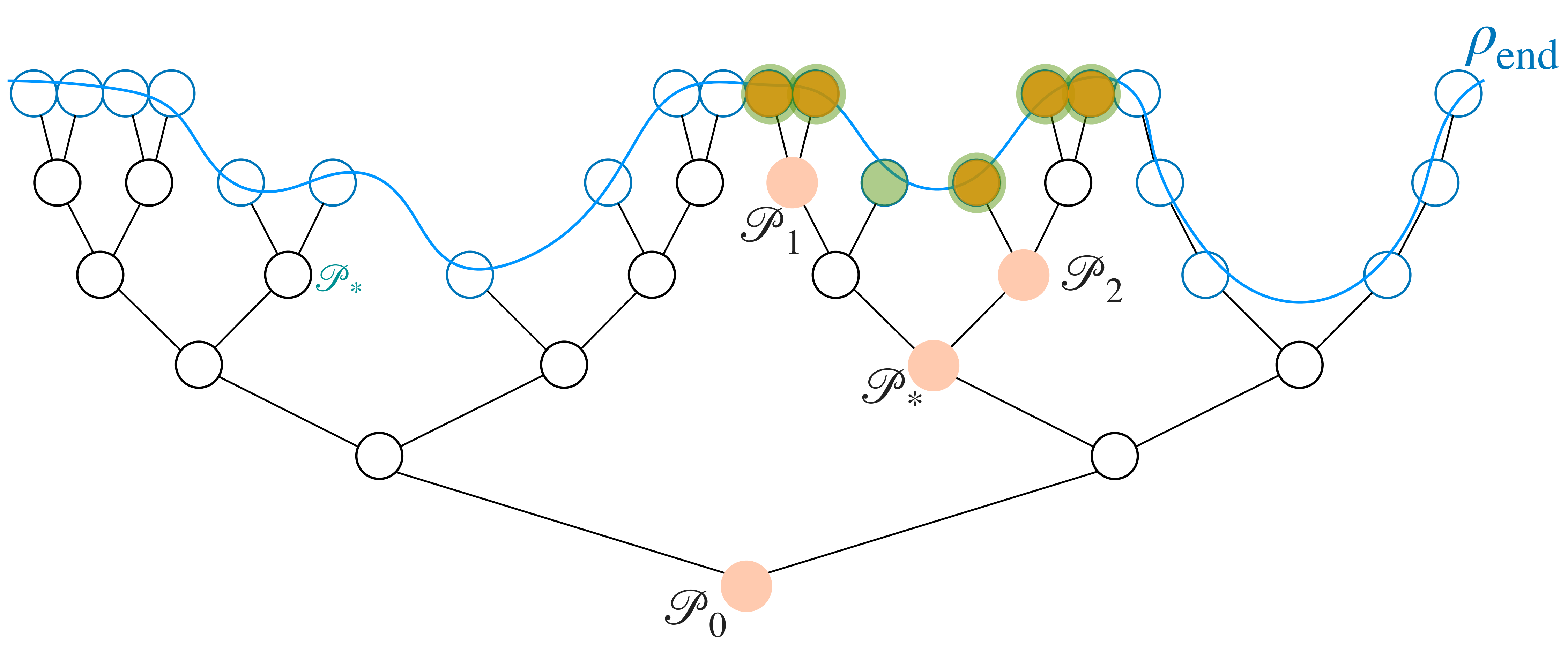

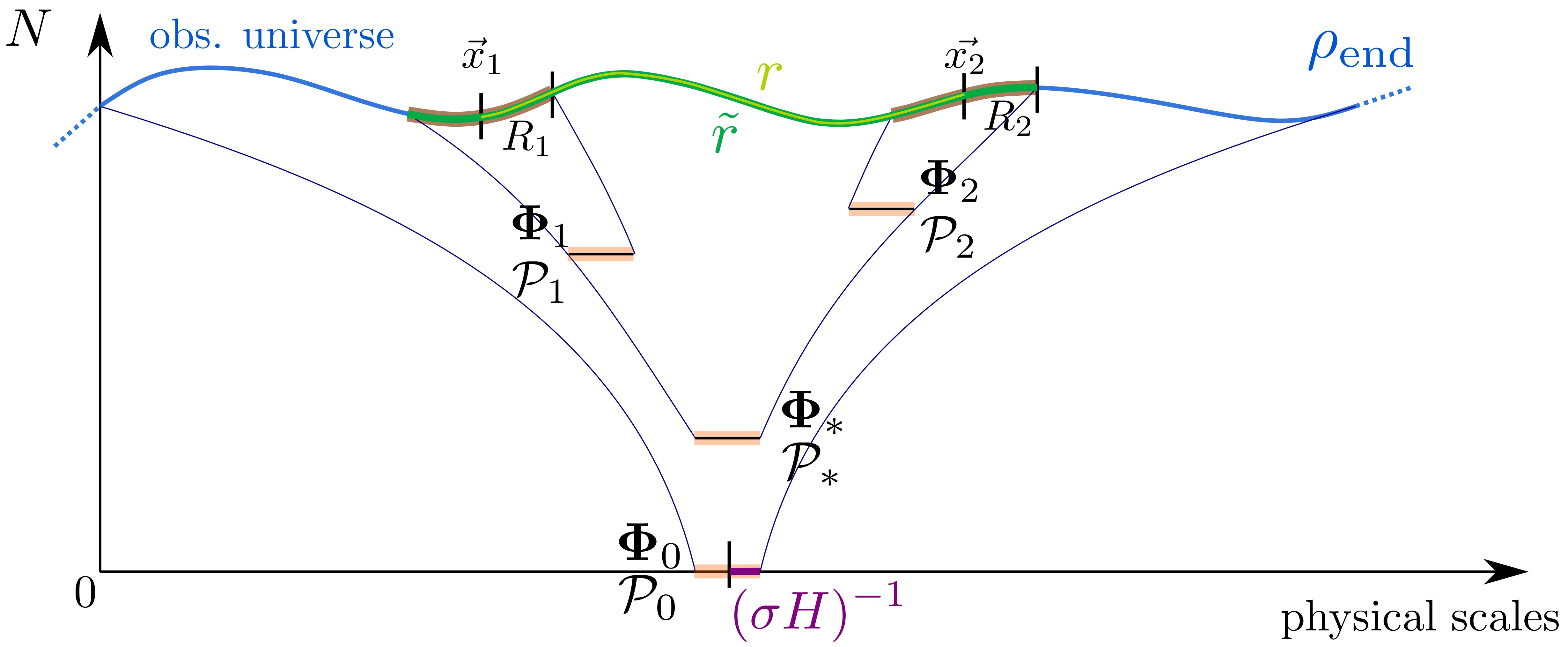

$$\zeta_{{\rm cg},R_i}(\overrightarrow{x}_i) \equiv \zeta_{R_i}(\overrightarrow{x}_i) = \mathbb{E}^V_{\mathscr{P}_i}[\mathscr{N}_{\mathscr{P}_0}(\overrightarrow{x})] - \mathbb{E}^V_{\mathscr{P}_0}[\mathscr{N}_{\mathscr{P}_0}(\overrightarrow{x})]$$

$$\mathscr{N}_{\mathscr{P}_0}(\overrightarrow{x}_i) = \underline{\mathscr{N}_{\mathscr{P}_0\to\mathscr{P}_*}(\overrightarrow{x})} + \mathscr{N}_{\mathscr{P}_*\to\mathscr{P}_i}(\overrightarrow{x}_i) + \mathscr{N}_{\mathscr{P}_i}(\overrightarrow{x}_i)$$

→ *Shared history*

Volume weighting

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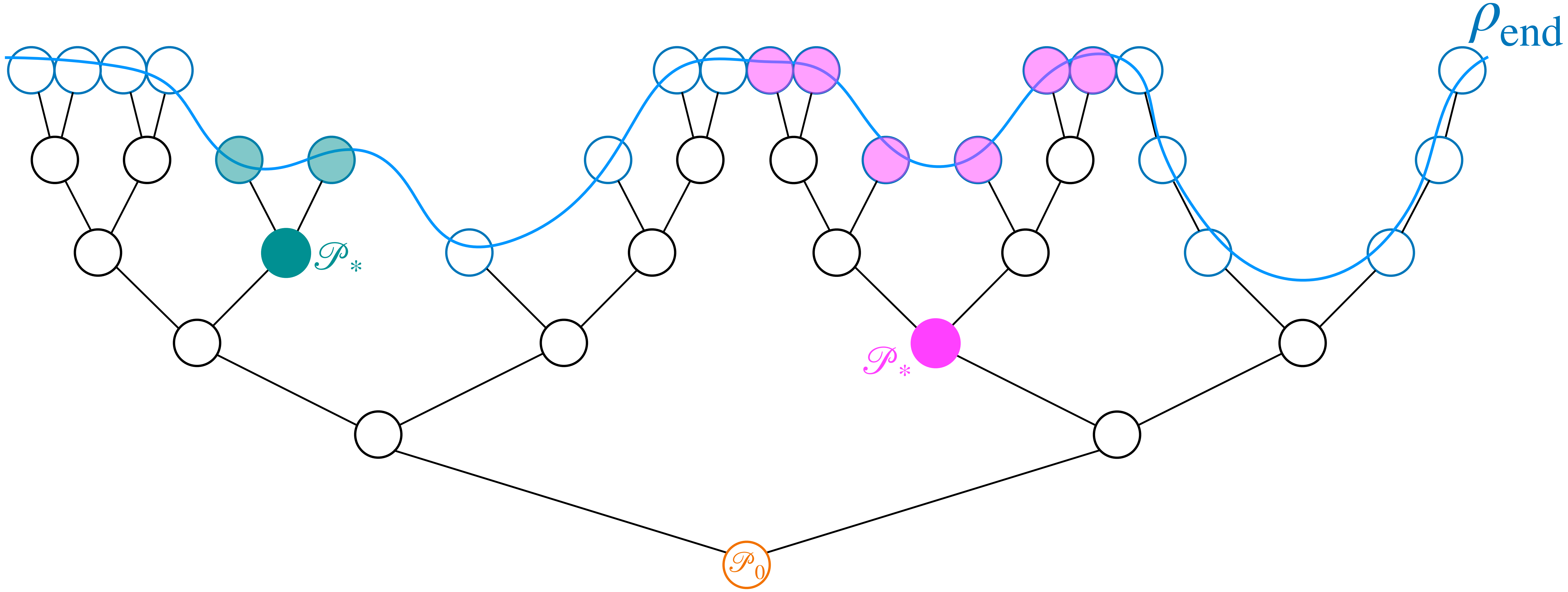
they contribute differently to ensemble averages computed by local observers on the end-of-inflation hypersurface



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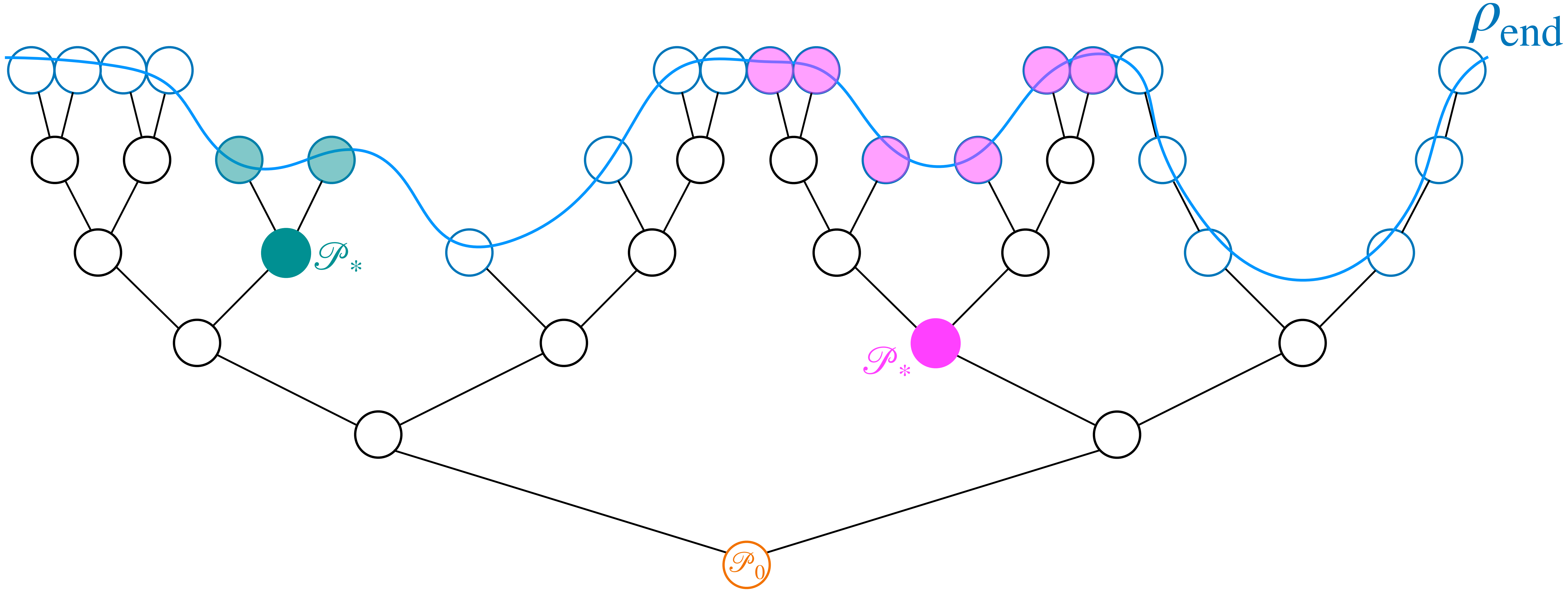

Distributions with respect to which observable quantities are defined should be **volume weighted**

Volume weighting

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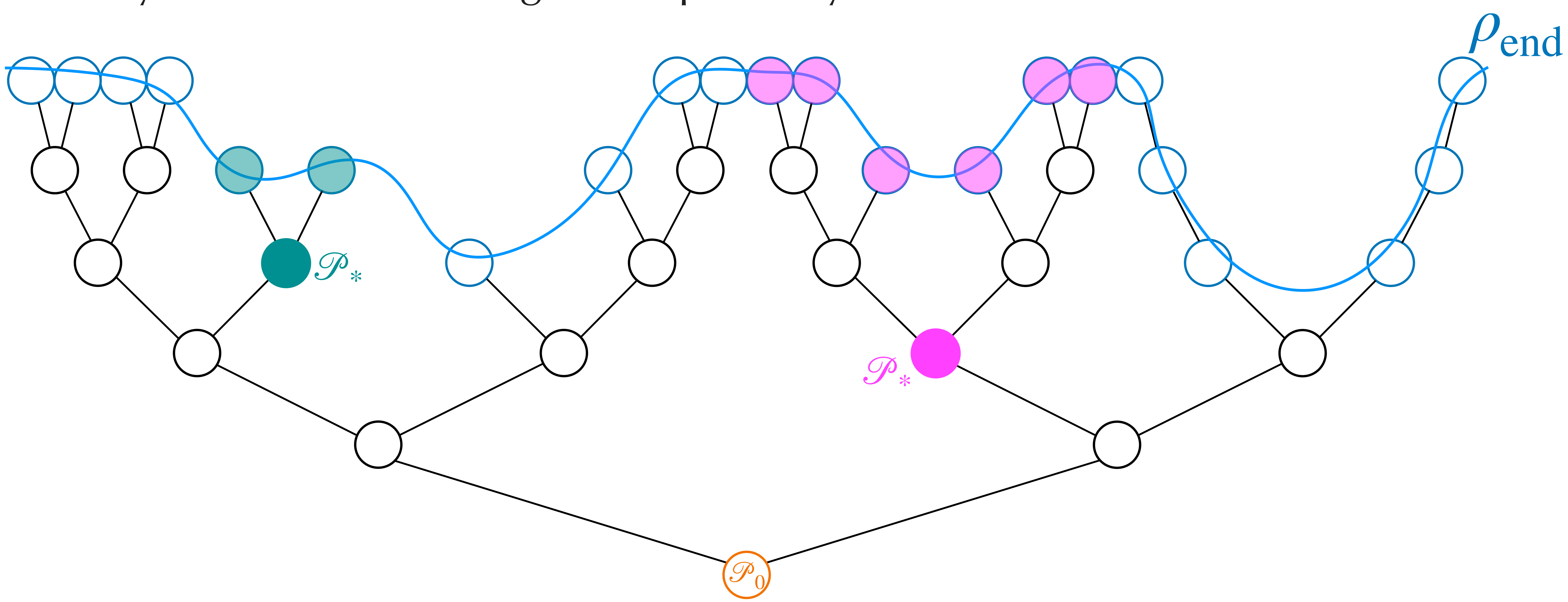

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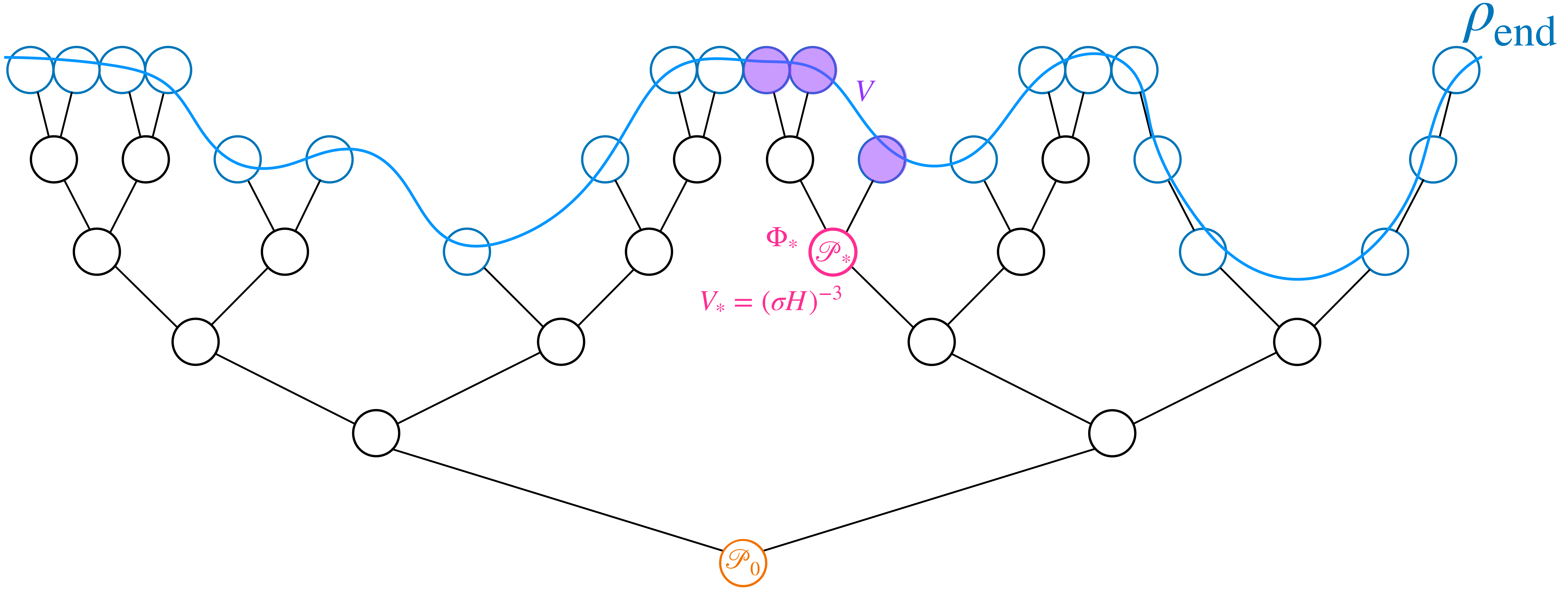
$$P^V_{\mathrm{FPT},\Phi_0}(\mathcal{N}) = \frac{P_{\mathrm{FPT},\Phi_0}(\mathcal{N})\, e^{3\mathcal{N}}}{\int_0^\infty \mathrm{d}\mathcal{N} P_{\mathrm{FPT},\Phi_0}(\mathcal{N})\, e^{3\mathcal{N}}}$$

$$\zeta_{\mathrm{cg}}(\vec{x}) = \mathcal{N}_{\mathscr{P}_0}(\vec{x}) - \mathbb{E}^V_{\mathscr{P}_0}(\mathcal{N}_{\mathscr{P}_0}) \qquad P(\zeta_{\mathrm{cg}}\,|\,\Phi_0) = P^V_{\mathrm{FPT},\Phi_0}(\zeta_{\mathrm{cg}} + \mathbb{E}^V_{\mathscr{P}_0}(\mathcal{N}_{\mathscr{P}_0}))$$

Extracting cosmological observables

Extracting cosmological observables

Relation between field values and physical distances encoded in the structure of a universe which inflates stochastically



Extracting cosmological observables

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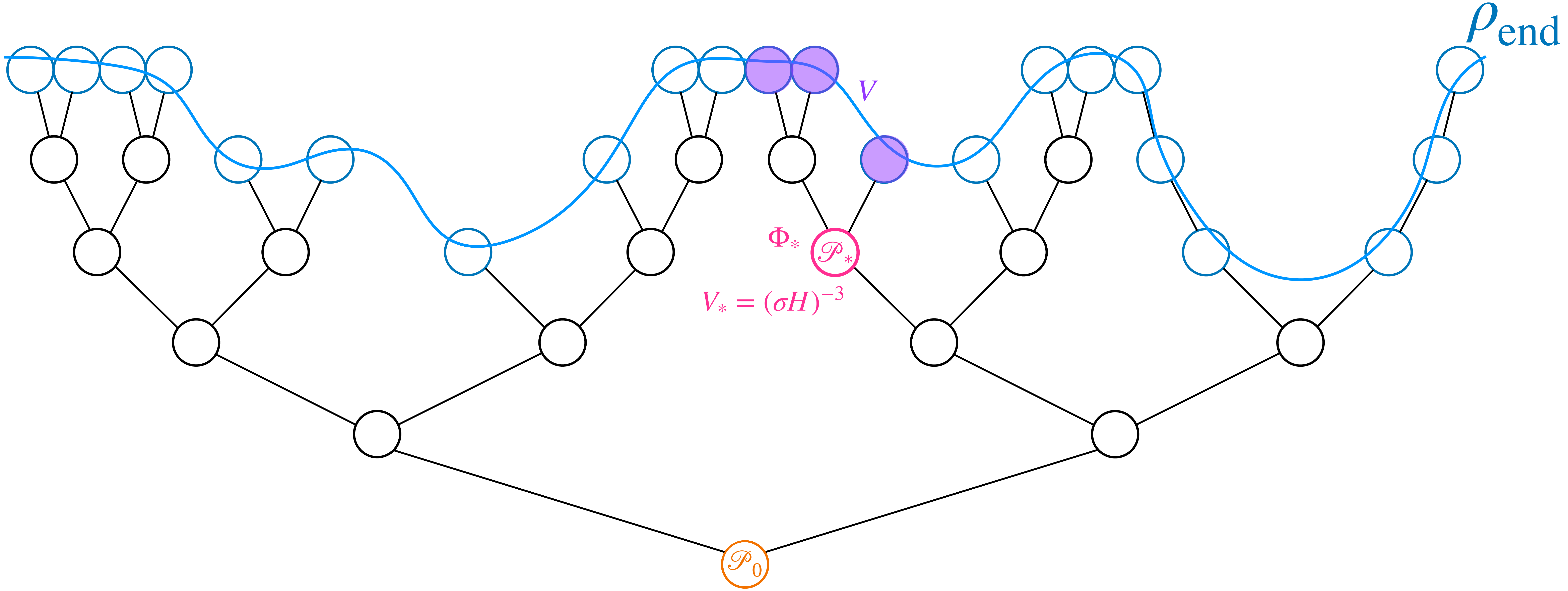

Final volume: $\dfrac{V}{V_*} = \dfrac{\int_{\mathscr{P}_*} \mathrm{d}\vec{x}\, e^{3\mathcal{N}_{\mathscr{P}_*}(\vec{x})}}{\int_{\mathscr{P}_*} \mathrm{d}\vec{x}} = \mathbb{E}_{\mathscr{P}_*}\left[e^{3\mathcal{N}_{\mathscr{P}_*}(\vec{x})}\right]$

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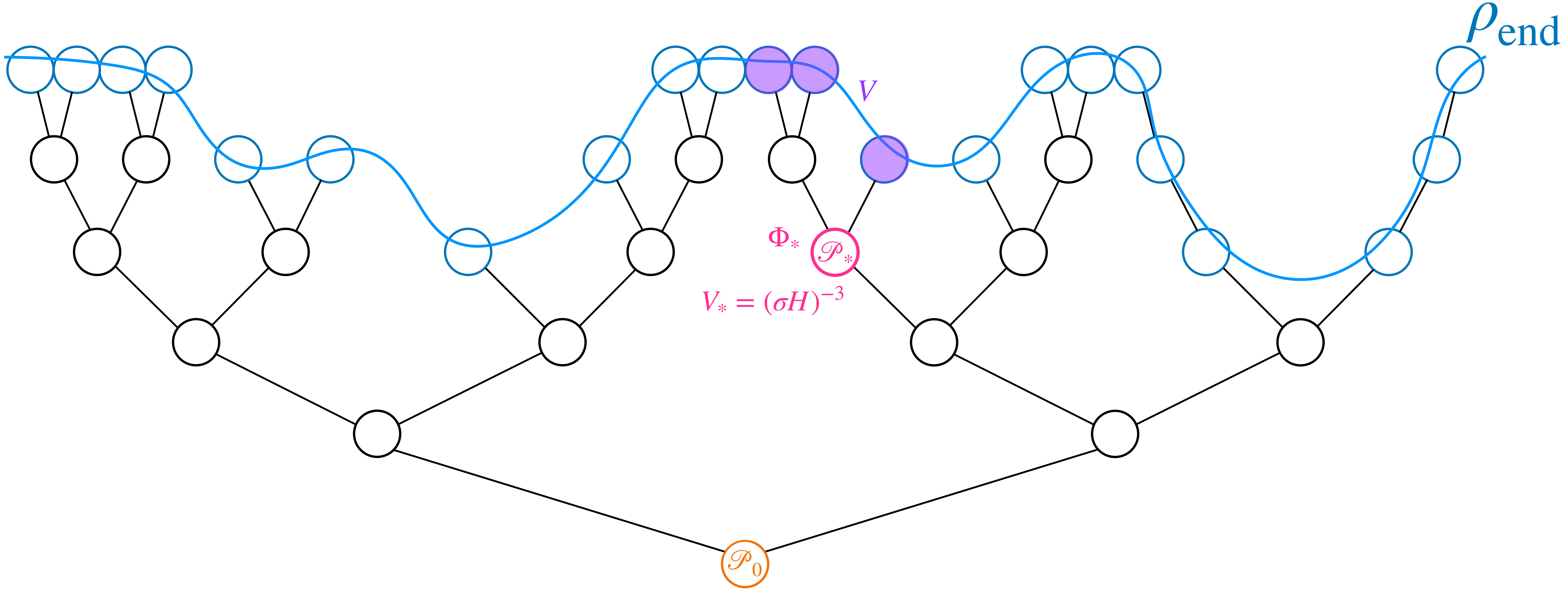

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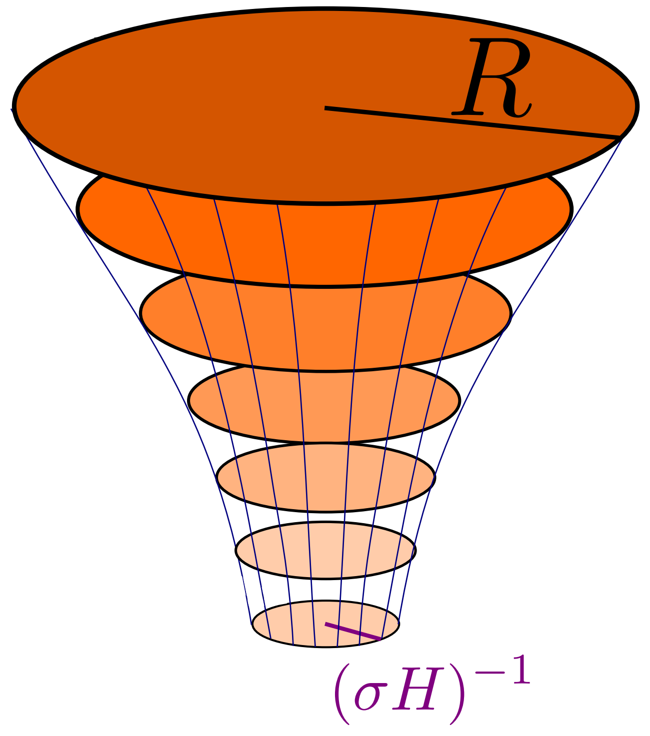
Backward distribution: $P(\Phi_* \mid V, \Phi_0) = \frac{P(V \mid \Phi_*) P(\Phi_* \mid \Phi_0)}{P(V)} = \frac{P(V \mid \Phi_*) P(\Phi_* \mid \Phi_0)}{\int \mathrm{d}\Phi_*\, P(V \mid \Phi_*) P(\Phi_* \mid \Phi_0)}$

Large-volume approximation



$$R^3 \gg (\sigma H)^{-3}$$

Ensemble average over the set of final leaves → Stochastic average of a single element within the ensemble

$$V \to \langle V \rangle \qquad P(V\,|\,\Phi_*) \simeq \delta_{\mathrm{D}}(V - V_* \langle e^{3\mathcal{N}_{\Phi_*}} \rangle) \qquad \langle e^{3\mathcal{N}_{\Phi_*}} \rangle = \int_0^\infty P_{\mathrm{FPT},\Phi_*}(\mathcal{N}) e^{3\mathcal{N}} \mathrm{d}\mathcal{N}$$

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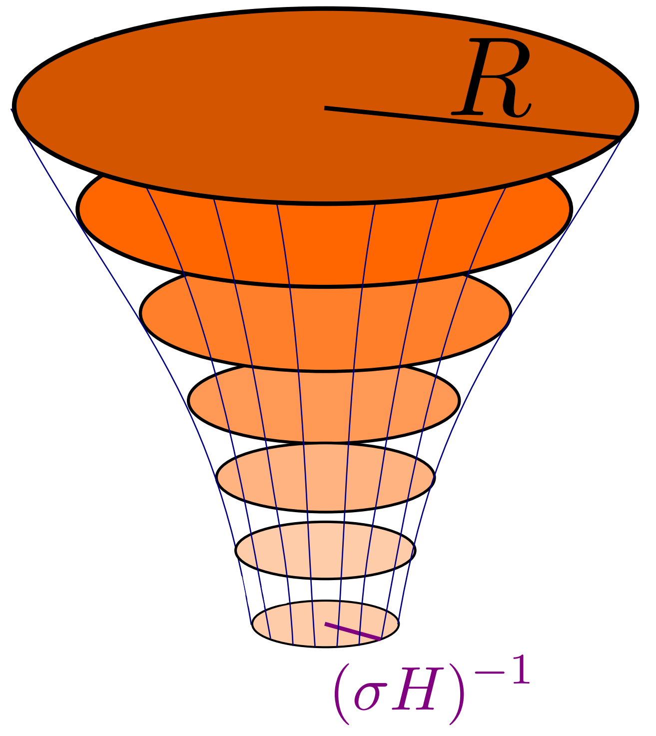

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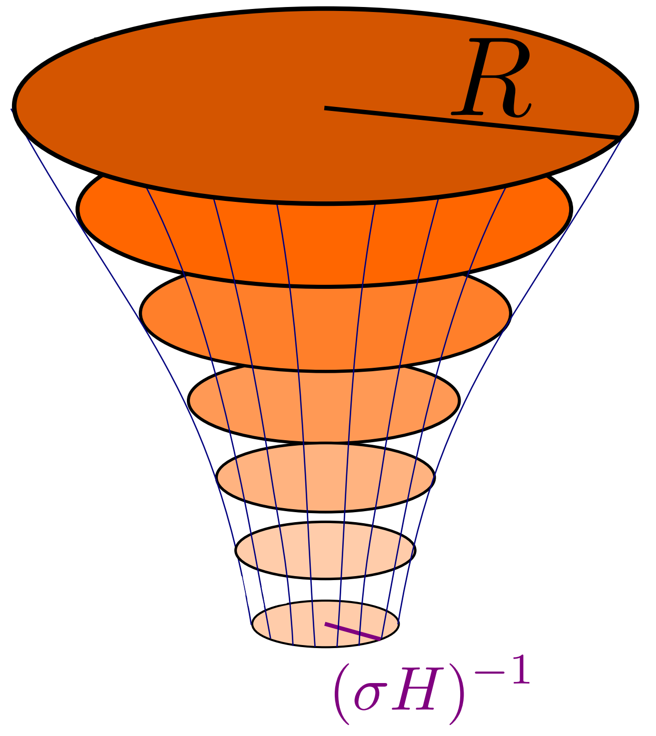
$$P(\zeta_{R_1}, \zeta_{R_2}) = \int \mathrm{d}\mathcal{N}_{\phi_0 \to \phi_*} (\mathcal{N}_{\phi_0 \to \phi_*}) P^V_{\mathrm{FPT},\phi_* \to \phi_1} \left(\zeta_{R_1} - \mathcal{N}_{\phi_0 \to \phi_*} + \langle \mathcal{N}_{\phi_0} \rangle_V - \langle \mathcal{N}_{\phi_1} \rangle_V \right) P^V_{\mathrm{FPT},\phi_* \to \phi_2} \left(\zeta_{R_2} - \mathcal{N}_{\phi_0 \to \phi_*} + \langle \mathcal{N}_{\phi_0} \rangle_V - \langle \mathcal{N}_{\phi_2} \rangle_V \right)$$

Large-volume approximation



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Ensemble average over the set of final leaves ⟶ Stochastic average of a single element within the ensemble

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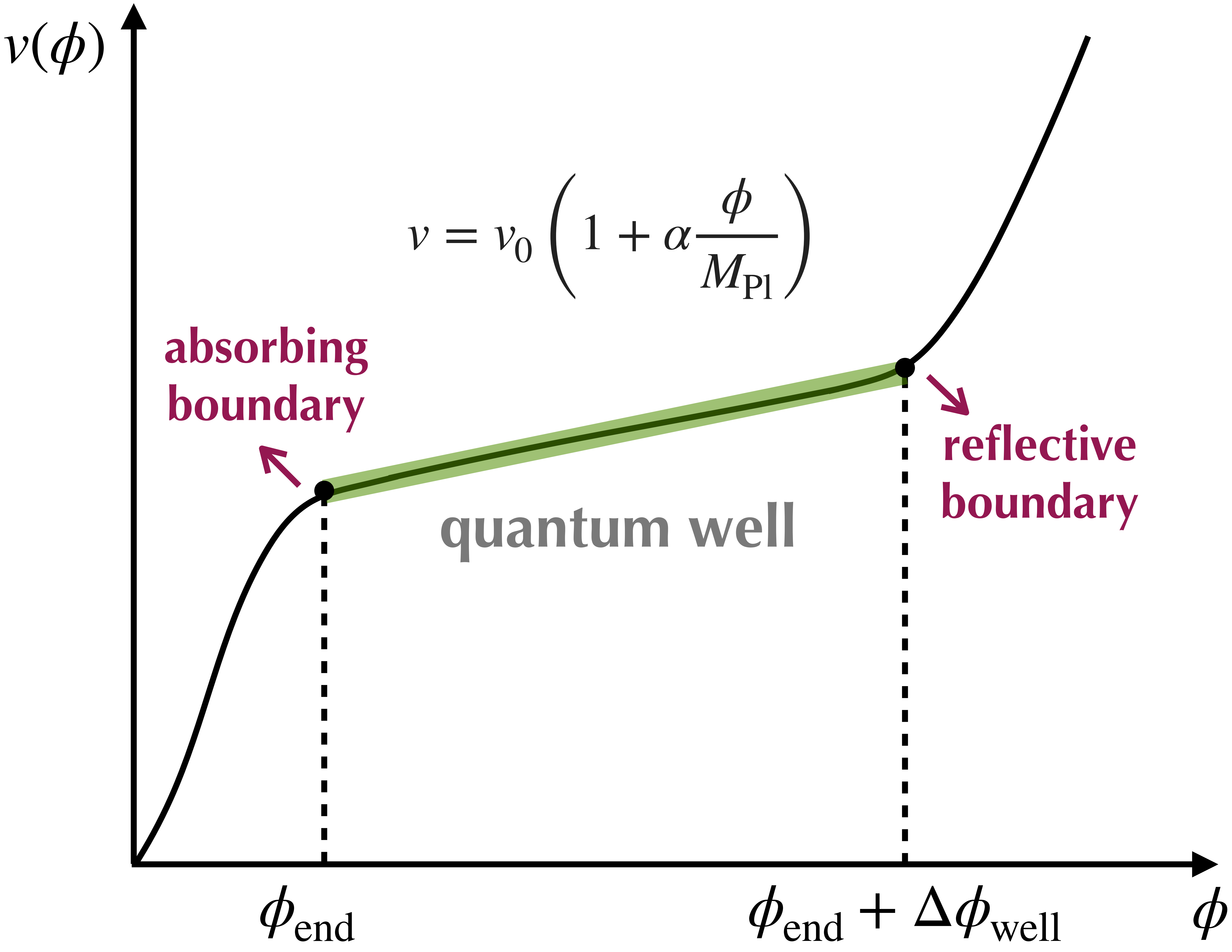
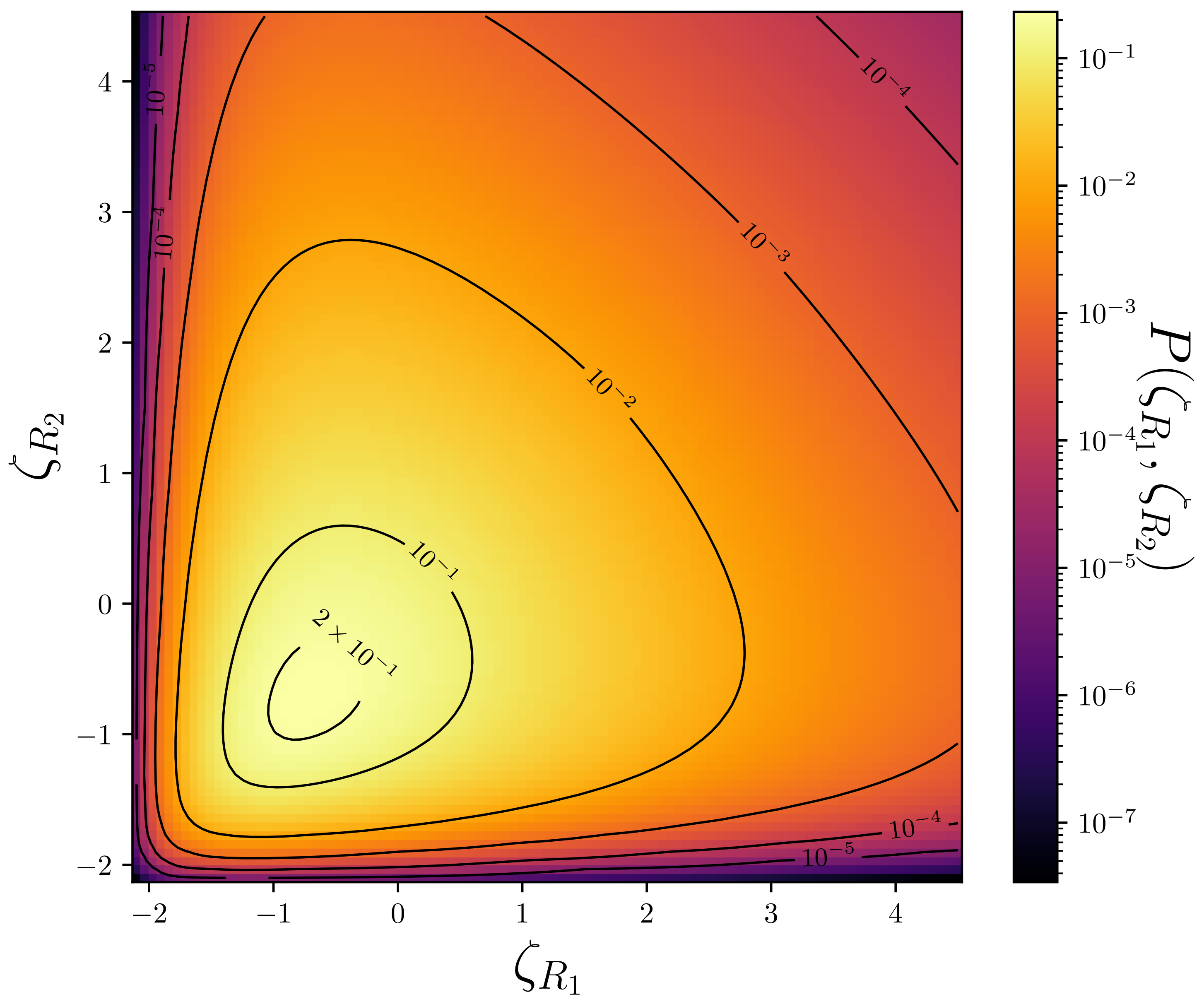
Applications: quantum well

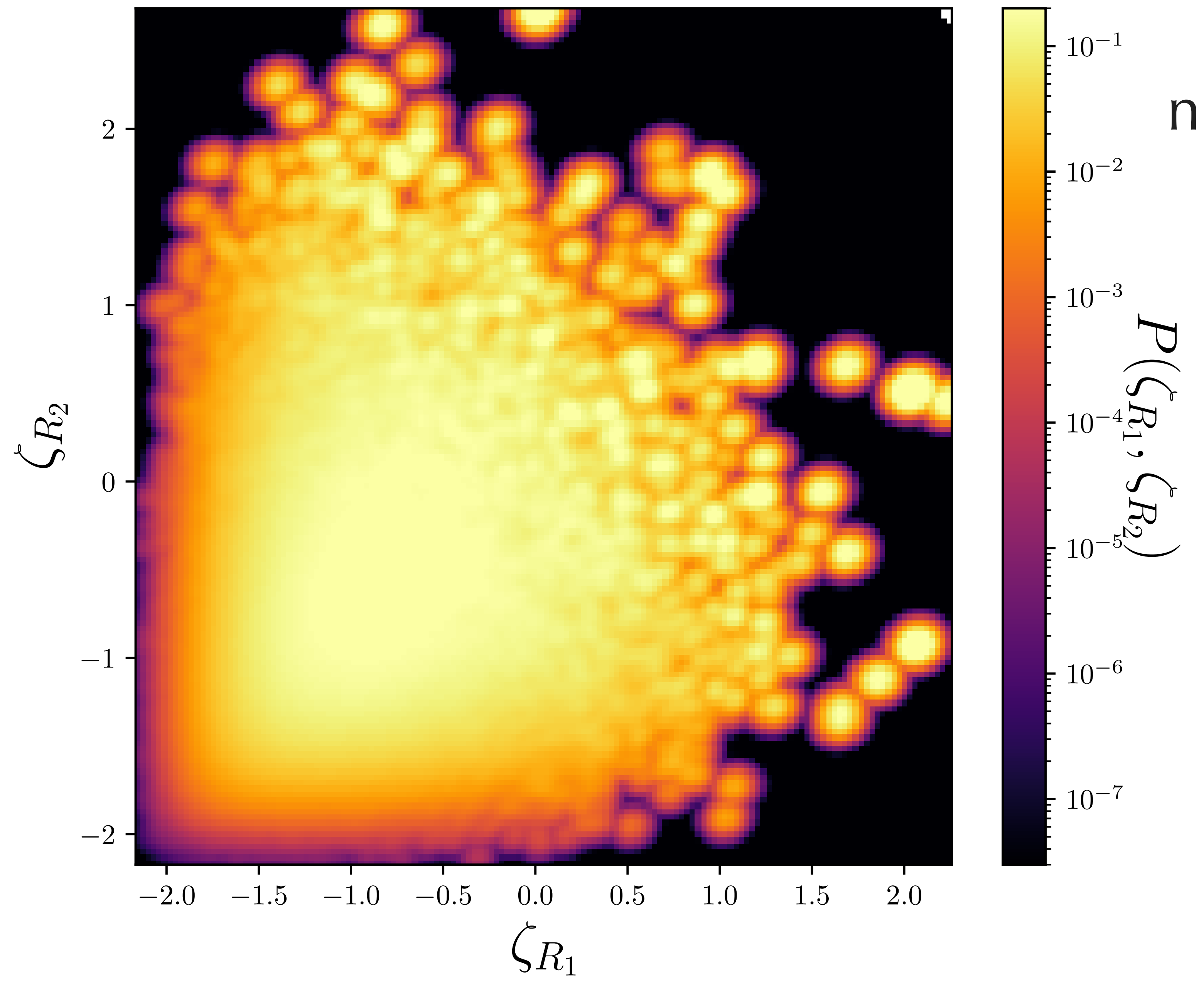


Two-point distributions: tilted-well model

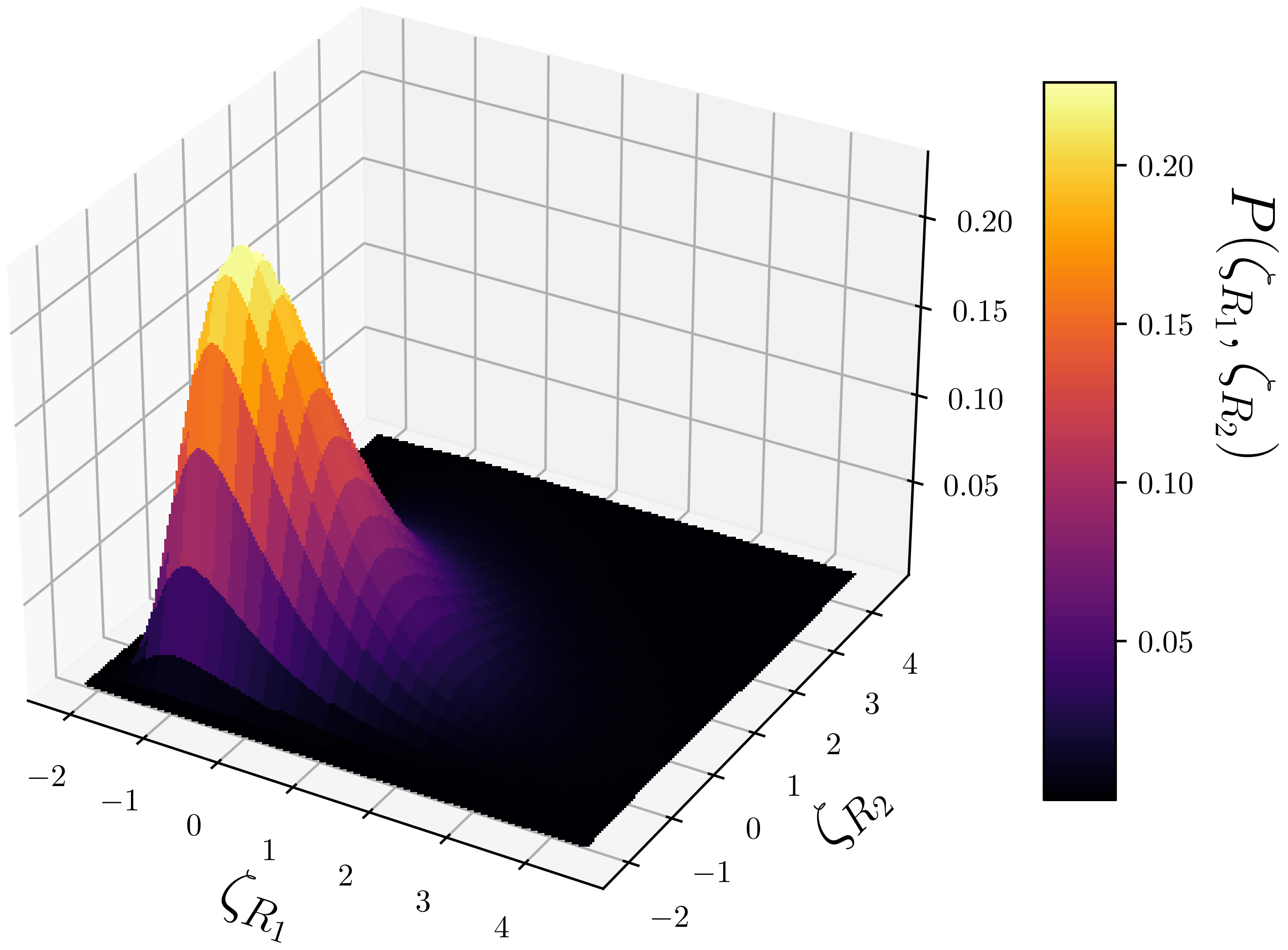
$\alpha \Delta\phi_{\mathrm{well}}/(v_0 M_{\mathrm{Pl}})$
$\equiv d\mu^2 \to \simeq 51$

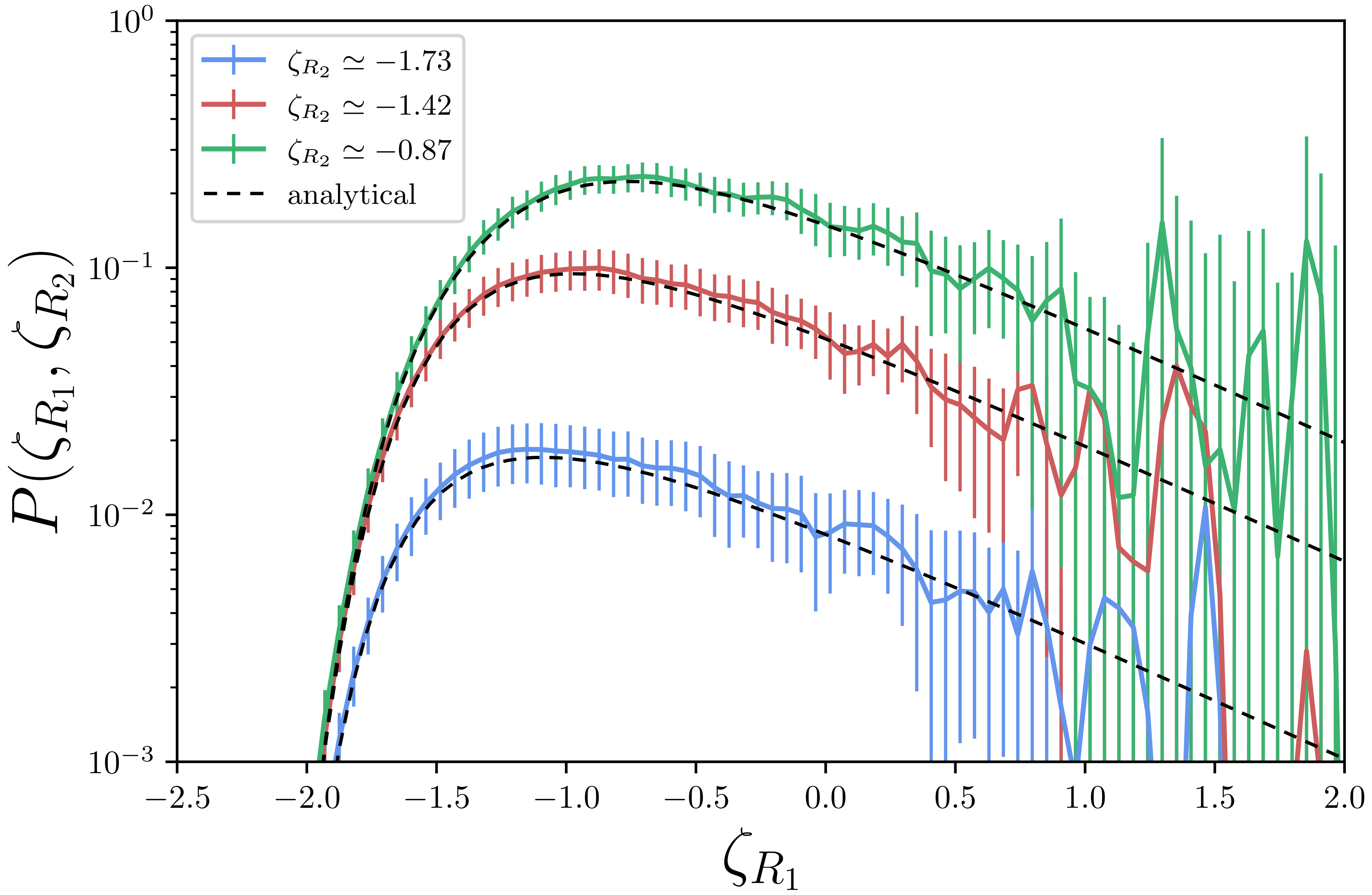


analytical approx. results



numerical simulations





Two-point distributions & clustering: tilted-well model

$$P(\zeta_{R_1}, \zeta_{R_2}) = P(\zeta_{R_1})\, P(\zeta_{R_1}) \frac{a_V(x_*, x_1)}{a_V(x_0, x_1)} \frac{a_V(x_*, x_2)}{a_V(x_0, x_2)} \int \mathrm{d}\mathcal{N}\, P^V_{\mathrm{FPT}, x_0 \to x_*}(\mathcal{N}_{x_0 \to x_*}) e^{\left[\frac{\mu^2 d^2}{2} + \frac{\pi^2}{\mu^2 (1 - x_1)^2} + \frac{\pi^2}{\mu^2 (1 - x_2)^2} - 6\right] \mathcal{N}_{x_0 \to x_*}}$$

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$$P^V_{\mathrm{FPT},x}(\mathcal{N}) \simeq a_V(x) e^{-(\Lambda_0 + 3)\mathcal{N}}$$

$$a_V(x) = a_0(x) / \langle e^{3\mathcal{N}_x} \rangle$$

lowest residue of the characteristic function

$$\Lambda_0 \simeq \mu^2 d^2/4 + \pi^2/\mu^2$$

lowest pole of the characteristic function

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Independent on the threshold of formation

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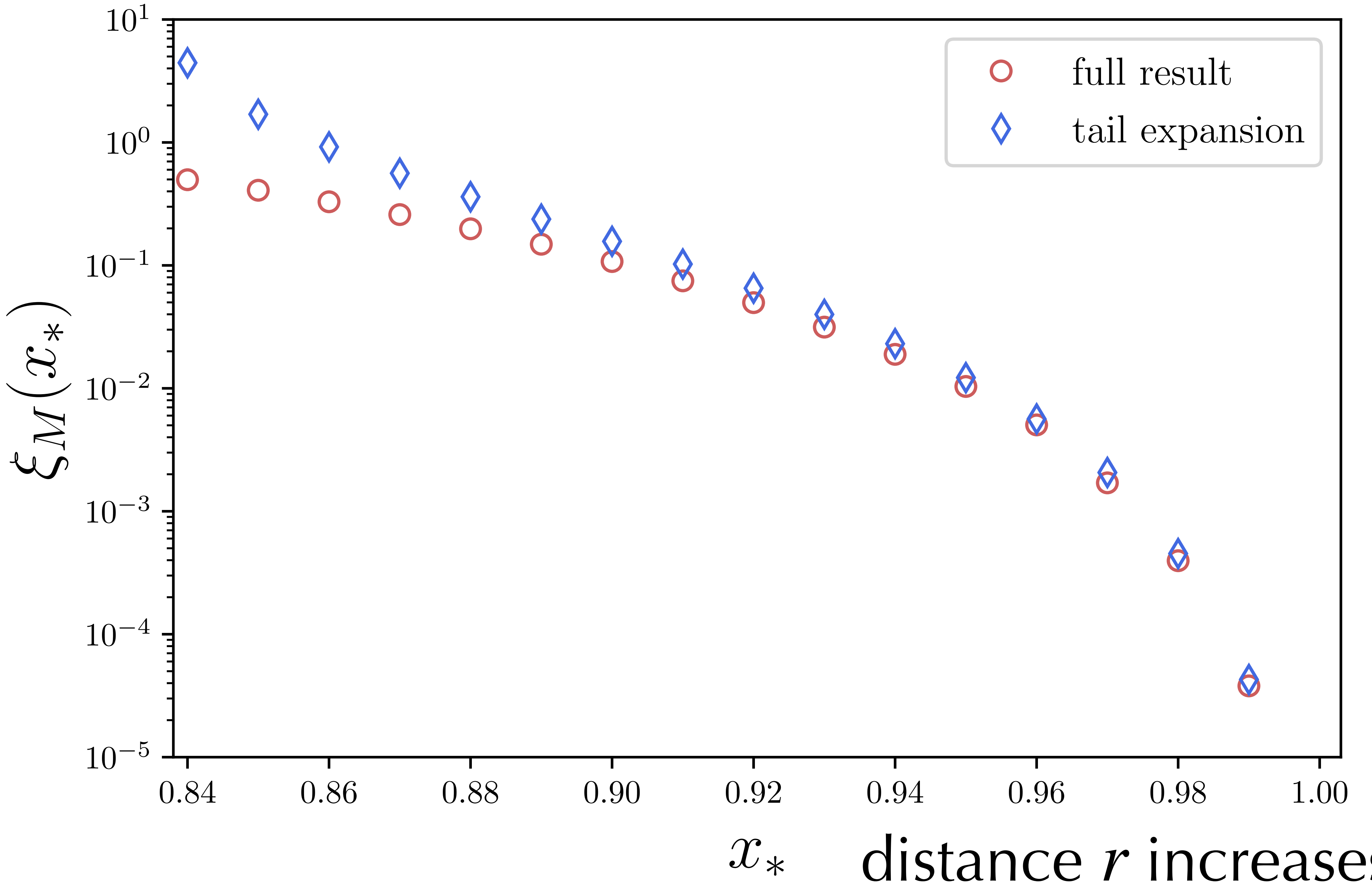
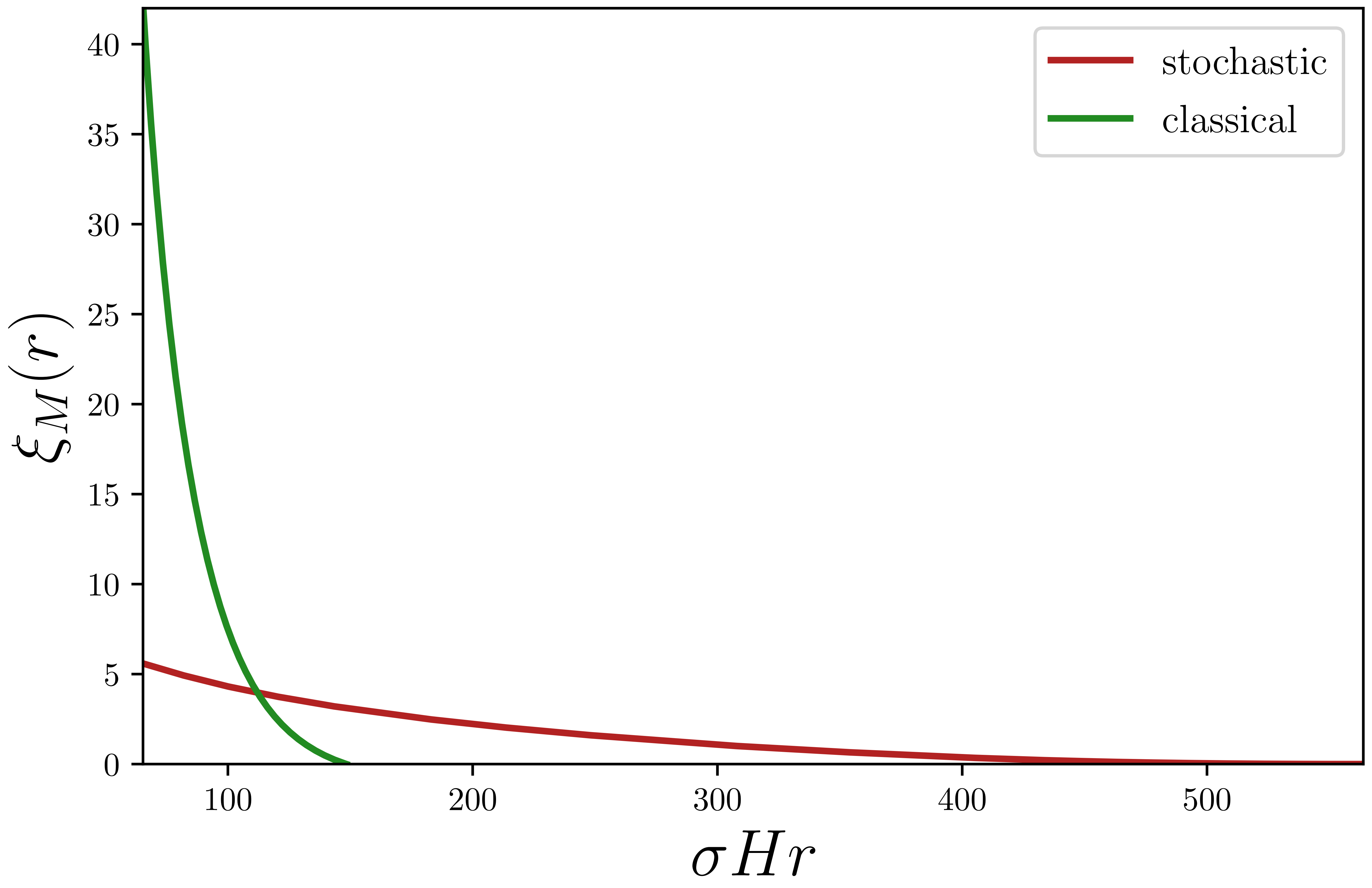


$d\mu^2 \simeq 51$

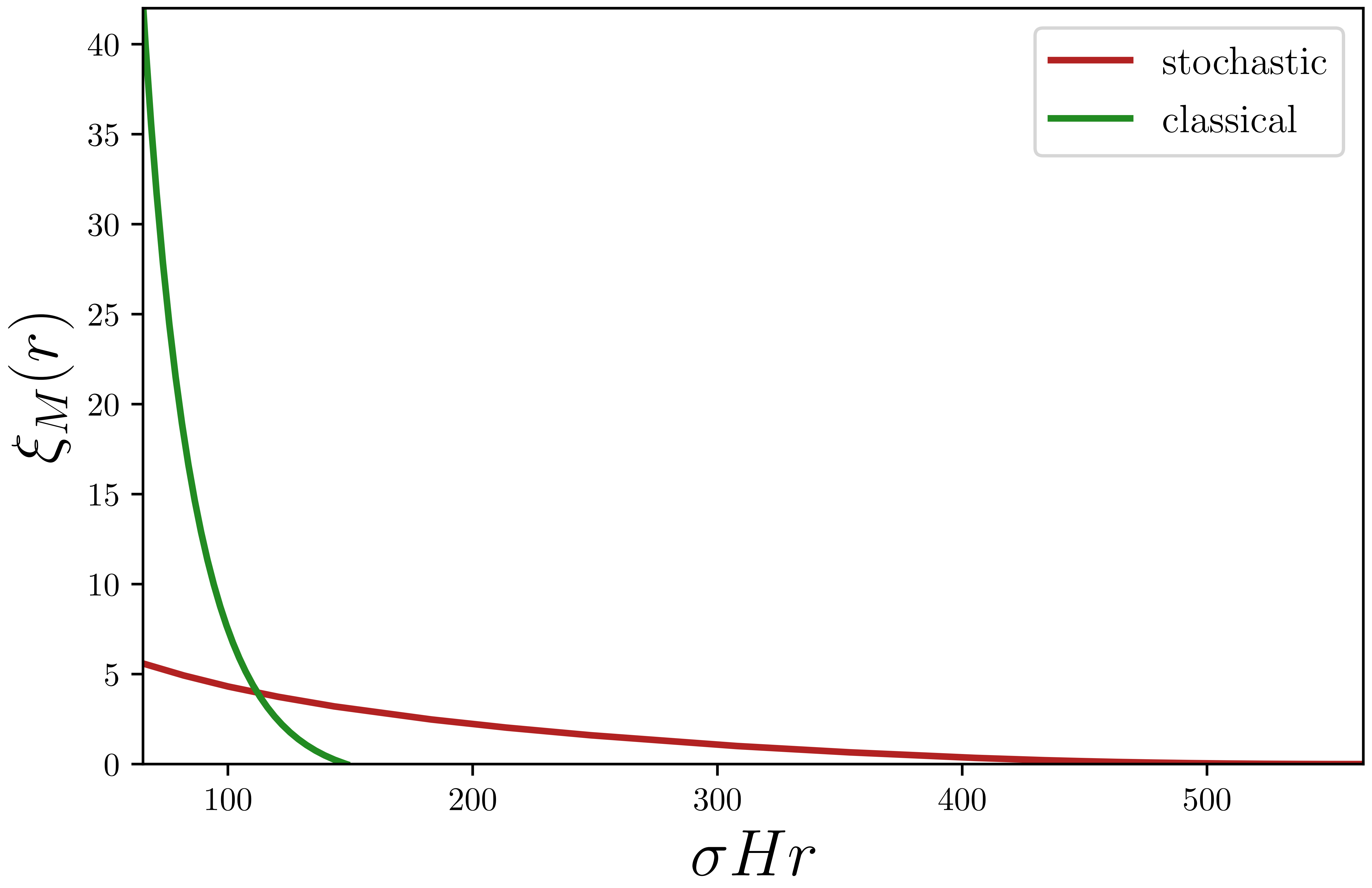
distance r increases

Clustering: comparison with the classical limit

Clustering: comparison with the classical limit



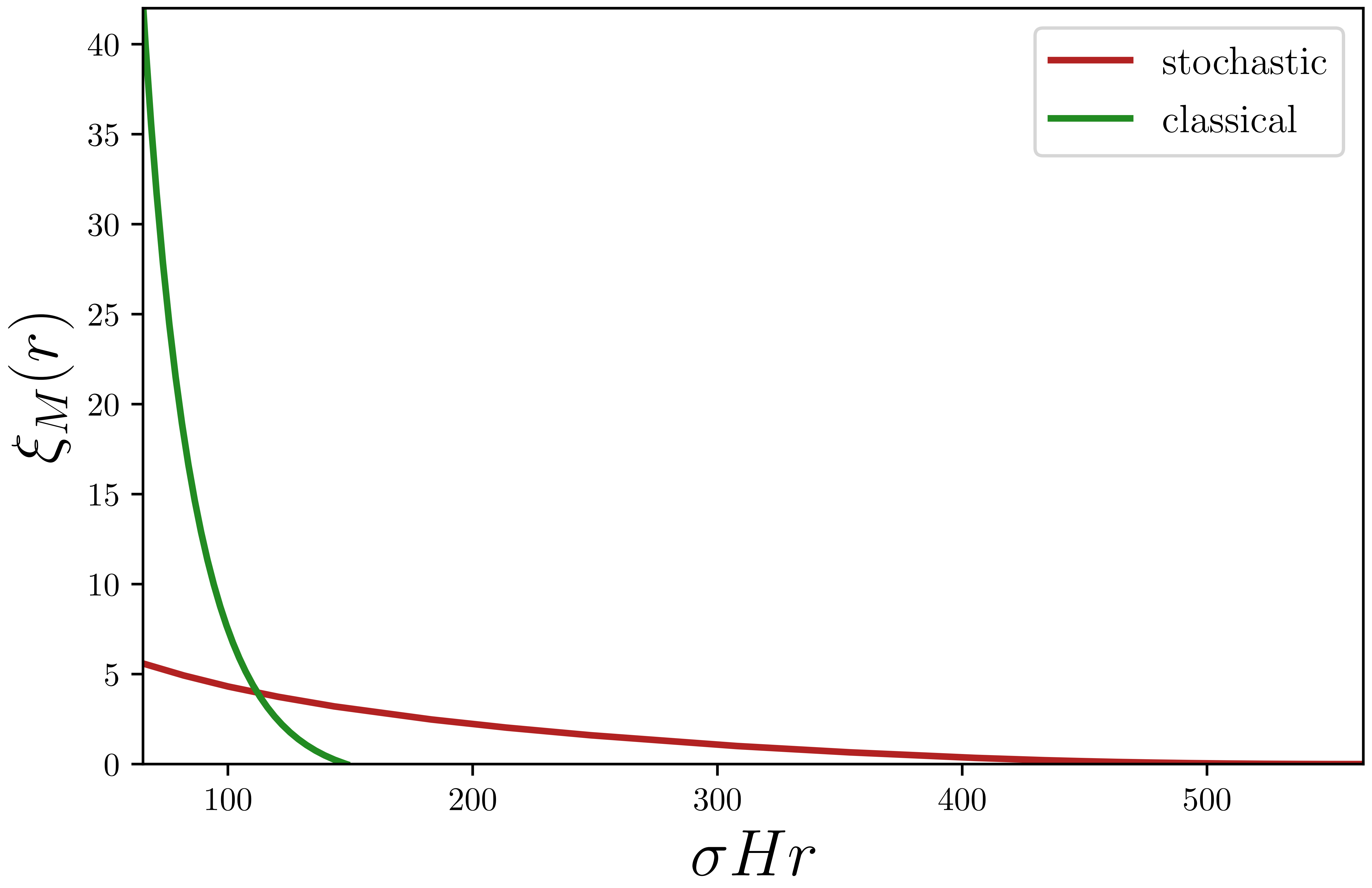
Clustering: comparison with the classical limit



→ **Larger distances** r are covered in the stochastic calculation than in its classical counterpart

different relation between scales and field values: $r_{\max}^{\text{class}} = e^{1/d}$ versus $\tilde{r}_{\max}^{\text{stoch}} = 2\langle e^{3\mathcal{N}}\rangle_{x=1}^{1/3}$

Clustering: comparison with the classical limit



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- PBHs are correlated over longer distances once quantum diffusion is taken into account

Final remarks

- **Physical distances** (measured by a local observer on the end-of-inflation hypersurface) and patches during inflation linked by the **emerging volume**.
- Different regions inflate by different amount: statistics are **volume weighted**.
- PBHs can be created with spatial correlation across **longer distances** if quantum diffusion is included.
- On the tail, the reduced correlation does not depend of the threshold of formation: **universal clustering profile**.

Next?

- Two-point distribution of the compaction function.
- Numerical approaches (recursive sampling algorithm).
- Phenomenological consequences, more realistic scenarios…

Final remarks

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- Numerical approaches (recursive sampling algorithm).

- Phenomenological consequences, more realistic scenarios…

Thanks for the attention!

Backup slides

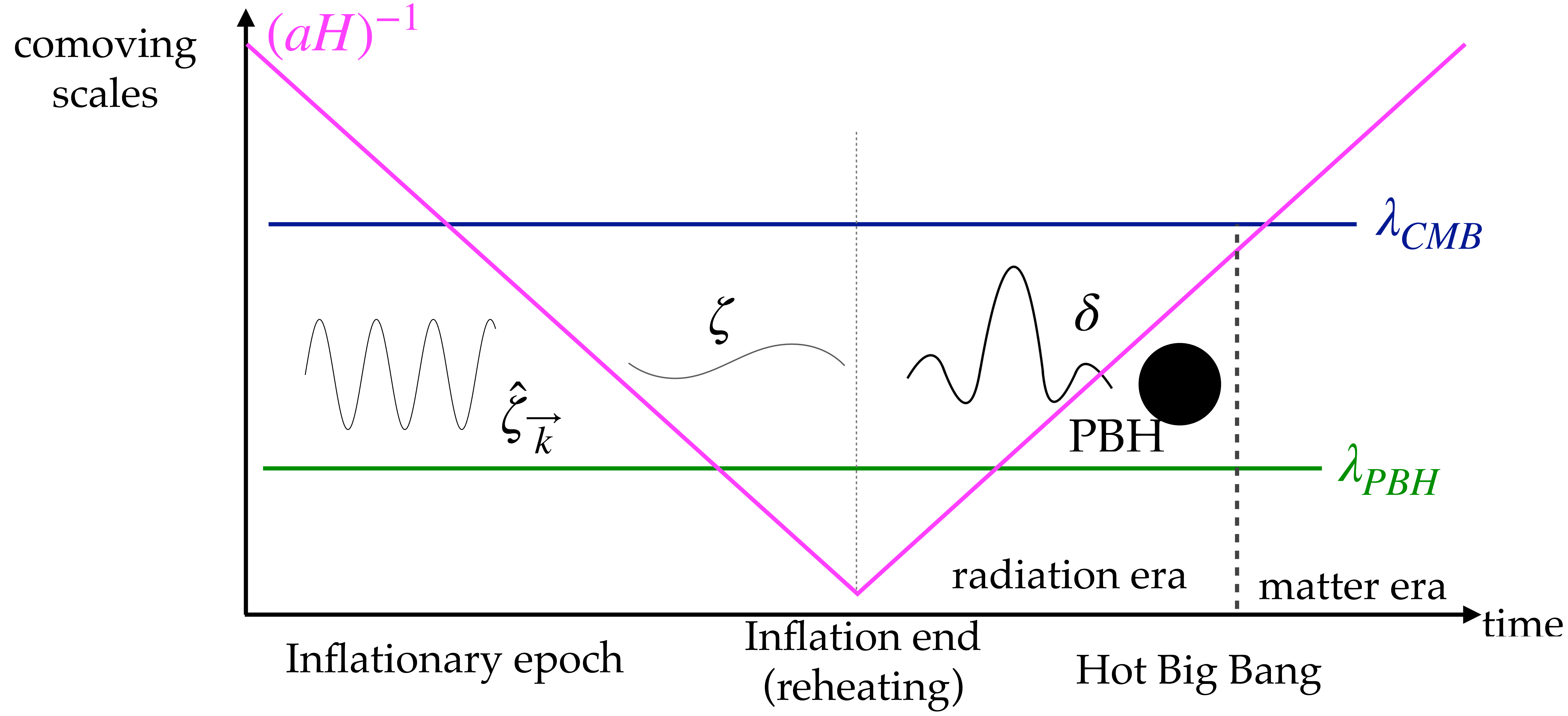
Primordial Black Holes (PBHs) from inflation

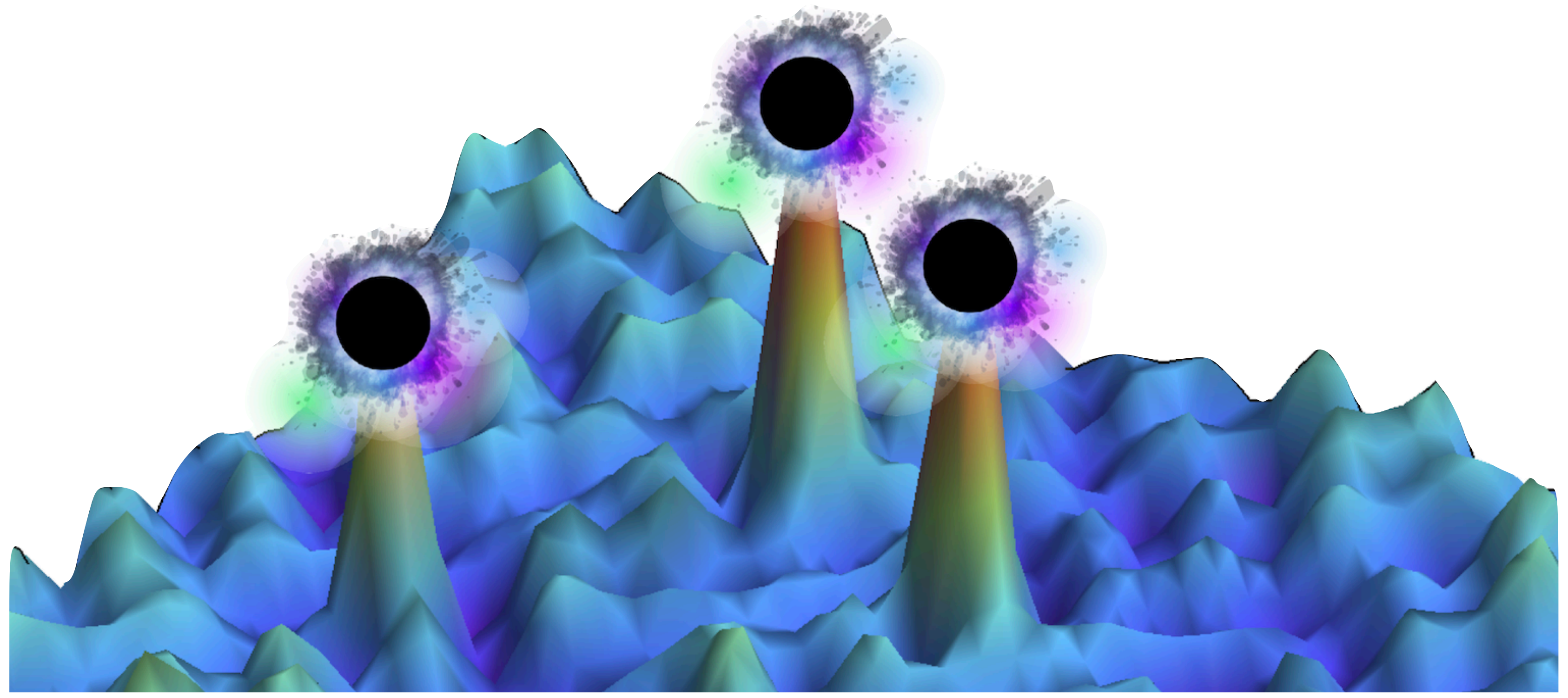
Zel'dovich & Novikov [1967]
Hawking [1971]
Carr & Hawking [1974]

Black holes which could have formed in the early Universe through a non-stellar way

B. Carr, A. Green,
The History of Primordial Black Holes
[2024]

PBHs may originate from peaks of the density perturbations generated in the early universe





$$\delta \sim \left.\frac{\delta\rho}{\rho}\right|_{k=aH} \sim \zeta > \zeta_c \sim \mathcal{O}(1)$$

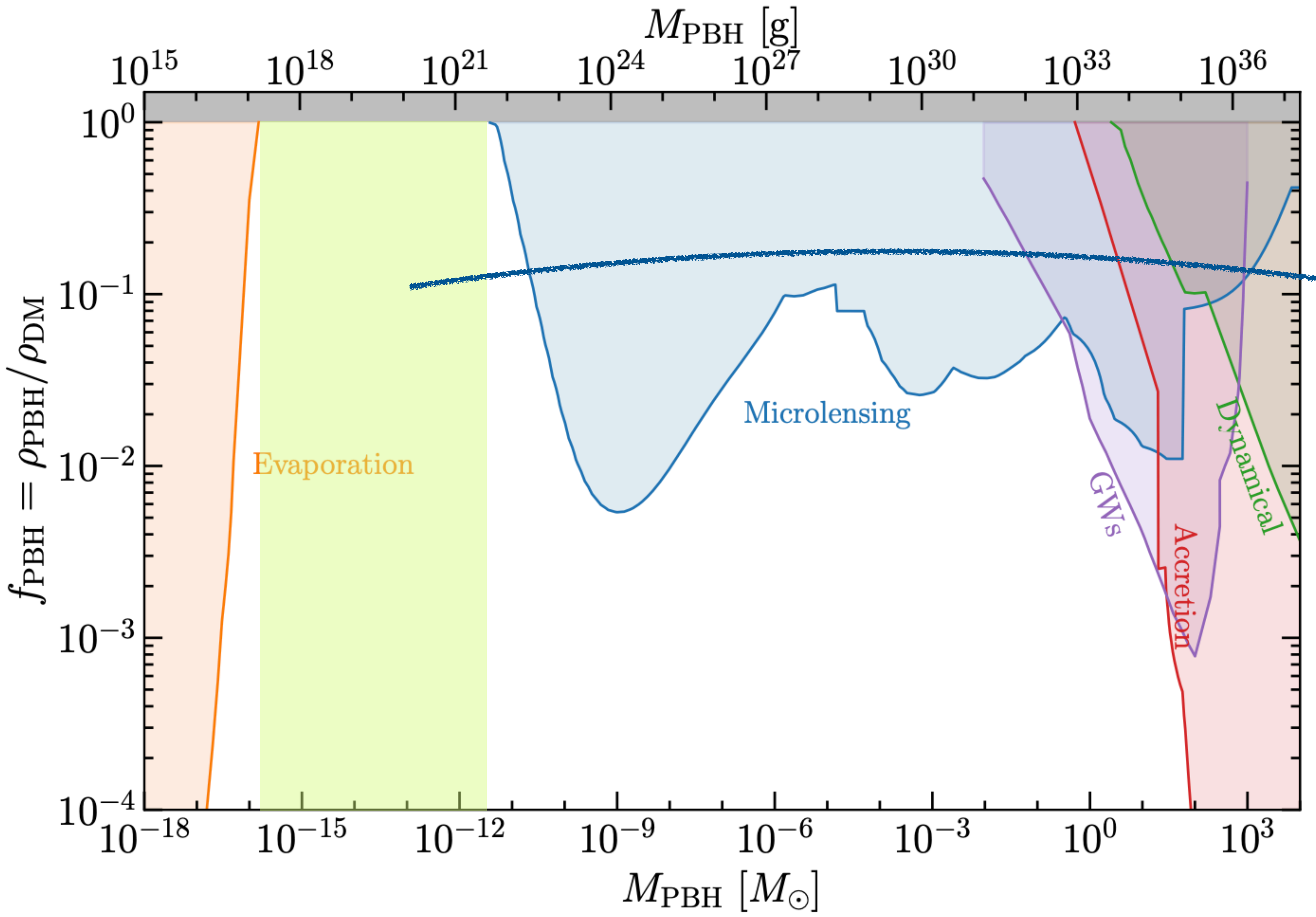
Primordial black holes as dark matter candidates

PBHs are good candidates of dark matter: stable, non-baryonic, cold, could be formed in the right abundance to be the dark matter

PBHs evaporate emitting Hawking radiation but they are stable if the initial mass $M_{\rm in} \gtrsim 10^{15}$g

Not a new particle, but they require some physics beyond the standard model (e.g. inflation)

Current constraints

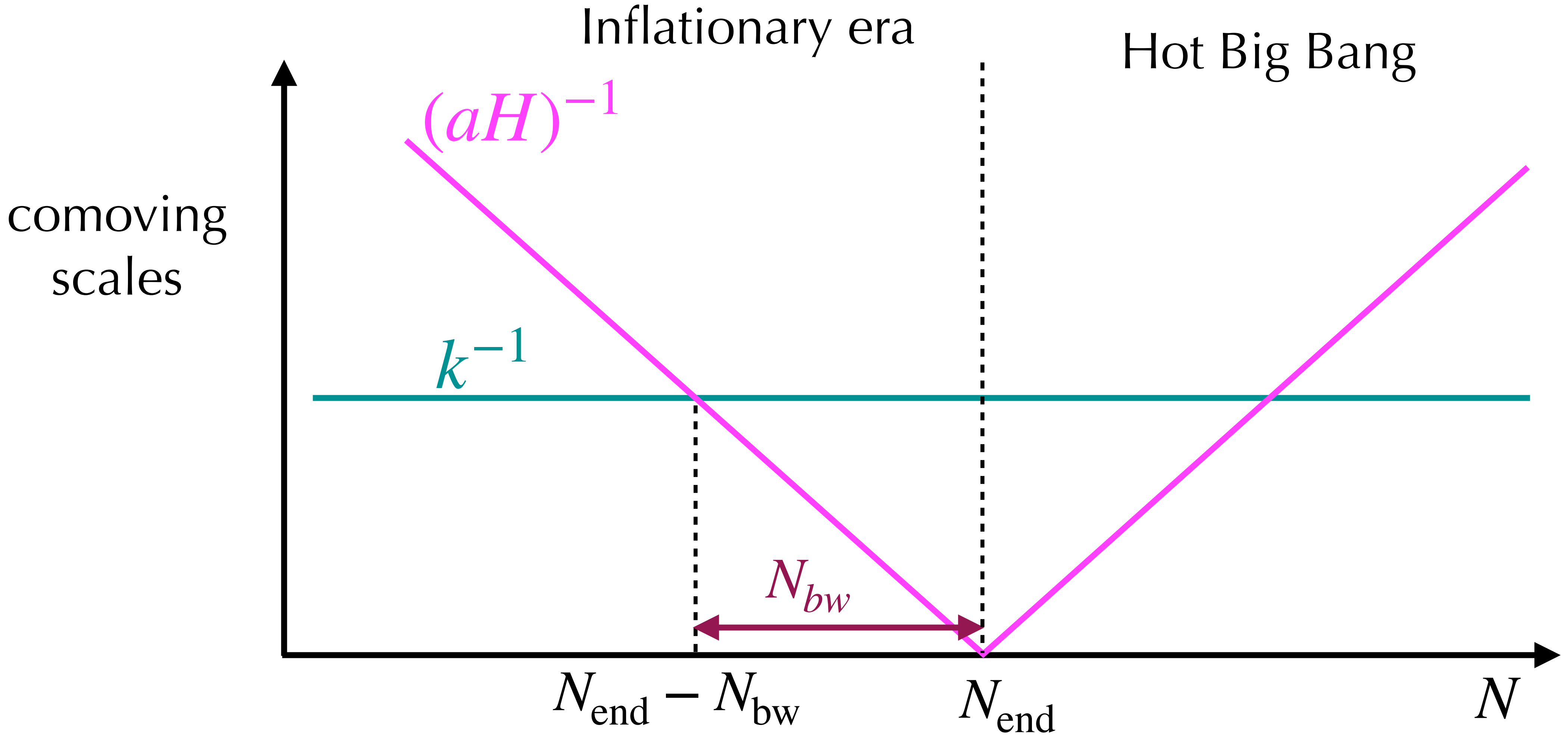


Asteroid mass range: $10^{17}\text{g} \lesssim M \lesssim 10^{22}\text{g}$

PBHs could be the totality of DM, i.e., $f_{\rm PBH} = 1$

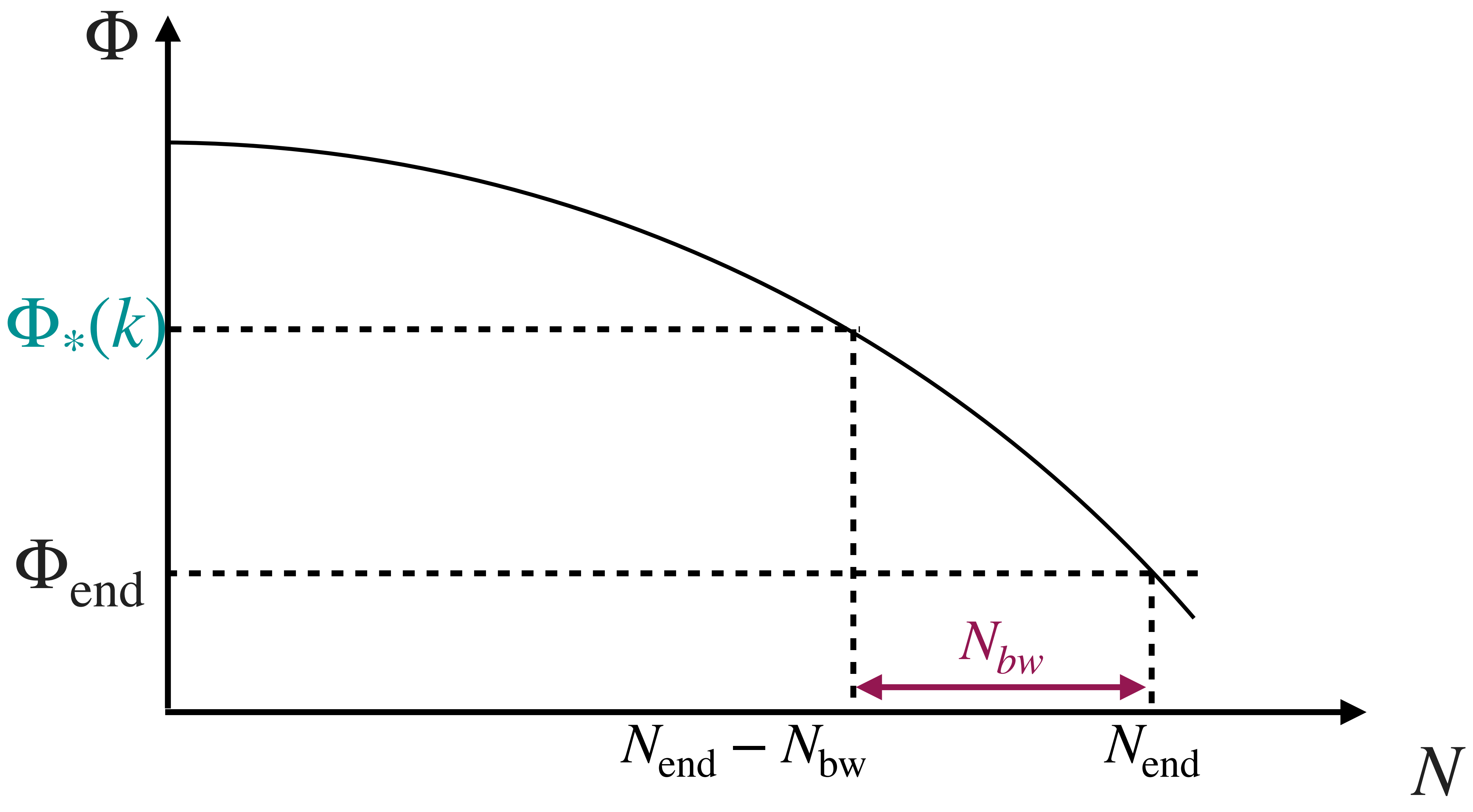
From A. Green 2024, Primordial black holes as a dark matter candidate - a brief overview

Extracting cosmological observables



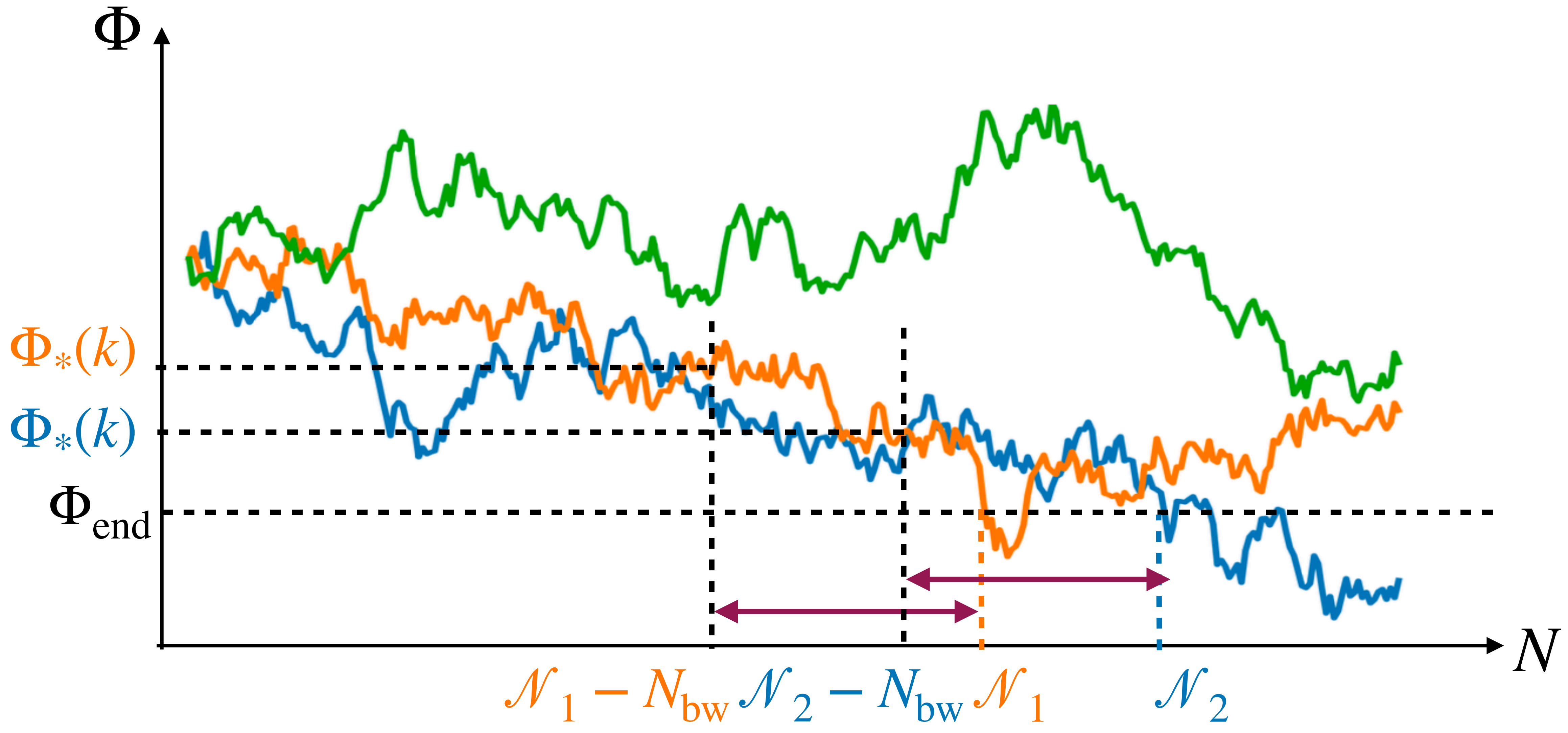
Scale k crosses the Hubble radius at
$N_* = N_{\rm end} - N_{\rm bw} = N_{\rm end} - \log(a_{\rm end} H/k)$

classical problem



one-to-one correspondence between k and $\Phi_*(k)$

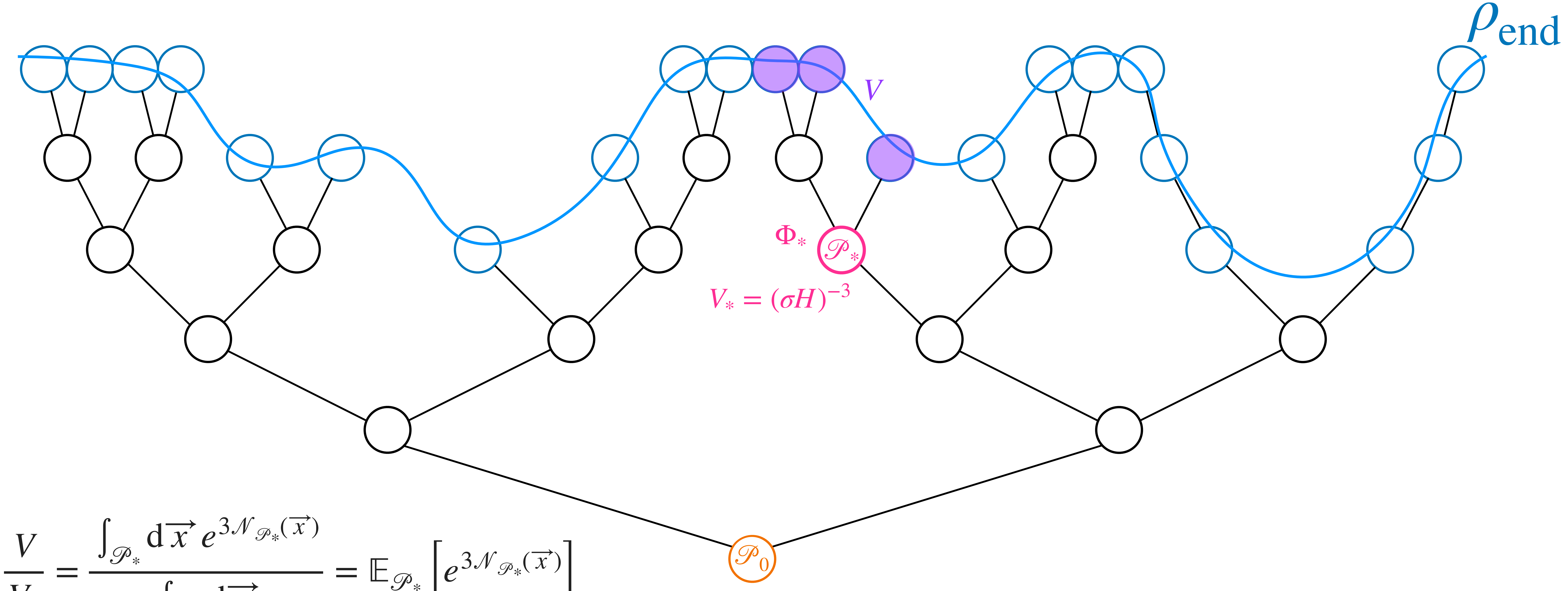
stochastic problem



$\Phi_*(k)$ is a random quantity endowed with a backward distribution

Extracting cosmological observables

Relation between field values and physical distances encoded in the structure of a universe which inflates stochastically



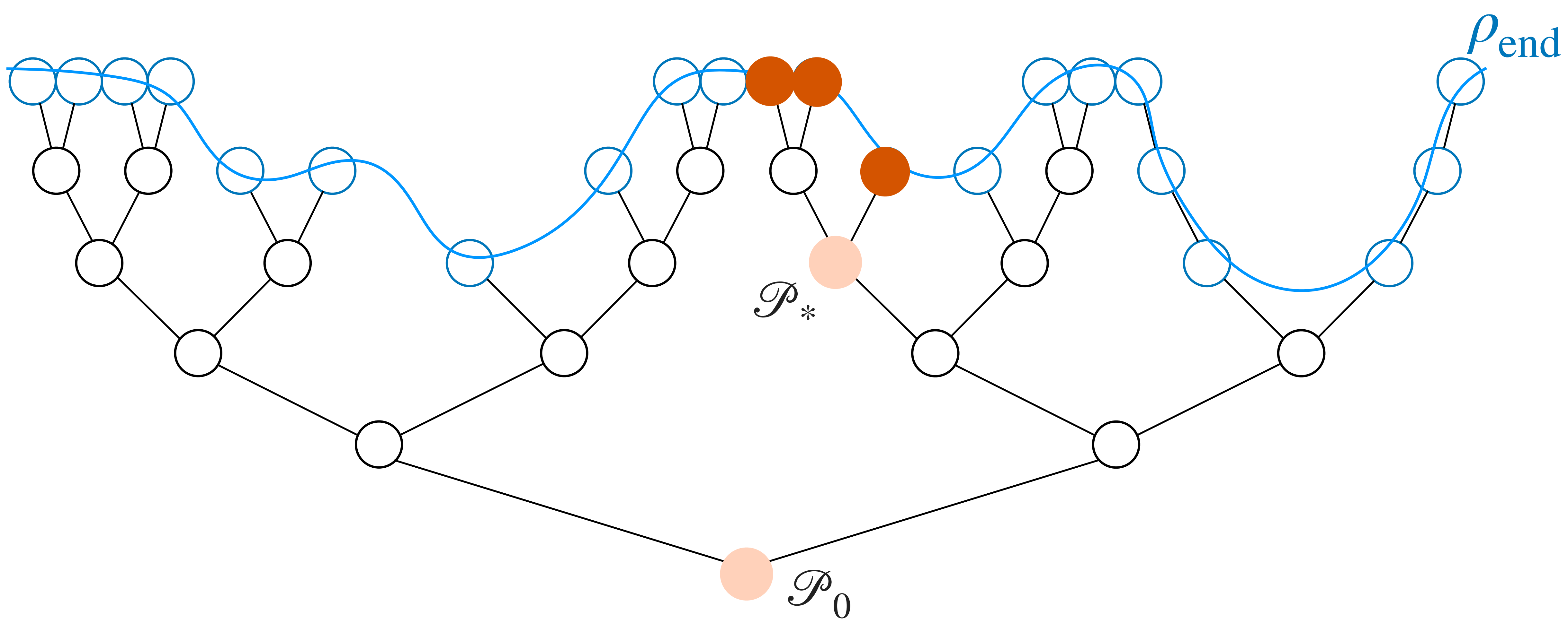
Final volume: $\frac{V}{V_*} = \frac{\int_{\mathscr{P}_*} \mathrm{d}\vec{x}\, e^{3\mathcal{N}_{\mathscr{P}_*}(\vec{x})}}{\int_{\mathscr{P}_*} \mathrm{d}\vec{x}} = \mathbb{E}_{\mathscr{P}_*}\left[e^{3\mathcal{N}_{\mathscr{P}_*}(\vec{x})}\right]$

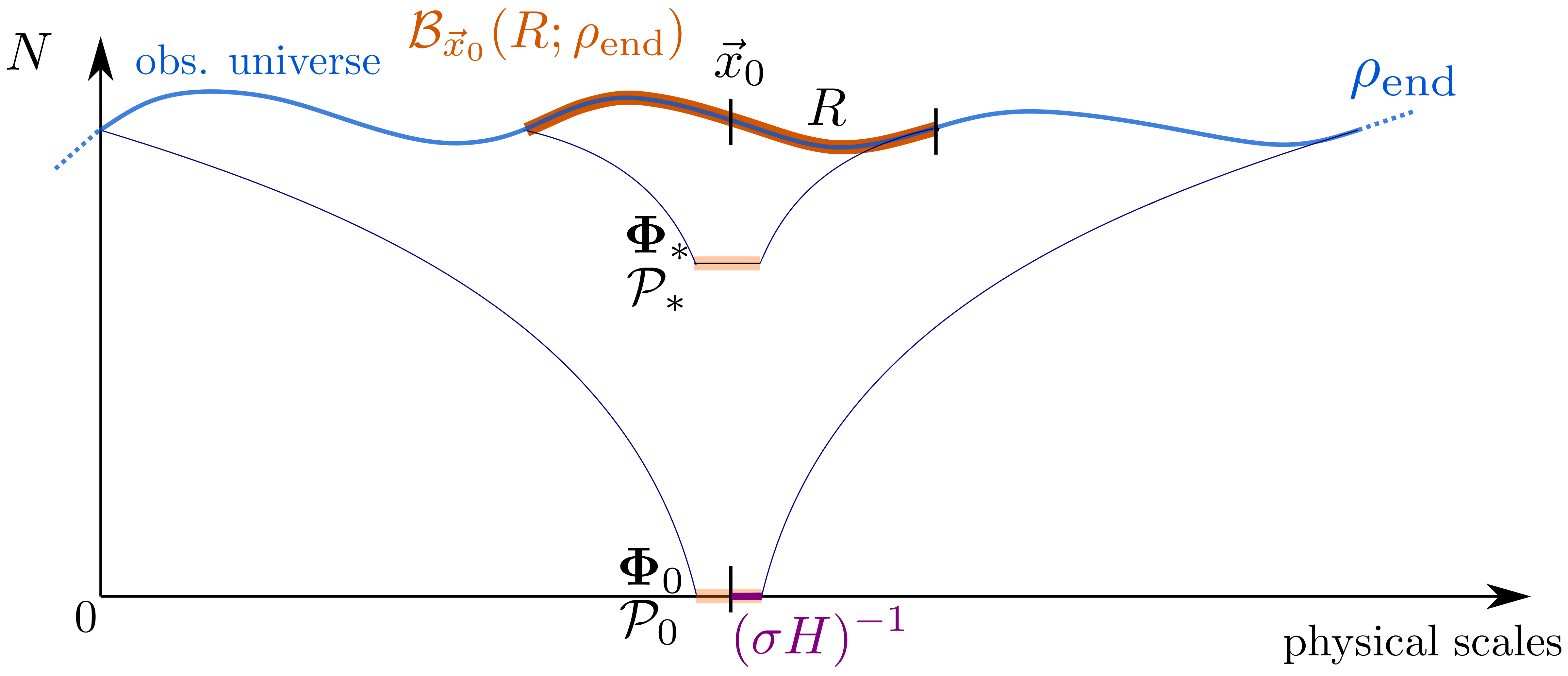
Volume-averaged number of e-folds: $W \equiv \mathbb{E}^{V}_{\mathscr{P}_*}\left[\mathcal{N}_{\mathscr{P}_*}(\vec{x})\right] = \frac{\int_{\mathscr{P}_*} e^{3\mathcal{N}_{\mathscr{P}_*}(\vec{x})}\mathcal{N}_{\mathscr{P}_*}(\vec{x})\mathrm{d}\vec{x}}{\int_{\mathscr{P}_*} e^{3\mathcal{N}_{\mathscr{P}_*}(\vec{x})}\mathrm{d}\vec{x}} = \frac{V_*}{V}\mathbb{E}_{\mathscr{P}_*}\left[e^{3\mathcal{N}_{\mathscr{P}_*}(\vec{x})}\mathcal{N}_{\mathscr{P}_*}(\vec{x})\right]$

Distributions $P(V\,|\,\Phi_*)$ and $P(V, W\,|\,\Phi_*)$ can be numerically sampled

Backward distribution: $P(\Phi_*\,|\,V, \Phi_0) = \frac{P(V\,|\,\Phi_*)P(\Phi_*\,|\,\Phi_0)}{P(V)} = \frac{P(V\,|\,\Phi_*)P(\Phi_*\,|\,\Phi_0)}{\int \mathrm{d}\Phi_*\, P(V\,|\,\Phi_*)P(\Phi_*\,|\,\Phi_0)}$

Stochastic-δN formalism: coarse-graining at arbitrary scale





$$\zeta_{\mathrm{cg},R}(\vec{x}_0) \equiv \zeta_R(\vec{x}_0) = \mathbb{E}^V_{\mathscr{P}_*}[\zeta_{\mathrm{cg}}(\vec{x})] = \mathbb{E}^V_{\mathscr{P}_*}[\mathscr{N}_{\mathscr{P}_0}(\vec{x})] - \mathbb{E}^V_{\mathscr{P}_0}[\mathscr{N}_{\mathscr{P}_0}(\vec{x})]$$

$$\mathscr{N}_{\mathscr{P}_0}(\vec{x}) = \underline{\mathscr{N}_{\mathscr{P}_0\to\mathscr{P}_*}(\vec{x})} + \mathscr{N}_{\mathscr{P}_*}(\vec{x})$$

Shared history

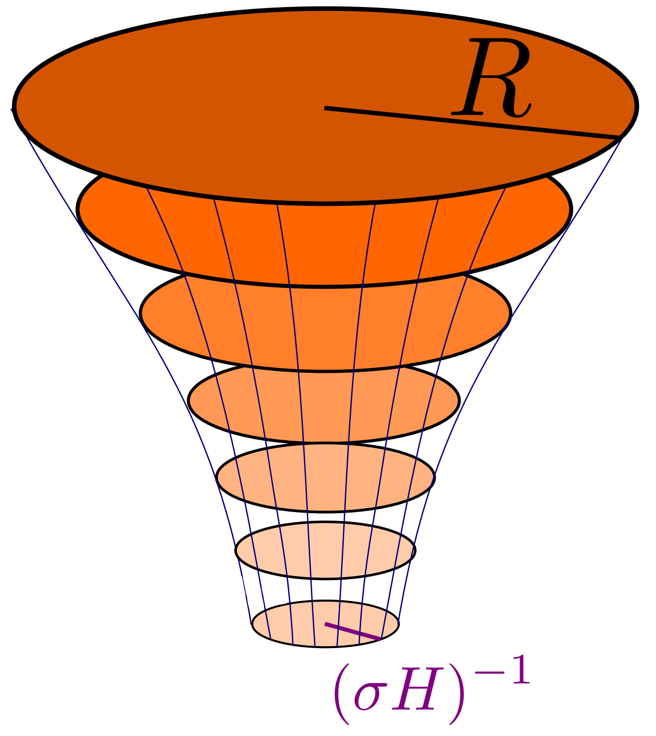
$$\zeta_{\mathrm{cg},R}(\vec{x}_0) \equiv \zeta_R(\vec{x}_0) = \mathscr{N}_{\mathscr{P}_0\to\mathscr{P}_*}(\vec{x}_0) + W(\mathscr{P}_*) - \mathbb{E}^V_{\mathscr{P}_0}[\mathscr{N}_{\mathscr{P}_0}(\vec{x})]$$

Solutions of Fokker-Planck, adjoint Fokker-Planck eqs., etc

$$P^V(\mathscr{N}_{\mathscr{P}_0\to\mathscr{P}_*}, W \mid V, \Phi_0) = \int \mathrm{d}\Phi_* P^V(\mathscr{N}_{\mathscr{P}_0\to\mathscr{P}_*}) P^V_{\mathrm{FP}}(\Phi_*, \mathscr{N}_{\mathscr{P}_0\to\mathscr{P}_*} \mid \Phi_0) \frac{P(V, W \mid \Phi_*)}{P(V)}$$

Can be numerically sampled

Large-volume approximation



$$R^3 \gg (\sigma H)^{-3}$$

Ensemble average over the set of final leaves ⟶ Stochastic average of a single element within the ensemble

$$V \to \langle V \rangle \qquad P(V \,|\, \Phi_*) \simeq \delta_\mathrm{D}(V - V_* \langle e^{3\mathcal{N}_{\Phi_*}} \rangle) \qquad \langle e^{3\mathcal{N}_{\Phi_*}} \rangle = \int_0^\infty P_{\mathrm{FPT},\Phi_*}(\mathcal{N}) e^{3\mathcal{N}} \mathrm{d}\mathcal{N}$$

$$W \to \langle W \rangle \qquad W \simeq \langle \mathcal{N}_{\Phi_*} \rangle_V = \frac{\langle \mathcal{N}_{\Phi_*} e^{3\mathcal{N}_{\Phi_*}} \rangle}{\langle e^{3\mathcal{N}_{\Phi_*}} \rangle}$$

$$\zeta_R(\vec{x}_0) = \mathcal{N}_{\mathscr{P}_0 \to \mathscr{P}_*}(\vec{x}_0) + W(\mathscr{P}_*) - \mathbb{E}^V_{\mathscr{P}_0}[\mathcal{N}_{\mathscr{P}_0}(\vec{x})] \longrightarrow \zeta_R \simeq \mathcal{N}_{\mathscr{P}_0 \to \mathcal{S}_*} + \langle \mathcal{N}_{\Phi_*} \rangle_V - \langle \mathcal{N}_{\Phi_0} \rangle_V$$

$$P(\zeta_R \,|\, \Phi_0) = \int_{\mathcal{S}_*} \mathrm{d}\Phi_* P^V_{\mathrm{FPTL},\Phi_0 \to \mathcal{S}_*}(\mathcal{N}_{\mathscr{P}_0 \to \mathcal{S}_*} = \zeta_R - \langle \mathcal{N}_{\Phi_*} \rangle_V + \langle \mathcal{N}_{\Phi_0} \rangle_V, \Phi_* \,|\, \Phi_0)$$

$\mathcal{S}_*$: *hypersurface of constant mean forward volume* $\langle e^{3\mathcal{N}_{\Phi_*}} \rangle = R^3$

first-passage time and location distribution

$$P^V_{\mathrm{FPTL},\Phi_0 \to \mathcal{S}_*}(\mathcal{N}_{\Phi_0 \to \mathcal{S}_*}, \Phi_* \,|\, \Phi_0) = P^V_{\mathrm{FPT},\Phi_0 \to \mathcal{S}_*}(\mathcal{N}_{\Phi_0 \to \mathcal{S}_*}) P(\Phi_* \,|\, \mathcal{N}_{\Phi_0 \to \mathcal{S}_*})$$

Two-point statistics of the coarse-grained curvature perturbation

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How the curvature perturbations coarse grained at two different locations are **correlated**?

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The distance between two patches is encoded in the time at which they become statistically independent K. Ando, V. Vennin [2021]

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Extension to **multiple-point statistics**

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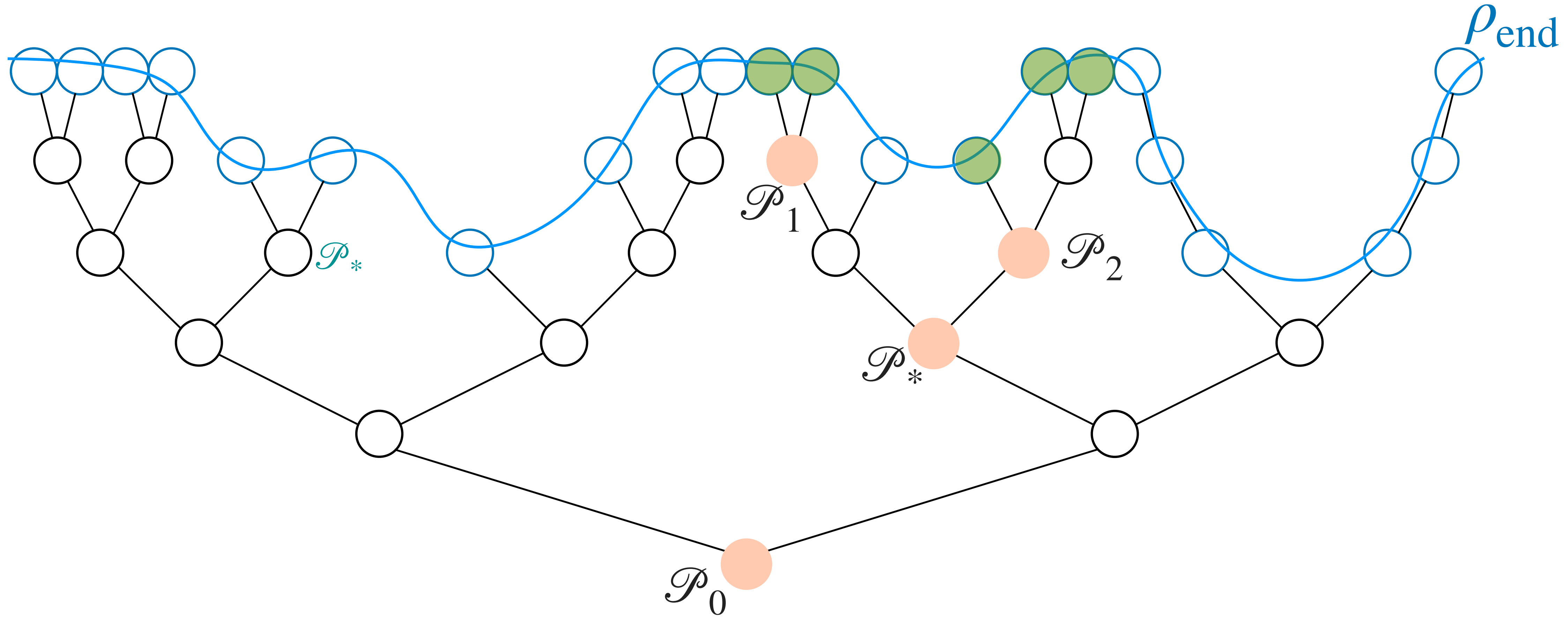
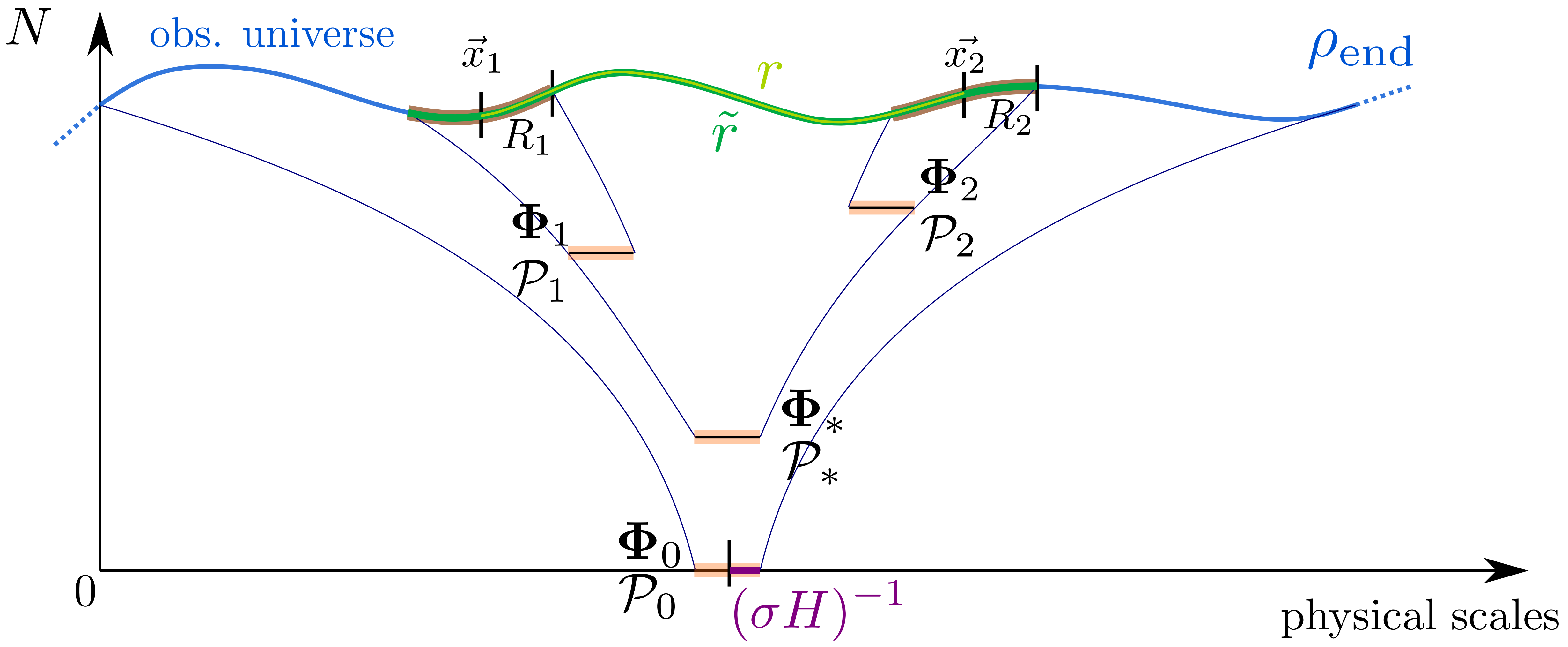
Extension to **multiple-point statistics**





Two-point statistics of the coarse-grained curvature perturbation

How the curvature perturbations coarse grained at two different locations are **correlated**?

The distance between two patches is encoded in the time at which they become statistically independent — K. Ando, V. Vennin [2021]

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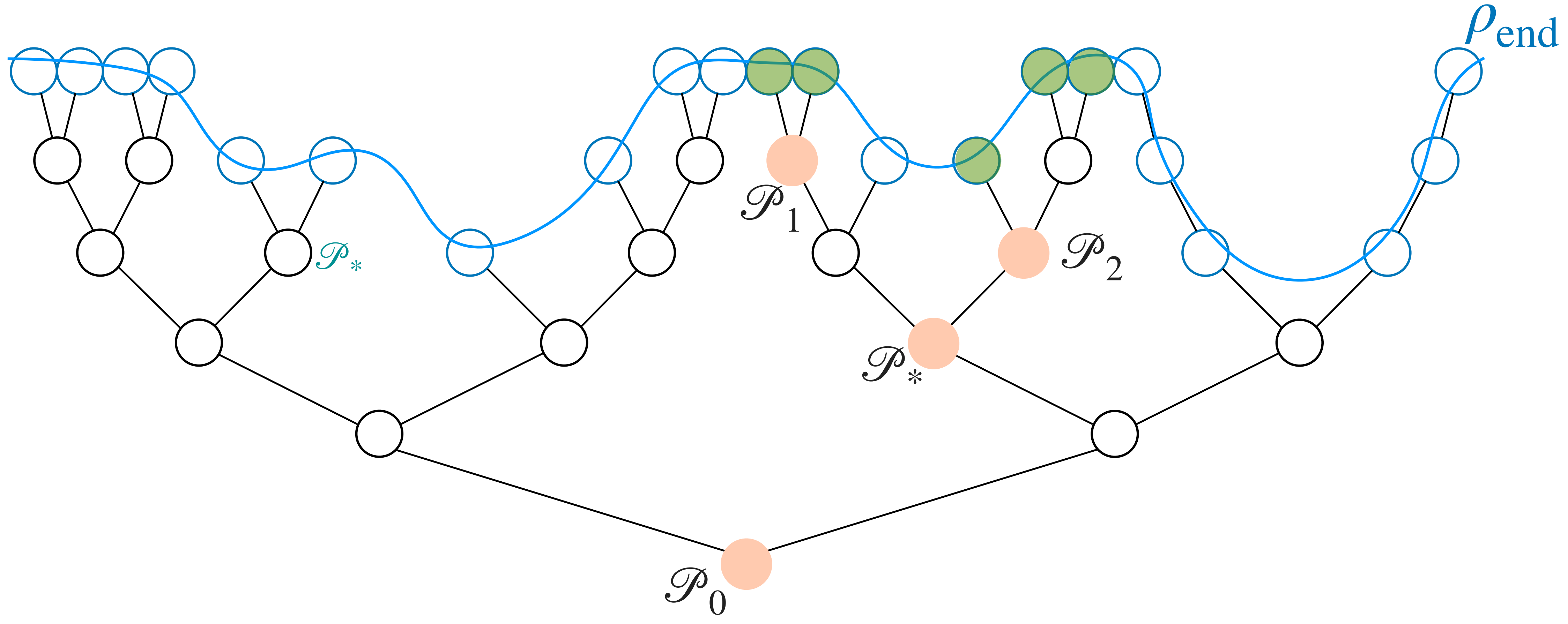
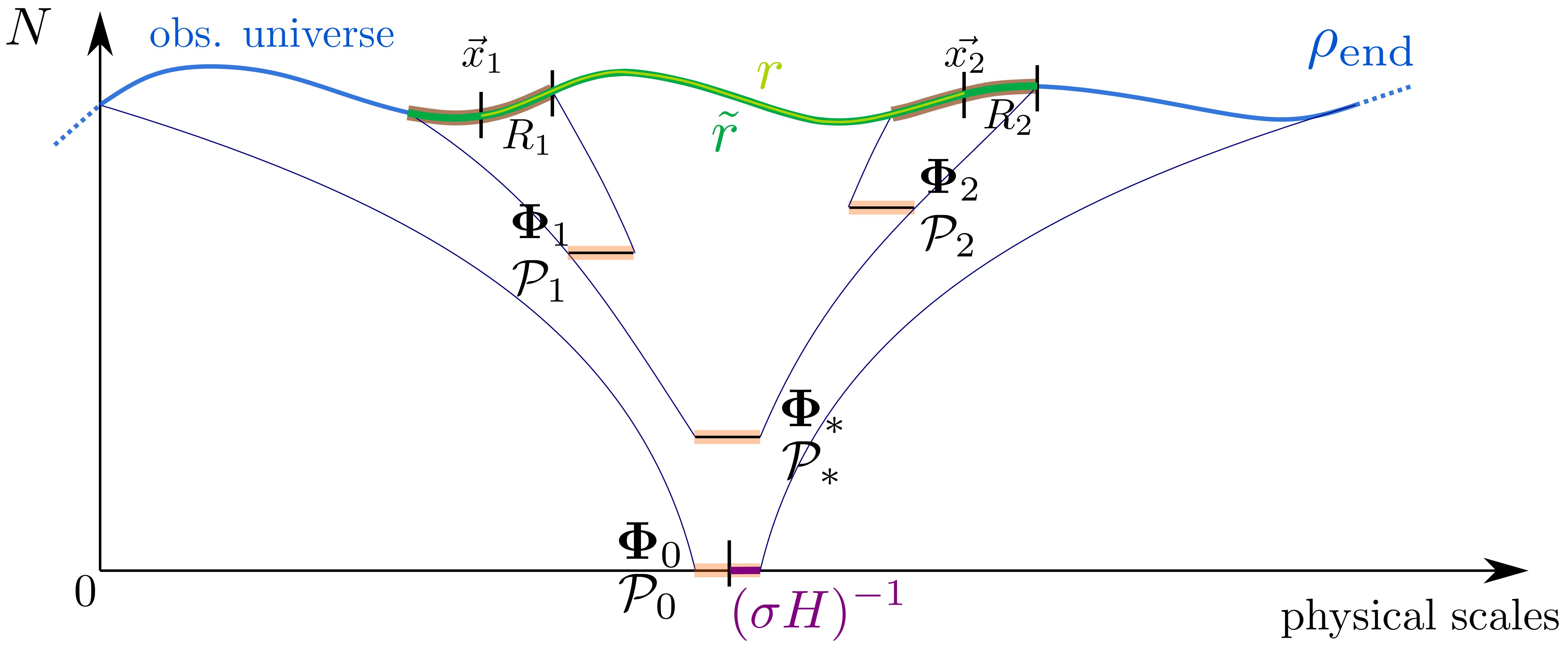




Large-volume approximation: $\zeta_{R_i} = \mathcal{N}_{\mathscr{P}_0 \to \mathscr{S}_i} + \langle \mathcal{N}_{\Phi_i} \rangle_V - \langle \mathcal{N}_{\Phi_0} \rangle_V = \mathcal{N}_{\mathscr{P}_0 \to \mathscr{S}_*} + \mathcal{N}_{\Phi_* \to \mathscr{S}_i} + \langle \mathcal{N}_{\Phi_i} \rangle_V - \langle \mathcal{N}_{\Phi_0} \rangle_V$

shared between the two regions: correlation

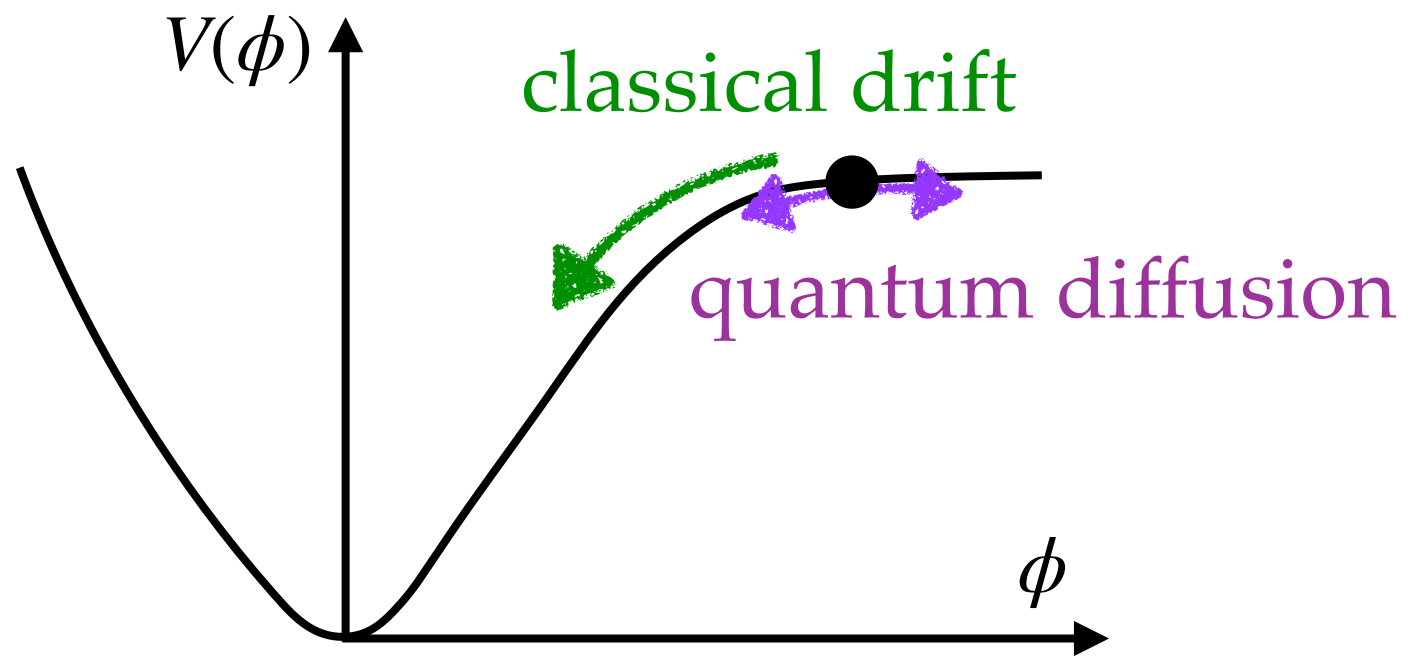
$$P(\zeta_{R_1}, \zeta_{R_2} \mid \Phi_0) = \int \mathrm{d}\Phi_* \, \mathrm{d}\Phi_1 \, \mathrm{d}\Phi_2 \, \mathrm{d}\mathcal{N}_{\Phi_0 \to \mathscr{S}_*} P^V_{\mathrm{FPTL}, \Phi_0 \to \mathscr{S}_*}(\mathcal{N}_{\Phi_0 \to \mathscr{S}_*}, \Phi_*)$$
$$P^V_{\mathrm{FPTL}, \Phi_* \to \mathscr{S}_1}(\zeta_{R_1} - \mathcal{N}_{\Phi_0 \to \mathscr{S}_*} + \langle \mathcal{N}_{\Phi_0} \rangle_V - \langle \mathcal{N}_{\Phi_1} \rangle_V, \Phi_1)$$
$$P^V_{\mathrm{FPTL}, \Phi_* \to \mathscr{S}_2}(\zeta_{R_2} - \mathcal{N}_{\Phi_0 \to \mathscr{S}_*} + \langle \mathcal{N}_{\Phi_0} \rangle_V - \langle \mathcal{N}_{\Phi_2} \rangle_V, \Phi_2)$$

$\mathscr{S}_*$: field-space hypersurface where $\langle e^{3\mathcal{N}_{\Phi_*}} \rangle = (\tilde{r}/2)^3$

$\mathscr{S}_i$: field-space hypersurfaces where $\langle e^{3\mathcal{N}_{\Phi_i}} \rangle = (R_i)^3$

Single-clock models

$\Phi \to \phi:$ single-field models of inflation along a dynamical attractor (slow roll)



Hypersurfaces $\mathcal{S}_*$ of fixed mean final volume reduce to **single points**

Backward fields become **deterministic** quantities

$$P(\zeta_R) = P^V_{\mathrm{FPT},\phi_0\to\phi_*}\left(\zeta_R - \langle\mathcal{N}_{\phi_*}\rangle_V + \langle\mathcal{N}_{\phi_0}\rangle_V\right)$$

$$P(\zeta_{R_1},\zeta_{R_2}) = \int \mathrm{d}\mathcal{N}_{\phi_0\to\phi_*}(\mathcal{N}_{\phi_0\to\phi_*})P^V_{\mathrm{FPT},\phi_*\to\phi_1}\left(\zeta_{R_1} - \mathcal{N}_{\phi_0\to\phi_*} + \langle\mathcal{N}_{\phi_0}\rangle_V - \langle\mathcal{N}_{\phi_1}\rangle_V\right)P^V_{\mathrm{FPT},\phi_*\to\phi_2}\left(\zeta_{R_2} - \mathcal{N}_{\phi_0\to\phi_*} + \langle\mathcal{N}_{\phi_0}\rangle_V - \langle\mathcal{N}_{\phi_2}\rangle_V\right)$$

Power spectrum from the two-point statistics

Two-point correlation function of coarse-grained fields:

$$\langle \zeta_{R_1}\zeta_{R_2}\rangle = \int \mathrm{d}\zeta_{R_1}\int \mathrm{d}\zeta_{R_2} P(\zeta_{R_1},\zeta_{R_2})\zeta_{R_1}\zeta_{R_2} = \langle \mathcal{N}^2_{\phi_0\to\phi_*}\rangle_V - \langle \mathcal{N}_{\phi_0\to\phi_*}\rangle_V^2 \equiv \langle \delta\mathcal{N}^2_{\phi_0\to\phi_*}\rangle_V = \langle \delta\mathcal{N}^2_{\phi_0}\rangle_V - \langle \delta\mathcal{N}^2_{\phi_*}\rangle_V$$

no dependence on the coarse-graining scales R_1, R_2

In Fourier space: $\zeta_{R_i}(\vec{x}_i) = \int \frac{\mathrm{d}\vec{k}}{(2\pi)^{3/2}}\zeta_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x}_i}\,\widetilde{W}\left(\frac{kR_i}{a}\right)$

$$\langle \zeta_{R_1}\zeta_{R_2}\rangle = \int_0^\infty \mathrm{d}\ln k\,\mathcal{P}_\zeta(k)\widetilde{W}\left(\frac{kR_1}{a}\right)\widetilde{W}\left(\frac{kR_2}{a}\right)\widetilde{W}\left(\frac{kr}{a}\right)$$

$r > R_1, R_2$ ⟶ $\langle \zeta_{R_1}\zeta_{R_2}\rangle = \int_0^\infty \mathrm{d}\ln k\,\mathcal{P}_\zeta(k)\widetilde{W}\left(\frac{kr}{a}\right)$

Differentiation w.r.t. r:

$$\mathcal{P}_\zeta(k) = -\frac{\partial}{\partial \ln r}\langle \zeta_{R_1}\zeta_{R_2}\rangle\Big|_{r=a_{\mathrm{end}}/k} = \frac{\partial}{\partial \ln r}\langle \delta\mathcal{N}_{\phi_*}\rangle^2\Big|_{r=a_{\mathrm{end}}/k}$$

$$\mathcal{P}_\zeta(k) = \frac{r}{\tilde{r}}\left[\frac{1}{3}\frac{\partial}{\partial\phi_*}\ln\langle e^{3\mathcal{N}_{\phi_*}}\rangle - \frac{\partial}{\partial\phi_*}\ln H(\phi_*)\right]^{-1}\frac{\partial}{\partial\phi_*}\langle \delta\mathcal{N}^2_{\phi_*}\rangle_V\Big|_{\langle e^{3\mathcal{N}_{\phi_*}}\rangle^{1/3}=\frac{1}{2}\frac{r}{\tilde{r}}\frac{a_{\mathrm{end}}\sigma H(\phi_*)}{k}}$$

$\tilde{r} = r + R_1 + R_2$

$r \gg R_1, R_2 \to \frac{r}{\tilde{r}} \simeq 1$

$\partial \ln N/\partial\phi \simeq \sqrt{\epsilon_1/2}/M_{\mathrm{Pl}}$

c.f.r. V. Vennin and A. A. Starobinsky [2015]
T. Fujita, M. Kawasaki, Y. Tada and T. Takesako [2013]

Same expression at l.o. in slow roll neglecting volume weighting and defining ϕ_* via $\langle\mathcal{N}\rangle$ and not via $\langle e^{3\mathcal{N}}\rangle$

Consistency checks

$\langle \zeta_R \rangle_V$ *vanishes*

Lemma: ϕ_1, ϕ_2, ϕ_3 such that $\phi_1 > \phi_2 > \phi_3$, then it is possible to split $\mathcal{N}_{\phi_1 \to \phi_3} = \mathcal{N}_{\phi_1 \to \phi_2} + \mathcal{N}_{\phi_2 \to \phi_3}$ where $\mathcal{N}_{\phi_1 \to \phi_2}, \mathcal{N}_{\phi_2 \to \phi_3}$ are first-passage times, and independent random variables (Markovianity)

$$P_{\mathrm{FPT},\phi_0}(\mathcal{N}_{\phi_0}) = \int_0^{\mathcal{N}_{\phi_0}} \mathrm{d}\mathcal{N}_{\phi_*} P_{\mathrm{FPT},\phi_0 \to \phi_*}(\mathcal{N}_{\phi_0} - \mathcal{N}_{\phi_*}) P_{\mathrm{FPT},\phi_*}(\mathcal{N}_{\phi_*})$$

Convolution structure also applies to the volume-weighted statistics:

$$P^V_{\mathrm{FPT},\phi_0}(\mathcal{N}_{\phi_0}) \propto P_{\mathrm{FPT},\phi_0}(\mathcal{N}_{\phi_0})\, e^{3\mathcal{N}_{\phi_0}} = \int_0^{\mathcal{N}_{\phi_0}} \mathrm{d}\mathcal{N}_{\phi_*} P^V_{\mathrm{FPT},\phi_0 \to \phi_*}(\mathcal{N}_{\phi_0} - \mathcal{N}_{\phi_*}) P^V_{\mathrm{FPT},\phi_*}(\mathcal{N}_{\phi_*})$$

Therefore:

$$\mathcal{N}_{\phi_0} = \mathcal{N}_{\phi_0 \to \phi_*} + \mathcal{N}_{\phi_*} \longrightarrow \langle \mathcal{N}_{\phi_0} \rangle = \langle \mathcal{N}_{\phi_0 \to \phi_*} \rangle + \langle \mathcal{N}_{\phi_*} \rangle \longrightarrow \langle \mathcal{N}_{\phi_0} \rangle_V = \langle \mathcal{N}_{\phi_0 \to \phi_*} \rangle_V + \langle \mathcal{N}_{\phi_*} \rangle_V$$

$$\longrightarrow \langle \zeta_R \rangle_V = 0 \quad ✓$$

Consistency checks:

Power spectrum from the one-point distribution

$$\langle \zeta_R^2 \rangle = \int \zeta_R^2 P(\zeta_R)\,\mathrm{d}\zeta_R = \int \mathrm{d}\zeta_R P^V_{\mathrm{FPT},\phi_0\to\phi_*}\left(\zeta_R + \langle \mathcal{N}_{\phi_0}\rangle_V - \langle \mathcal{N}_{\phi_*}\rangle_V\right)\zeta_R^2 = \langle \mathcal{N}^2_{\phi_0}\rangle_V - \langle \mathcal{N}^2_{\phi_*}\rangle_V - \langle \mathcal{N}_{\phi_0}\rangle_V^2 + \langle \mathcal{N}_{\phi_*}\rangle_V^2$$

$$= \langle \delta\mathcal{N}^2_{\phi_0}\rangle_V - \langle \delta\mathcal{N}^2_{\phi_*}\rangle_V$$

In Fourier space: $\langle \zeta_R^2 \rangle = \int \mathcal{P}_\zeta(k)\widetilde{W}^2\left(\frac{kR}{a}\right)\mathrm{d}\ln k$

differentiation w.r.t. R:

$$\mathcal{P}_\zeta(k) = -\frac{\partial}{\partial \ln R}\langle \zeta_R^2\rangle\bigg|_{R=a_{\mathrm{end}}/k} = \frac{\partial}{\partial \ln R}\langle \delta\mathcal{N}^2_{\phi_*}\rangle\bigg|_{R=a_{\mathrm{end}}/k}$$

✔

Second moment of ζ_R is consistent with the calculation of the power spectrum from the two-point statistics

Consistency checks

Marginalisation

One-point statistics can be obtained from the two-point statistics upon marginalisation:

$$\int \mathrm{d}\zeta_{R_2} P(\zeta_{R_1}, \zeta_{R_2}) = \int \mathrm{d}\zeta_{R_2} \int \mathrm{d}\mathcal{N}_{\phi_0\to\phi_*} P^V_{\mathrm{FPT},\phi_0\to\phi_*}(\mathcal{N}_{\phi_0\to\phi_*}) P^V_{\mathrm{FPT},\phi_*\to\phi_1}(\zeta_{R_1} - \mathcal{N}_{\phi_0\to\phi_*} + \langle\mathcal{N}_{\phi_0}\rangle_V - \langle\mathcal{N}_{\phi_1}\rangle_V)$$

$$\times P^V_{\mathrm{FPT},\phi_*\to\phi_2}(\zeta_{R_2} - \mathcal{N}_{\phi_0\to\phi_*} + \langle\mathcal{N}_{\phi_0}\rangle_V - \langle\mathcal{N}_{\phi_2}\rangle_V) =$$

$$\int \mathrm{d}\mathcal{N}_{\phi_0\to\phi_*} P^V_{\mathrm{FPT},\phi_0\to\phi_*}(\mathcal{N}_{\phi_0\to\phi_*}) P^V_{\mathrm{FPT},\phi_*\to\phi_1}(\zeta_{R_1} - \mathcal{N}_{\phi_0\to\phi_*} + \langle\mathcal{N}_{\phi_0}\rangle_V - \langle\mathcal{N}_{\phi_1}\rangle_V)$$

normalisation of the FPT distribution

Lemma: $\phi_1 < \phi_* < \phi_0 \longrightarrow \mathcal{N}_{\phi_0\to\phi_1} = \mathcal{N}_{\phi_0\to\phi_*} + \mathcal{N}_{\phi_*\to\phi_1}$ independent

$$\int \mathrm{d}\mathcal{N}_{\phi_0\to\phi_*} P^V_{\mathrm{FPT},\phi_0\to\phi_*}(\mathcal{N}_{\phi_0\to\phi_*}) P^V_{\mathrm{FPT},\phi_*\to\phi_1}(\mathcal{N}_{\phi_0\to\phi_1} - \mathcal{N}_{\phi_0\to\phi_*}) = P^V_{\mathrm{FPT},\phi_0\to\phi_1}(\mathcal{N}_{\phi_0\to\phi_1})$$

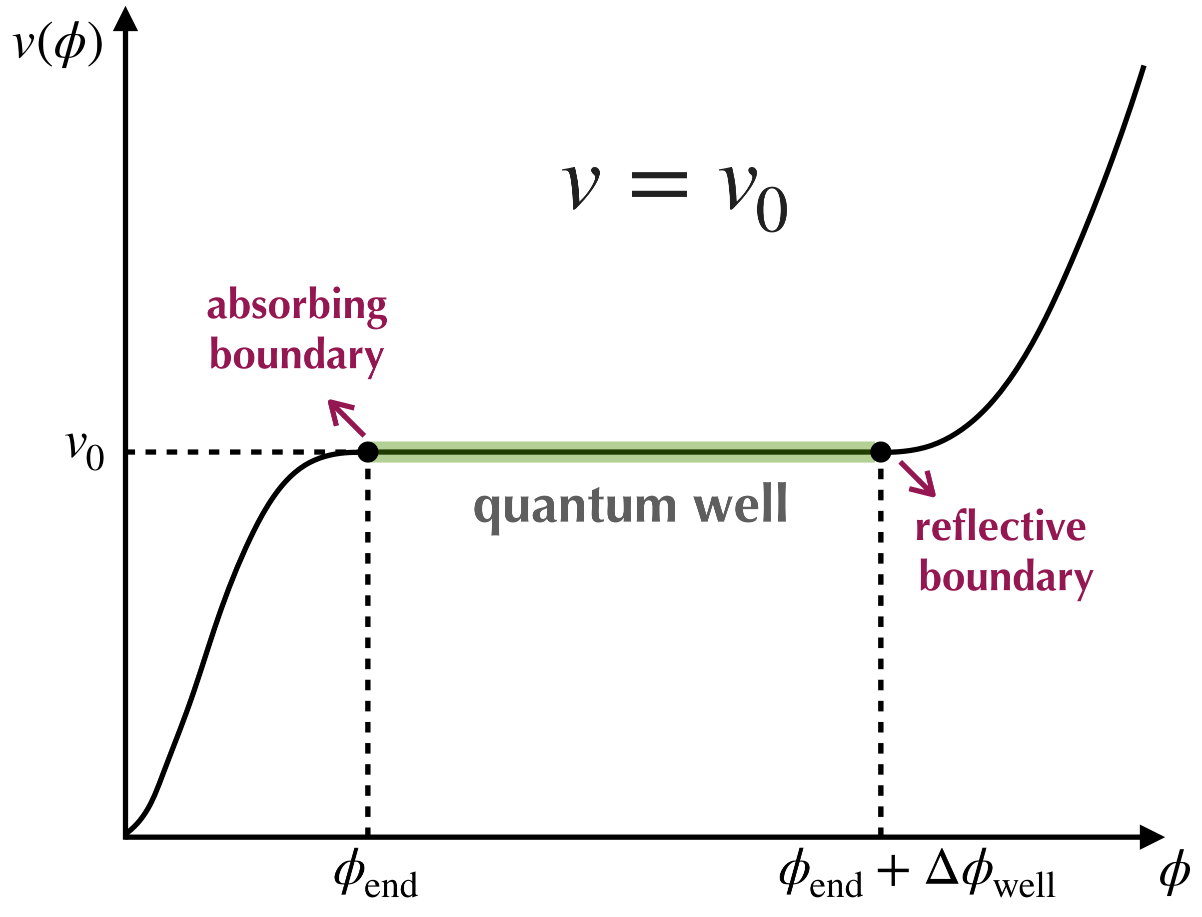
$$\boxed{\int \mathrm{d}\zeta_{R_2} P(\zeta_{R_1}, \zeta_{R_2}) = P^V_{\mathrm{FPT},\phi_0\to\phi_1}\left[\zeta_{R_1} + \langle\mathcal{N}_{\phi_0}\rangle_V - \langle\mathcal{N}_{\phi_1}\rangle_V\right] \equiv P(\zeta_{R_1})}$$

✓

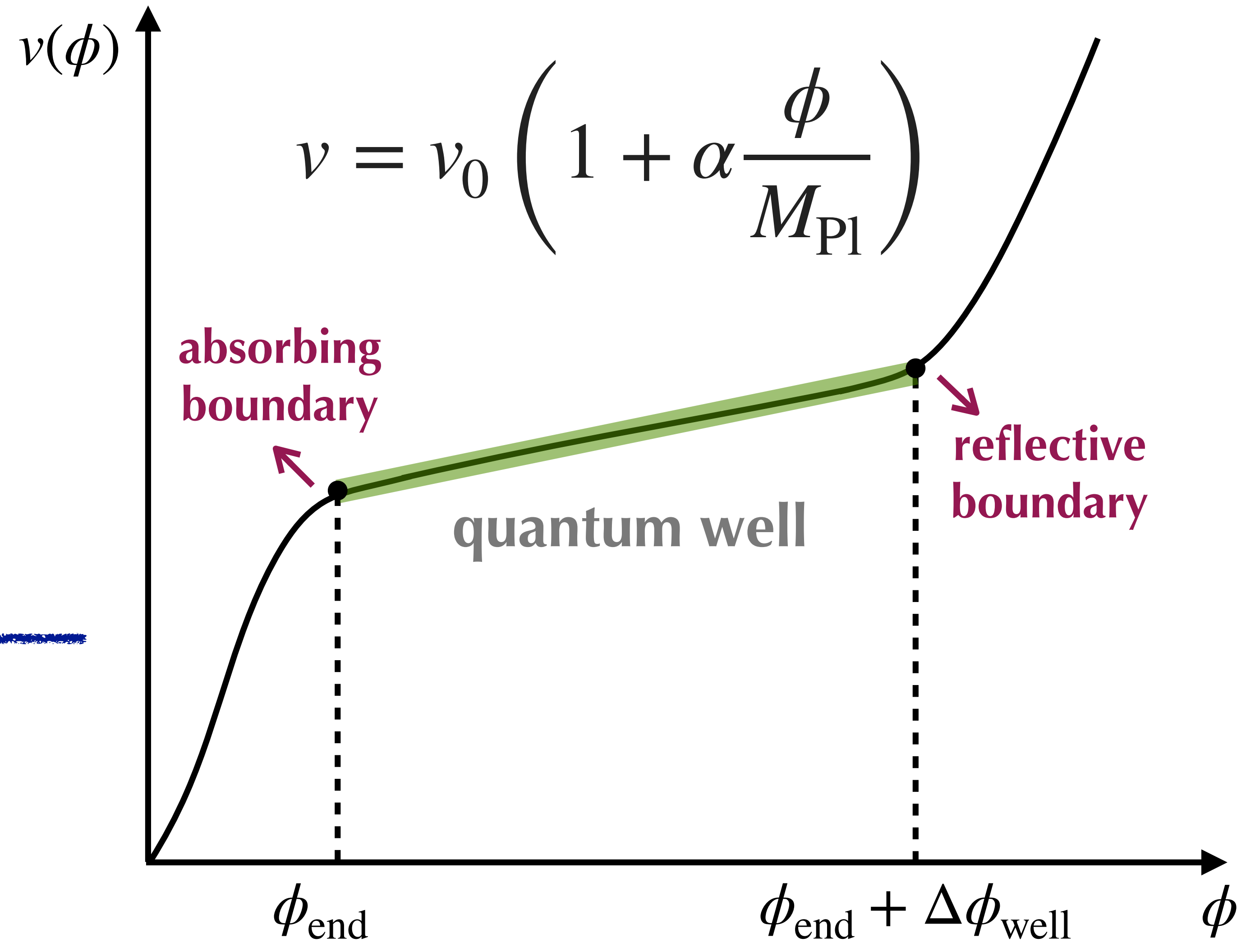
Applications

Single-field slow-roll models of inflation $\frac{\partial\phi}{\partial N} = -\frac{V'}{3H^2} + \frac{H}{2\pi}\xi$ $\quad \mathcal{L}^\dagger_{\mathrm{FP}}(\phi) = -M_{\mathrm{Pl}}^2 \frac{v'}{v}\frac{\partial}{\partial\phi} + v\frac{\partial^2}{\partial\phi^2}$ $\quad v \equiv V/(24\pi^2 M_{\mathrm{Pl}}^4)$

quantum wells:



→ exact analytical solution



approx. solution in the almost-constant regime: ←
$\alpha\Delta\phi_{\mathrm{well}}/M_{\mathrm{Pl}} \ll 1$

$$\chi_{\mathcal{N}}(t,\phi) = \langle e^{it\mathcal{N}}\rangle = \int_{-\infty}^{\infty} \mathrm{d}\mathcal{N}\, e^{it\mathcal{N}} P_{\mathrm{FPT},\phi}(\mathcal{N})$$

$$\mathcal{L}^\dagger_{\mathrm{FP}}(\phi)\chi_{\mathcal{N}}(t,\phi) = -it\chi_{\mathcal{N}}(t,\phi)$$

$$\chi(t,\phi_{\mathrm{end}}) = 1 \quad \frac{\partial}{\partial\phi}\chi(t,\phi)\Big|_{\phi_{\mathrm{end}+\Delta\phi_{\mathrm{well}}}} = 0$$

$$\chi_{\mathcal{N}}(t,\phi) = \frac{\cos[\sqrt{it}\,\mu\,(x-1)]}{\cos[\sqrt{it}\,\mu]}$$

$$P_{\mathrm{FPT},\phi}(\mathcal{N}) = -\frac{\pi}{2\mu^2}\vartheta_2'\left(\frac{\pi}{2}x, e^{-\frac{\pi^2}{\mu^2}\mathcal{N}}\right)$$

$$\chi^V_{\mathcal{N}(t,\phi)} = \frac{\chi_{\mathcal{N}}(t-3i,\phi)}{\chi_{\mathcal{N}}(-3i,\phi)}$$

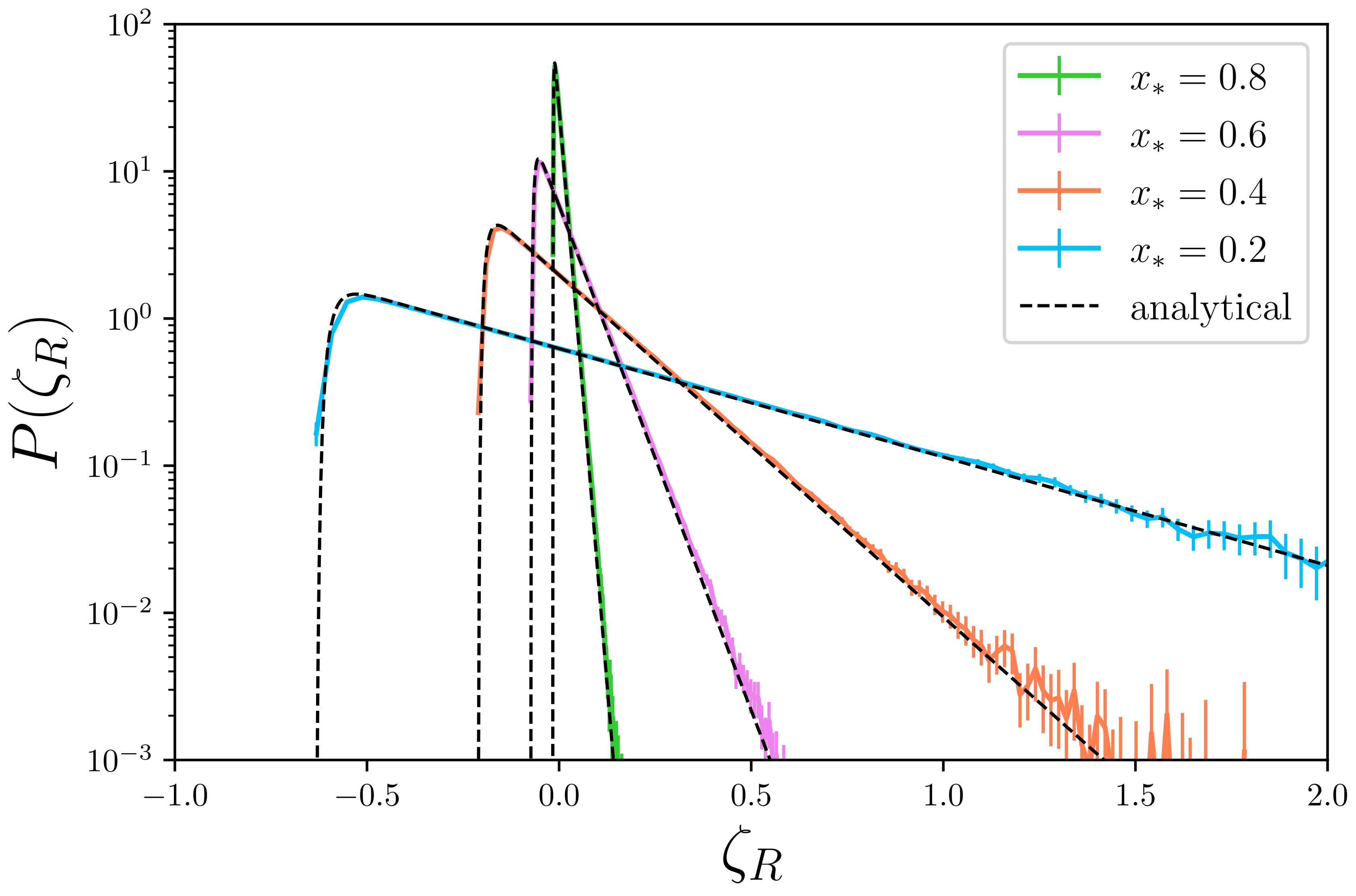
$$\langle\mathcal{N}^n_\phi\rangle_V = \frac{i^{-n}}{\chi_{\mathcal{N}}(-3i,\phi)}\frac{\partial}{\partial t}\chi_{\mathcal{N}}(t,\phi)\Big|_{t=-3i}$$

$$x = (\phi - \phi_{\mathrm{end}})/\Delta\phi_{\mathrm{well}}$$

$$\mu^2 = \frac{\Delta\phi^2_{\mathrm{well}}}{v_0 M_{\mathrm{Pl}}^2}$$

One-point distributions

$v = v_0$



R decreases

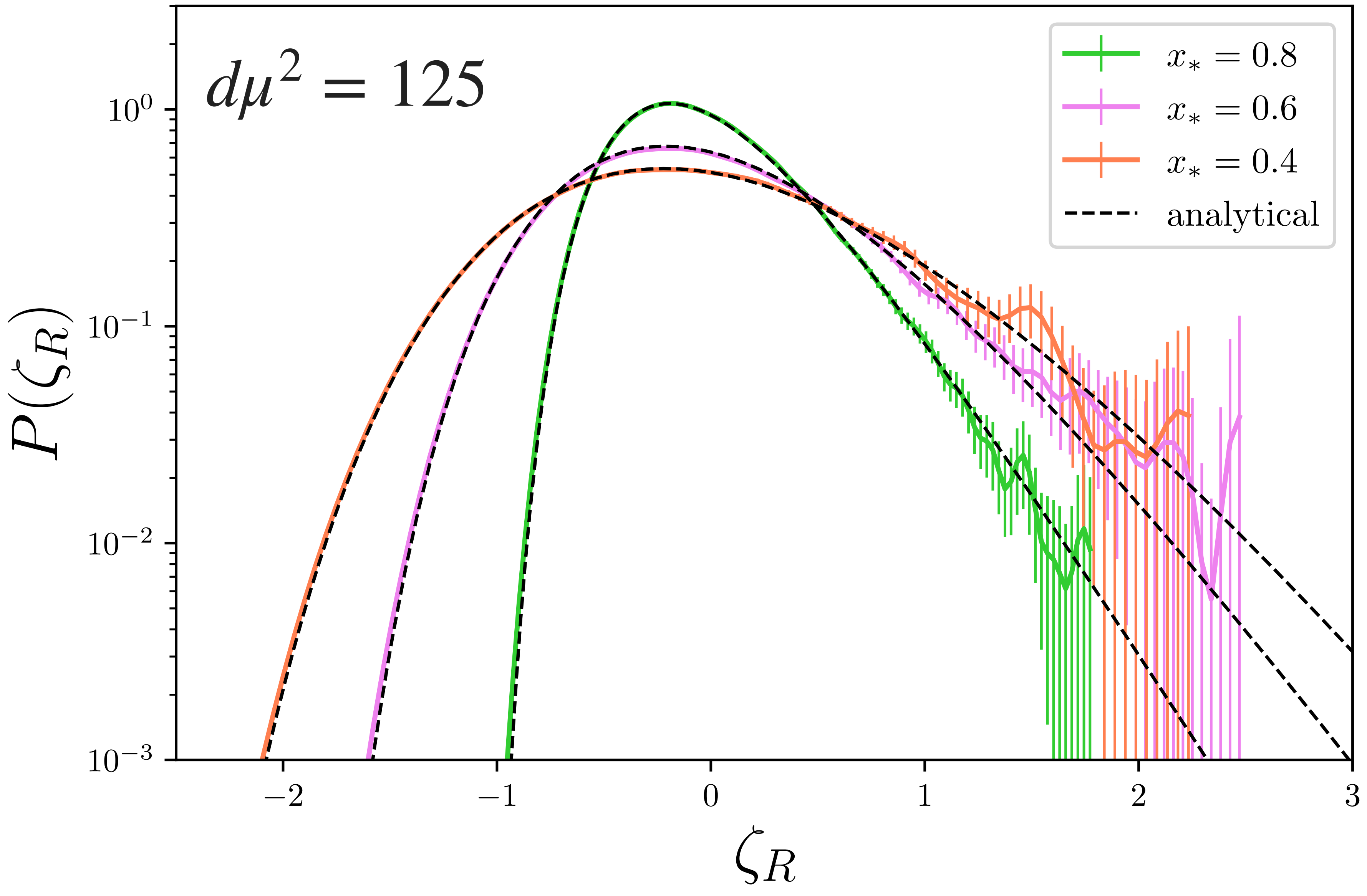
Tails become heavier

PBHs mostly form at scales emerging close to the end of the well (mass fraction tilted towards smaller masses)

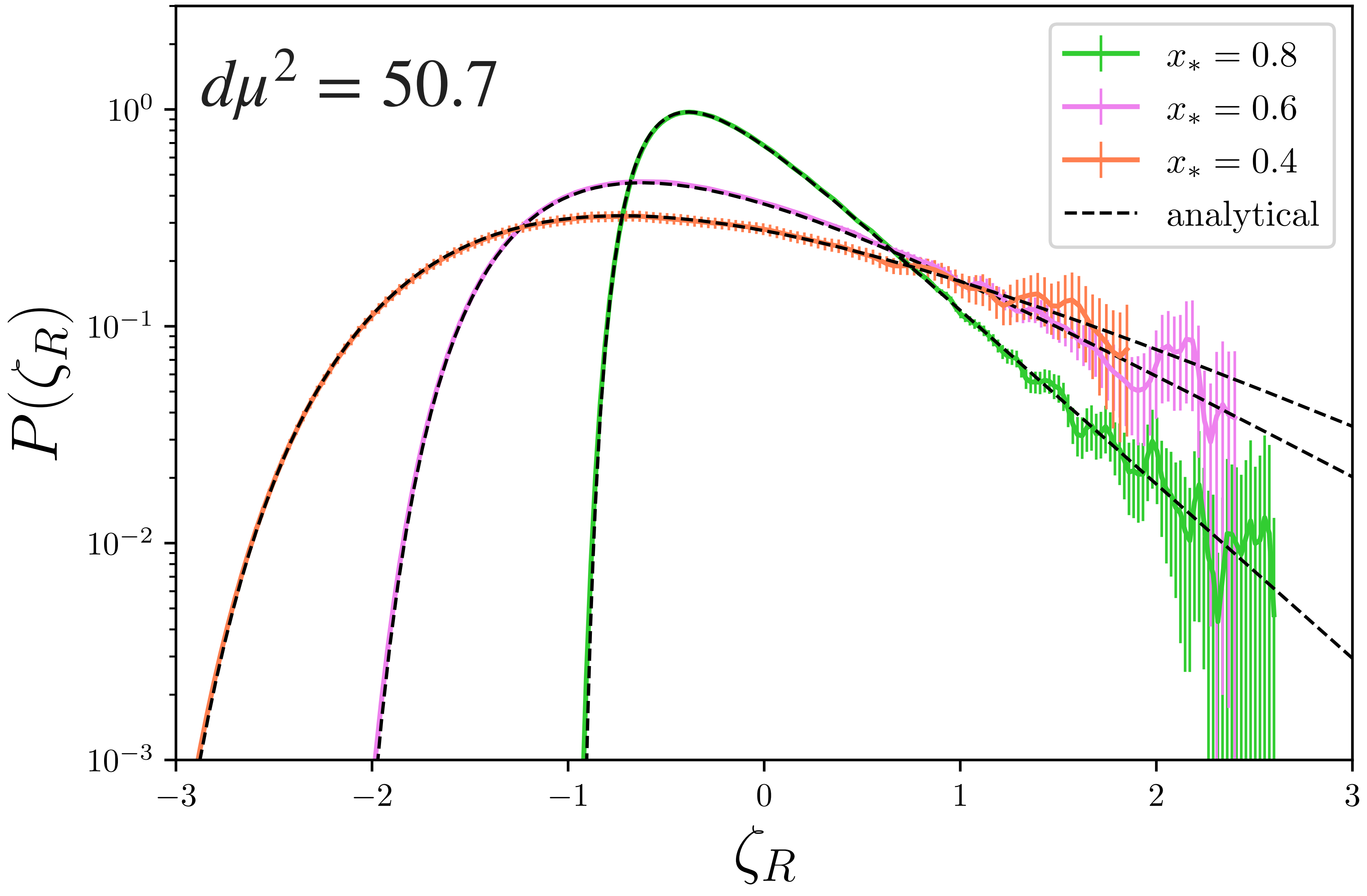
Tail behaviour: $$P(\zeta_R) \simeq \frac{\pi \cos\left[\sqrt{3}(1-x_*)\,\mu\right]}{(1-x_*)^2\,\mu^2} e^{\left[3-\frac{\pi^2}{4(1-x_*)^2\mu^2}\right]\left\{\zeta_R + \frac{\mu}{2\sqrt{3}}(1-x_*)\tan\left[\sqrt{3}\mu\,(1-x_*)\right]\right\}}$$

exponential-tail profile

$v = v_0(1 + \alpha\phi/M_{\rm Pl})$



$v = v_0(1 + \alpha\phi/M_{\rm Pl})$

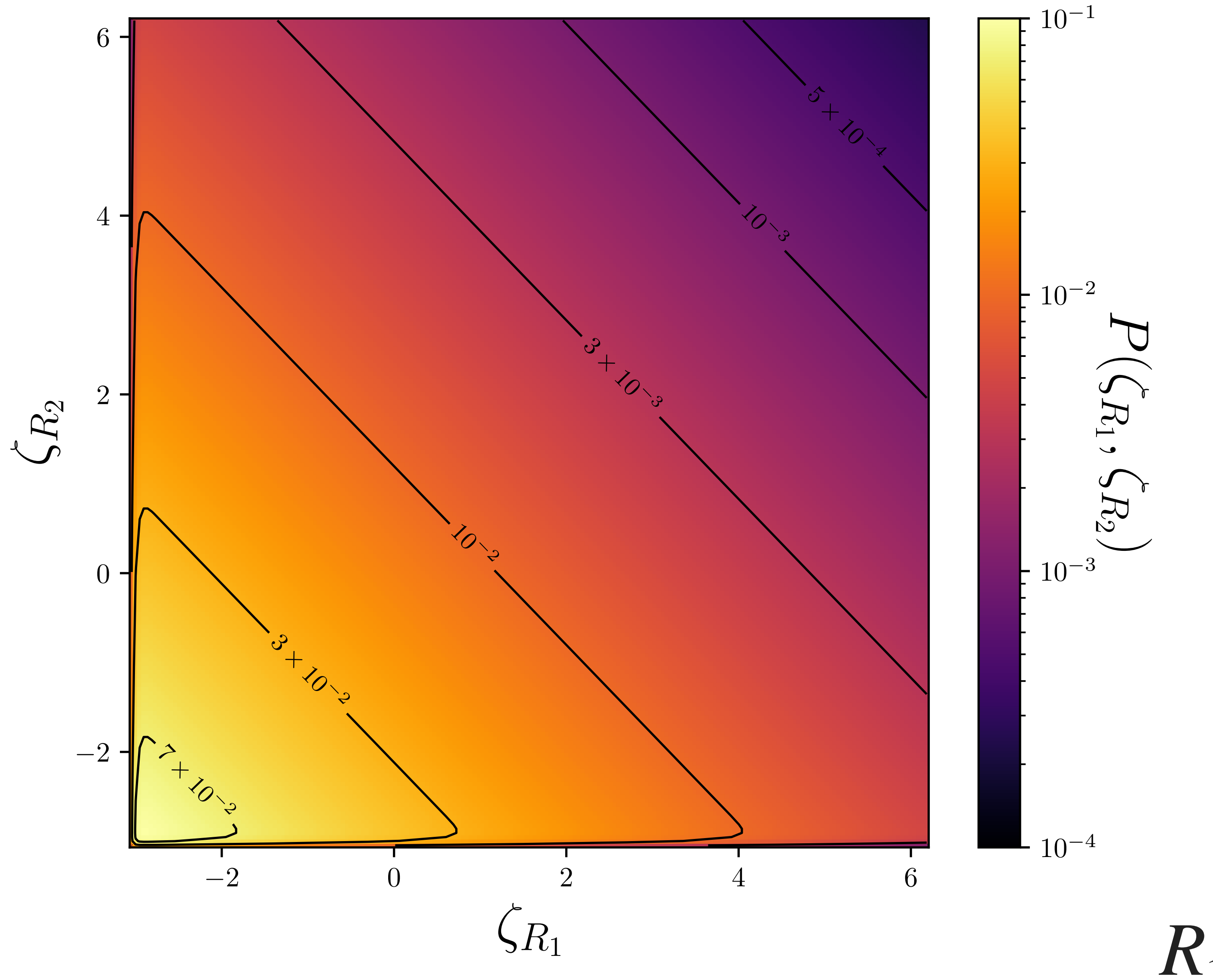


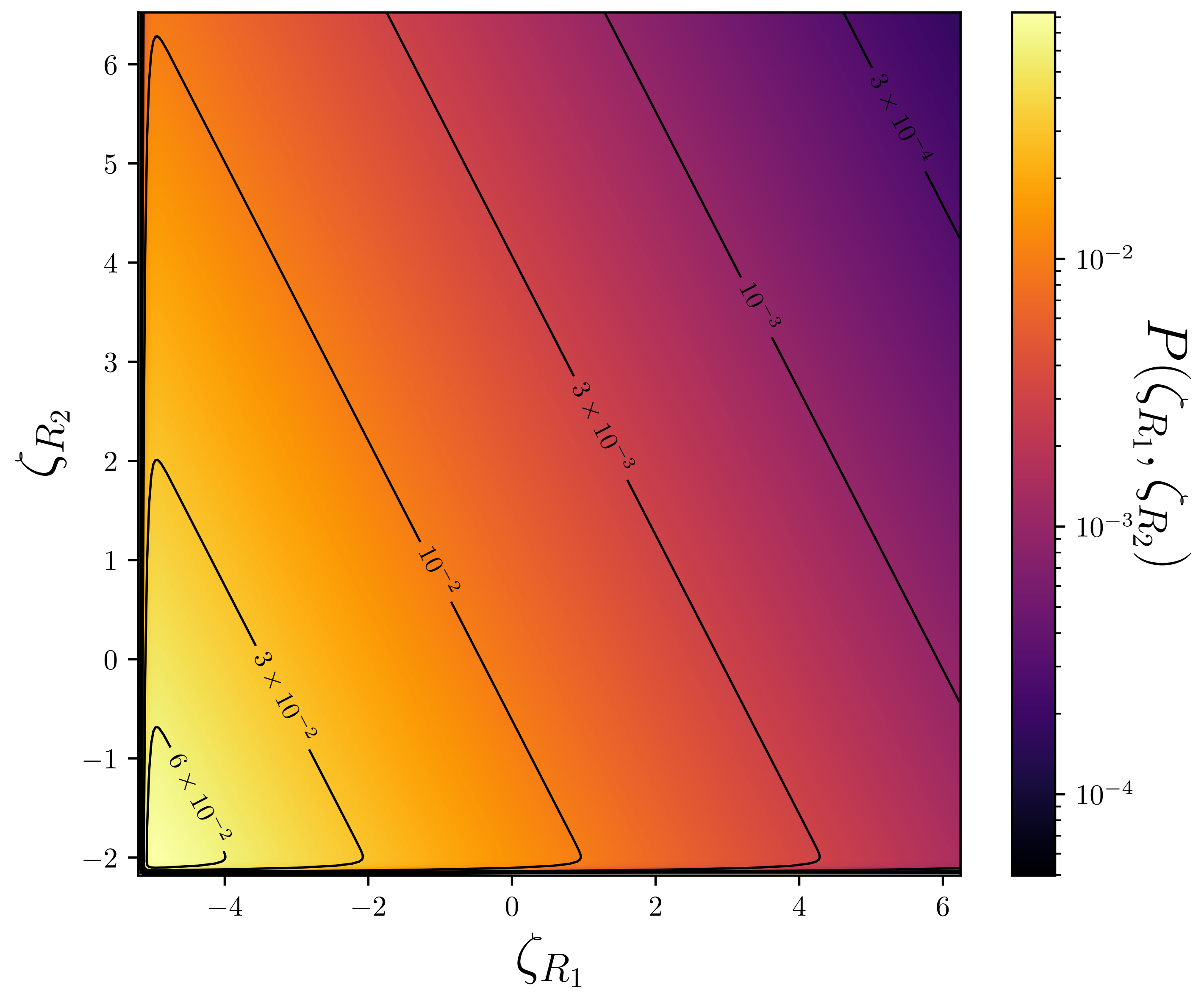
$\alpha\Delta\phi_{\rm well}/(v_0 M_{\rm Pl}) \equiv d\mu^2 \to \infty$:

classical limit

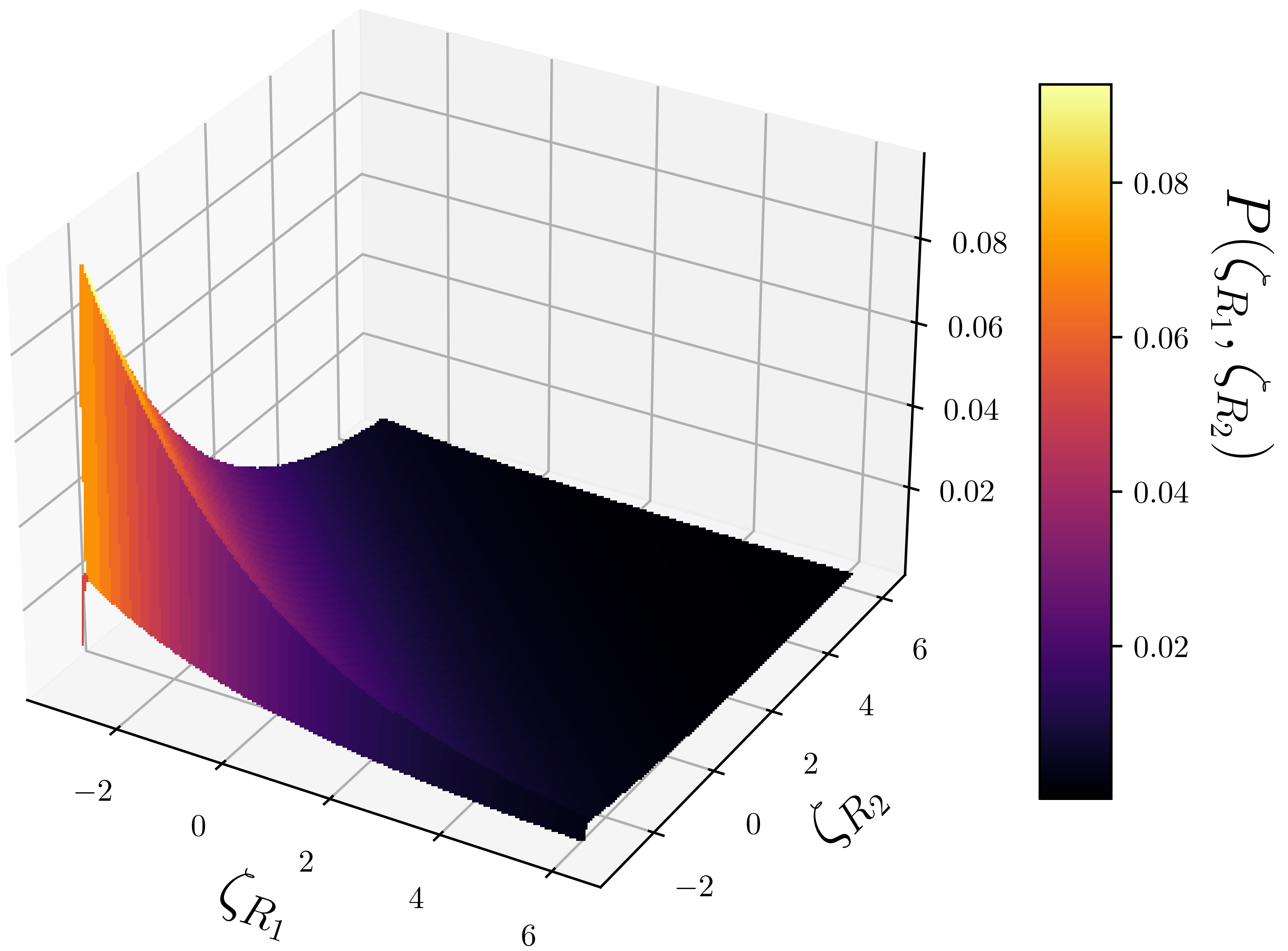
Two-point distributions: flat-well model

From numerical integration of exact analytical formula of the 2-pt distribution

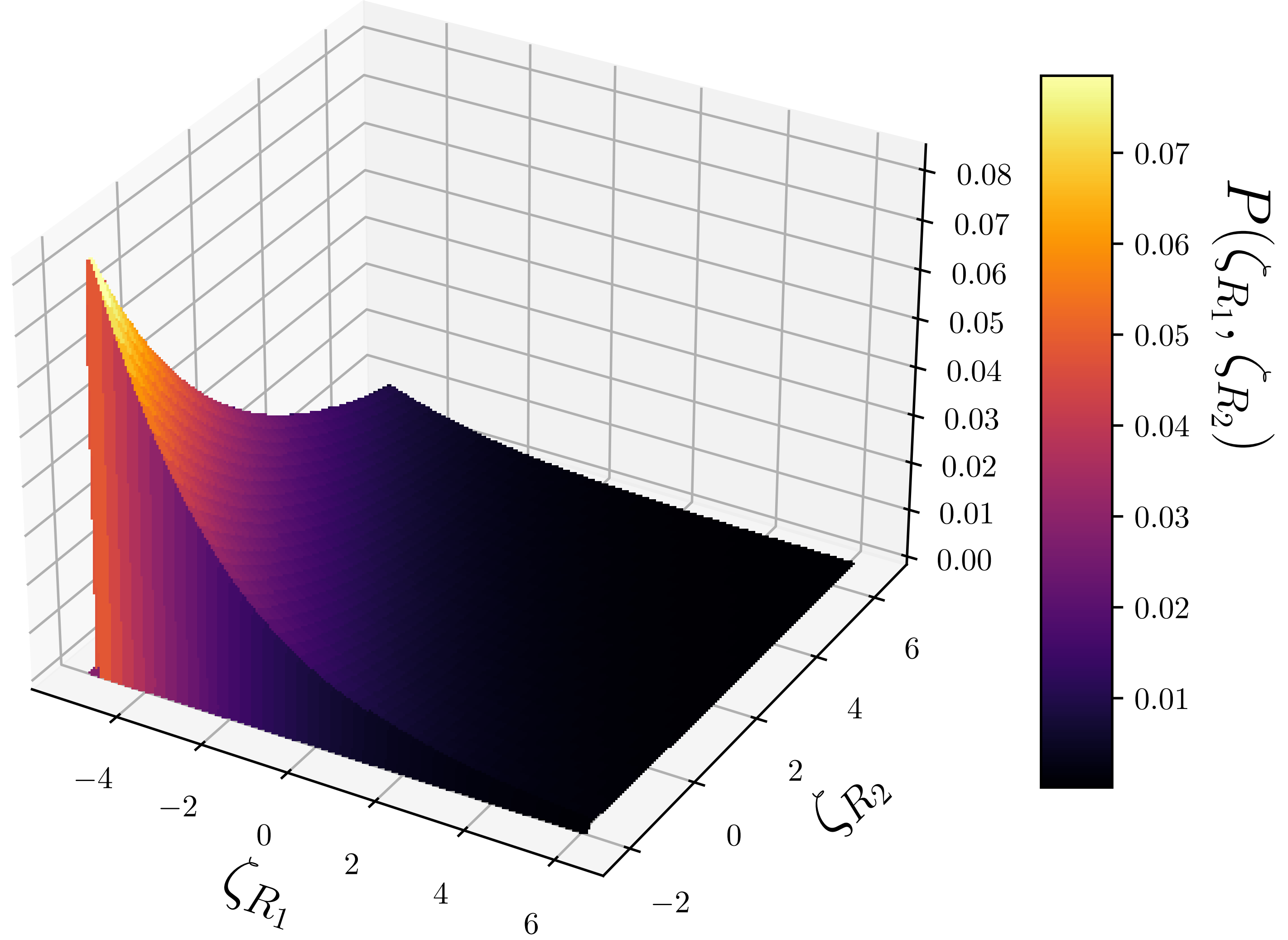




$R_1 = R_2$

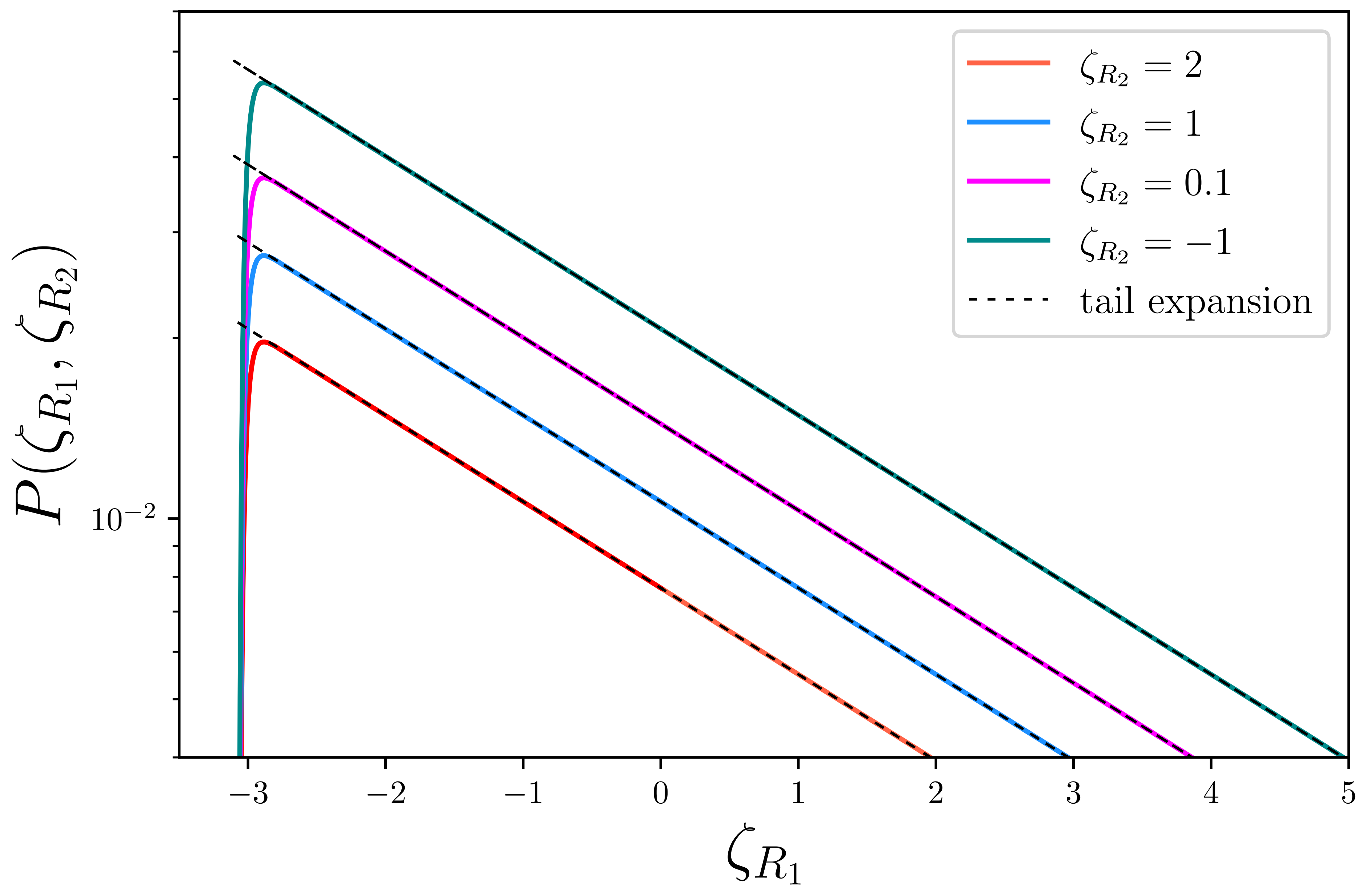


$R_1 < R_2$

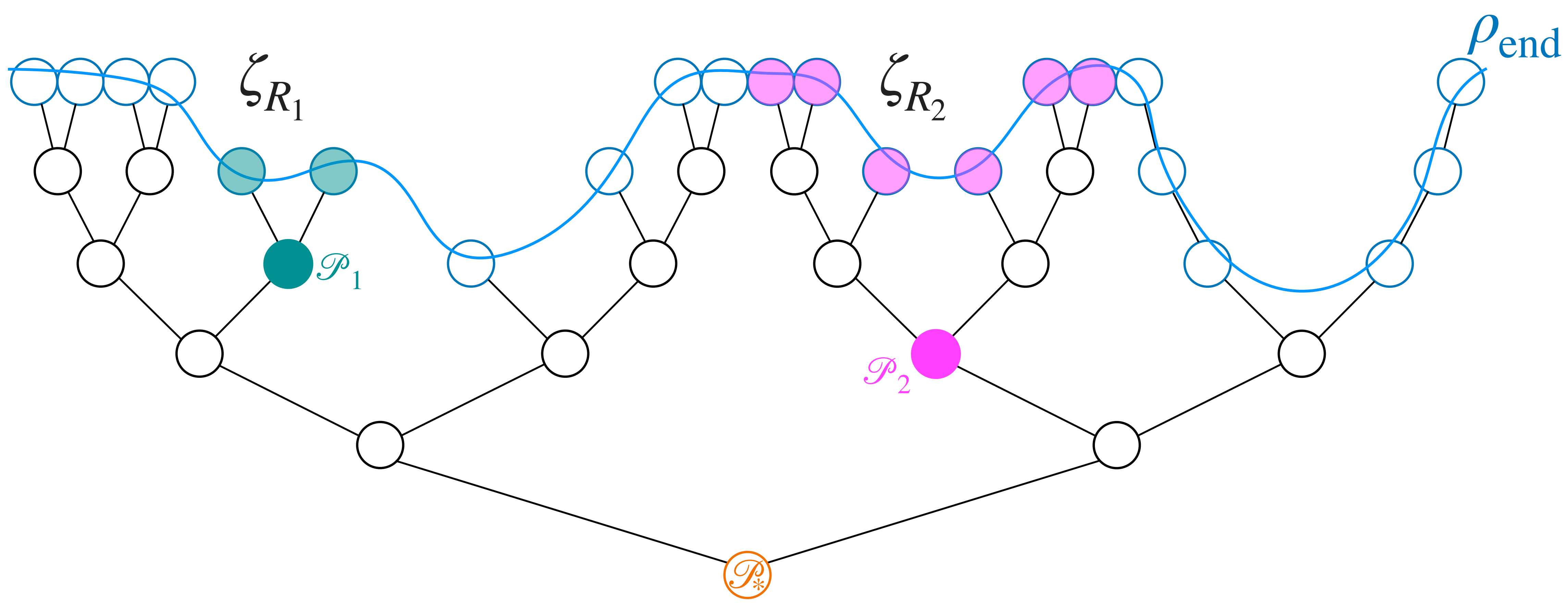


Two-point distributions: flat-well model

Tail behaviour: $$P(\zeta_{R_1},\zeta_{R_2}) \simeq P(\zeta_{R_1})P(\zeta_{R_2})\frac{\cos\left(\frac{\pi}{2}\frac{1-x_*}{1-x_1}\right)\cos\left(\frac{\pi}{2}\frac{1-x_*}{1-x_2}\right)}{\cos\left[\sqrt{3}\mu(1-x_*)\right]\cosh\left\{\sqrt{3}\mu(1-x_*)\sqrt{1-\frac{\pi^2}{12\mu^2}\left[\frac{1}{(1-x_1)^2}+\frac{1}{(1-x_2)^2}\right]}\right\}}$$



The two final regions do not share any parent node :
they cannot be correlated



For $x_* \to 1$ the joint distribution factorises: $P(\zeta_{R_1},\zeta_{R_2}) = P(\zeta_{R_1})\,P(\zeta_{R_1})$

Clustering: flat-well model

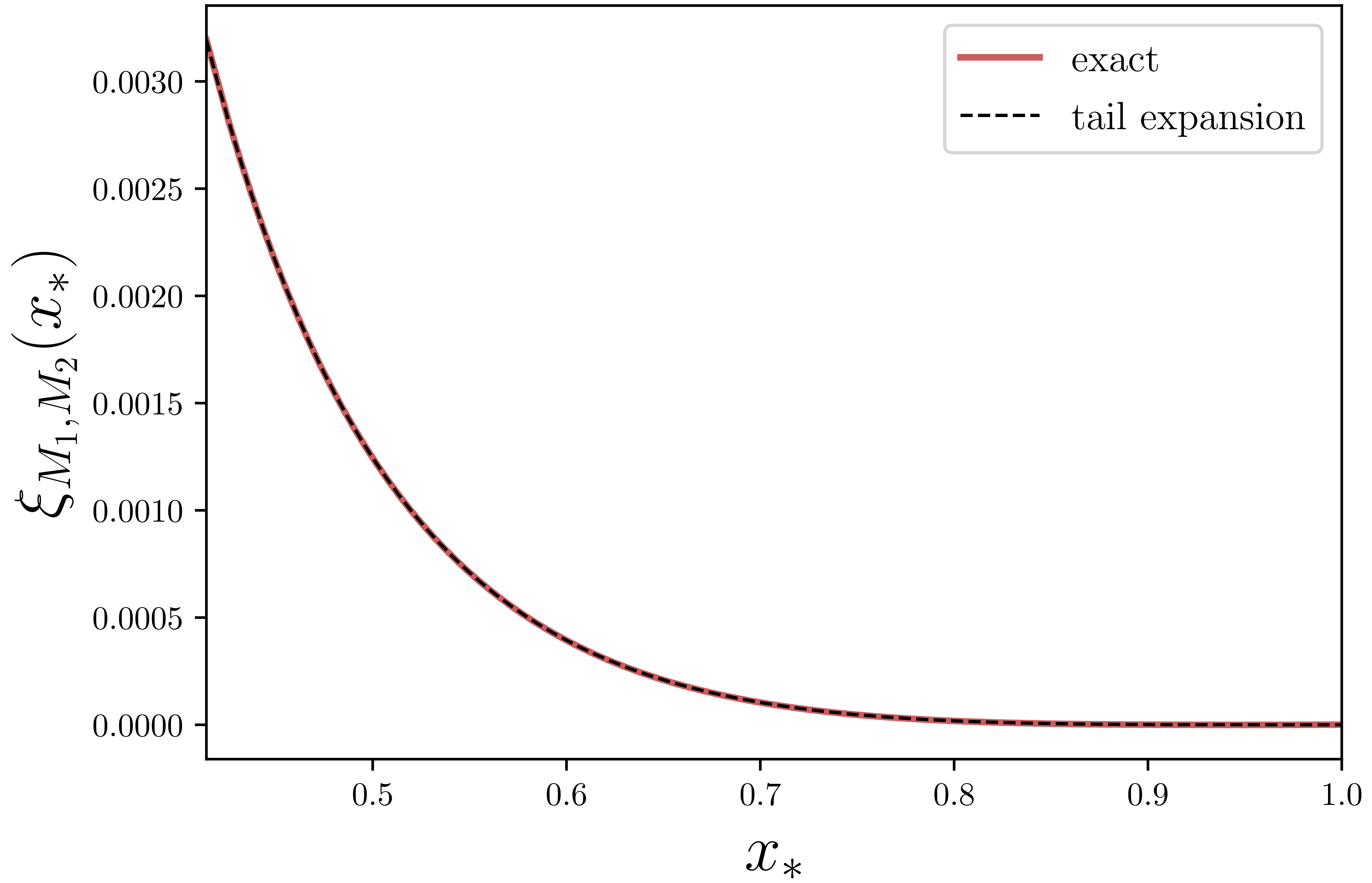
For simplicity, we consider that a PBH forms when $\zeta_R > \zeta_c$, where ζ_c is a threshold value of order unity

1-pt probability: $p_M = \int_{\zeta_c}^{\infty} P(\zeta_R)\,\mathrm{d}\zeta_R$

2-pt probability: $p_{M_1,M_2}(r) = \int_{\zeta_c}^{\infty} P(\zeta_{R_1},\zeta_{R_2})\,\mathrm{d}\zeta_{R_1}\mathrm{d}\zeta_{R_2}$

reduced correlation: $\xi_{M_1,M_2}(r) = \dfrac{p(M_1,M_2,r)}{p_{M_1}\,p_{M_2}} - 1$

$$\xi_{M_1,M_2}(r) \simeq \frac{\cos\left(\frac{\pi}{2}\frac{1-x_*}{1-x_1}\right)\cos\left(\frac{\pi}{2}\frac{1-x_*}{1-x_2}\right)}{\cos\left[\sqrt{3}\mu(1-x_*)\right]\cosh\left\{\sqrt{3}\mu(1-x_*)\sqrt{1-\frac{\pi^2}{12\mu^2}\left[\frac{1}{(1-x_1)^2}+\frac{1}{(1-x_2)^2}\right]}\right\}} - 1$$

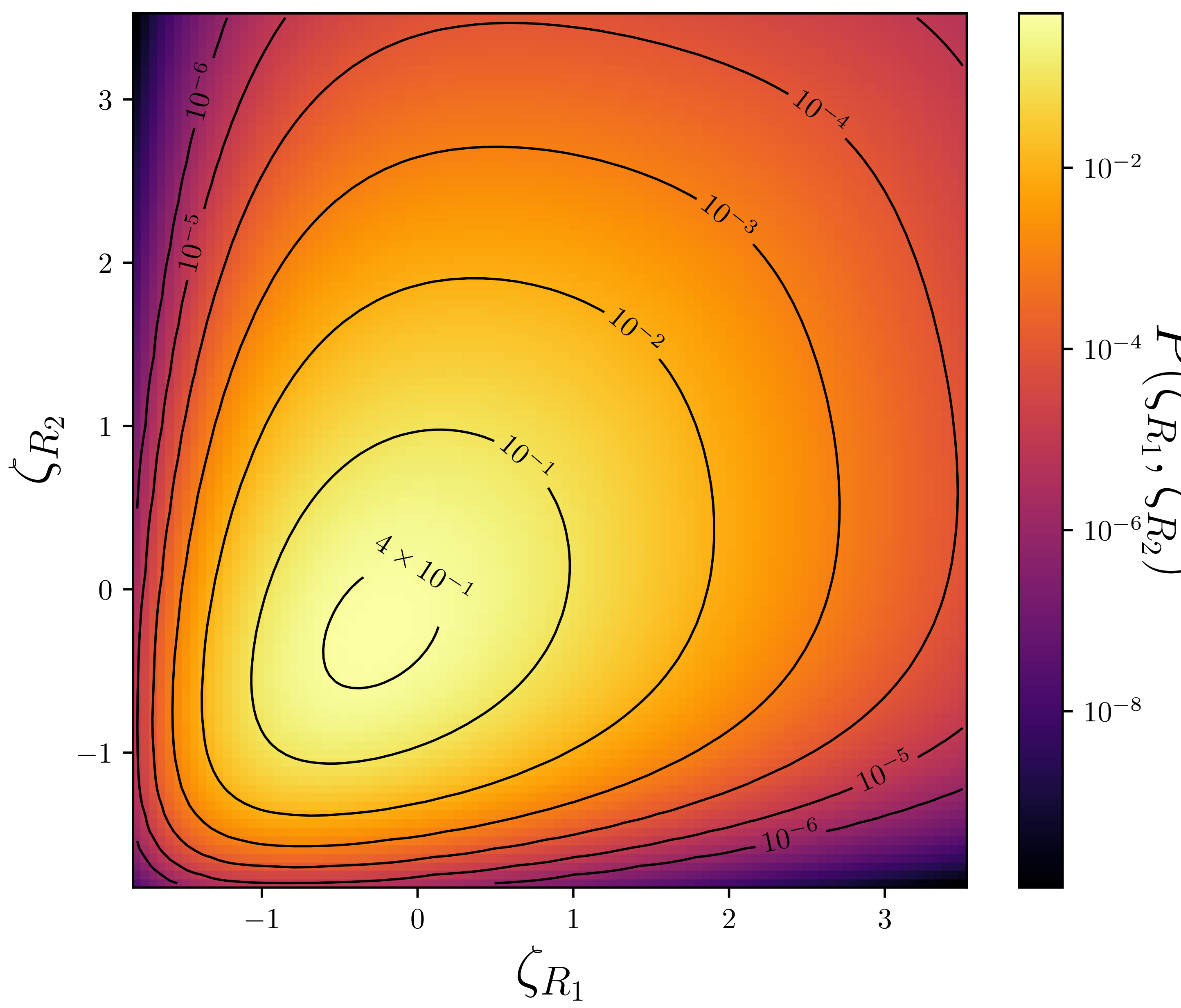


For $x_* \to 1$ the two-point distribution factorises: $\xi_{M_1,M_2} \to 0$

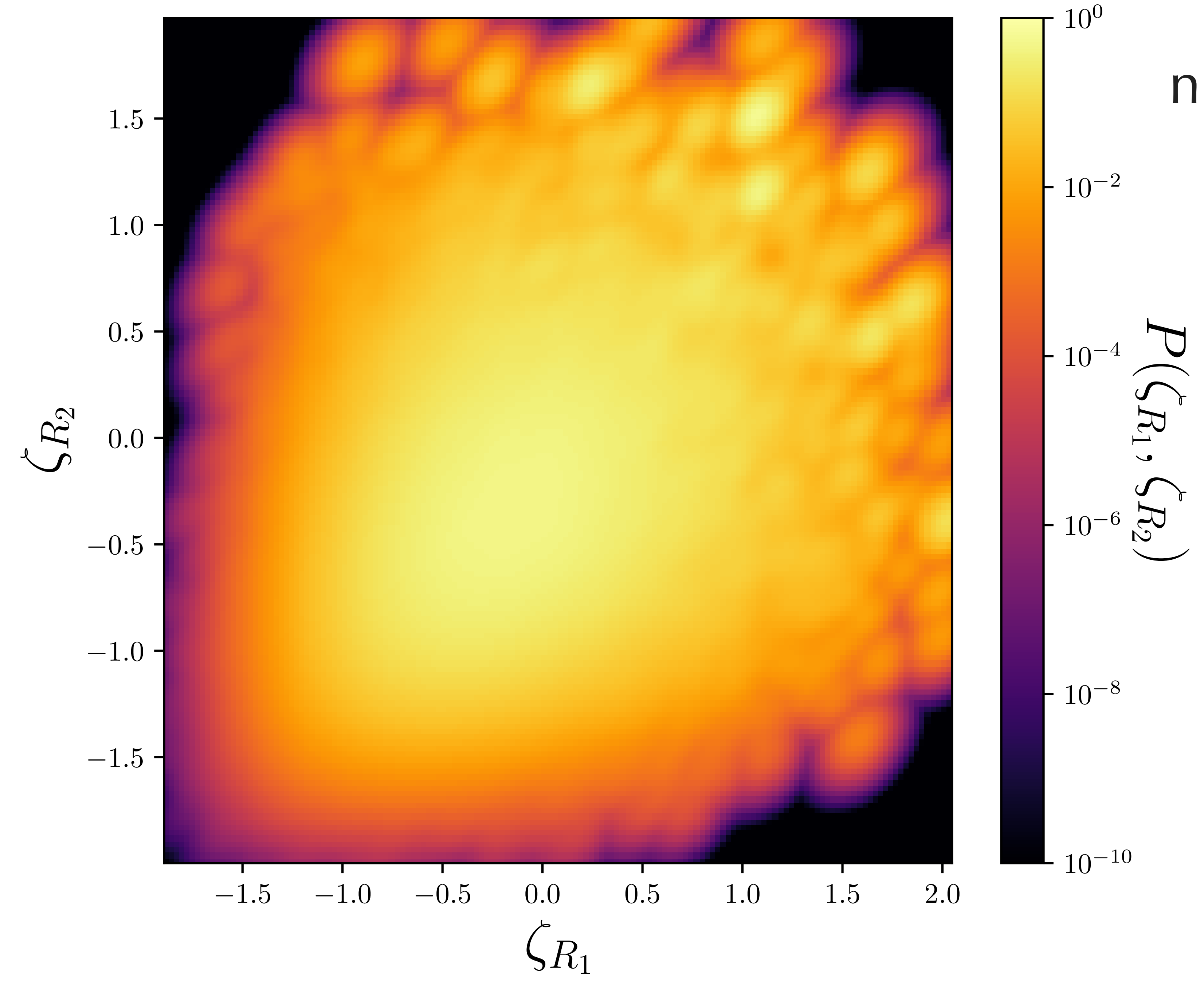
For small values of x_*: ξ_{M_1,M_2} reaches a maximum when $r \simeq R_1 + R_2$
(for smaller values one enters the exclusion zone)

Two-point distributions: tilted-well model

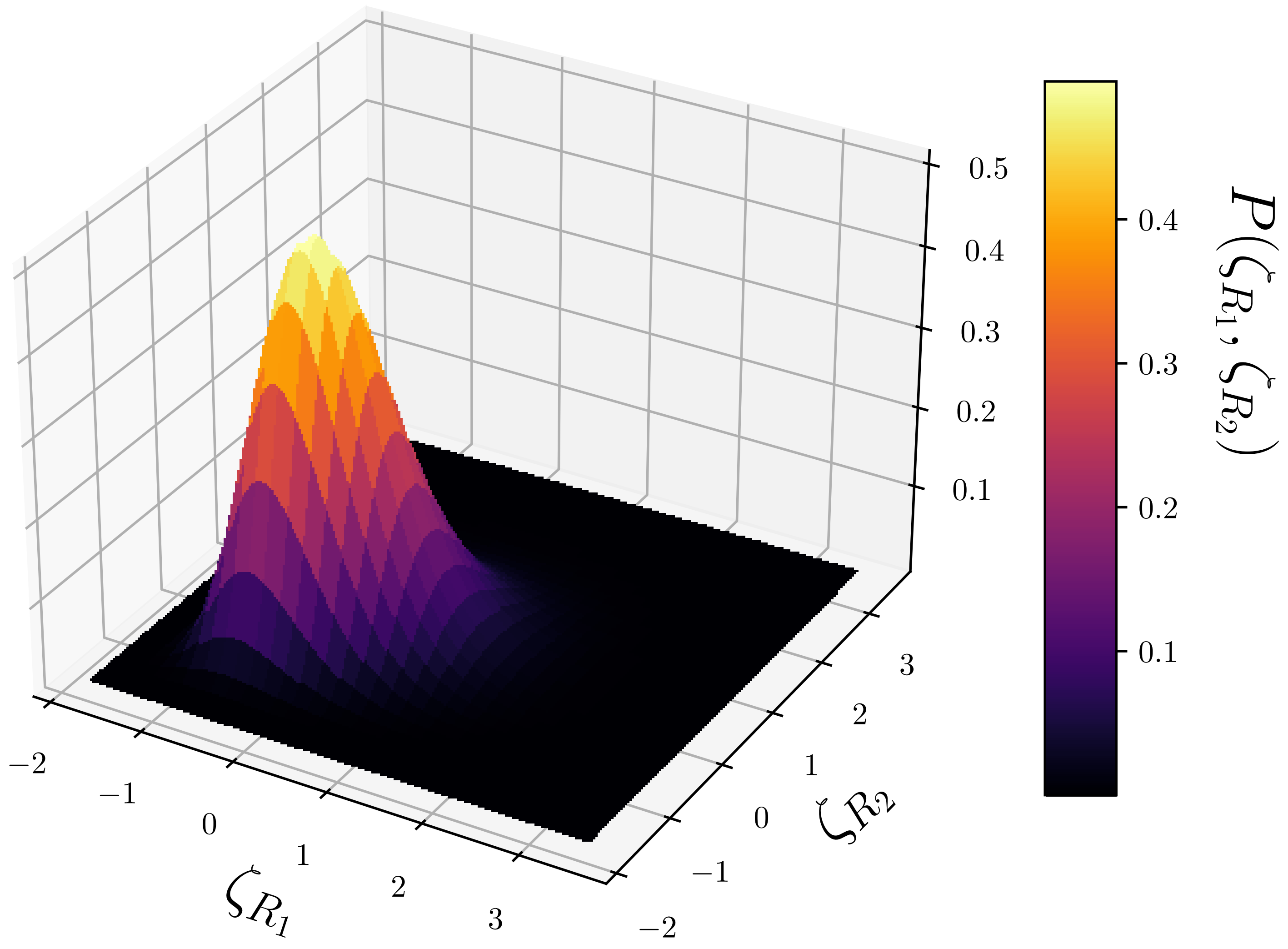
$d\mu^2 = 125$

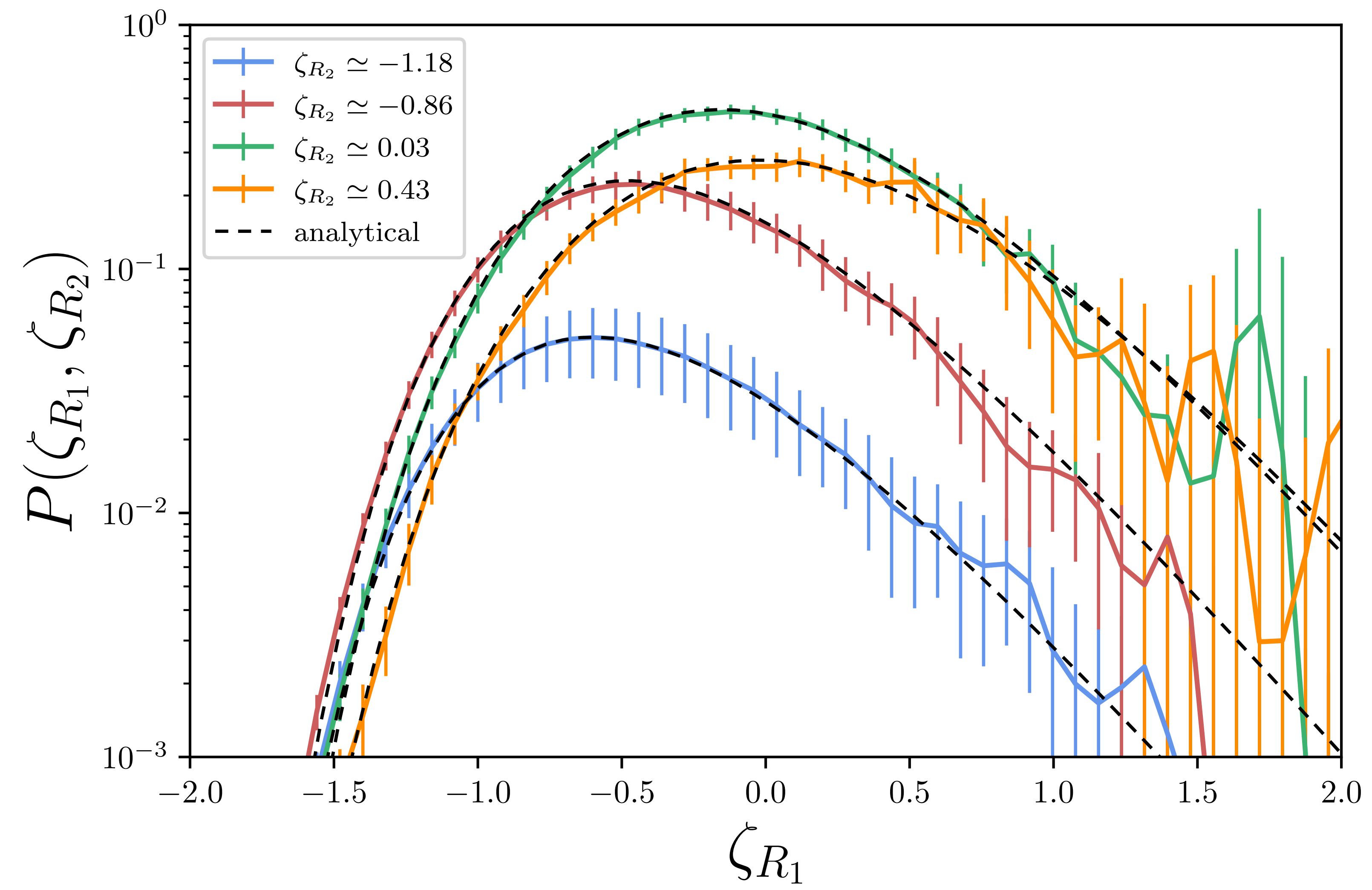


analytical approx. results



numerical simulations





Stochastic-δN formalism

Full PDF of the first passage time: characteristic function

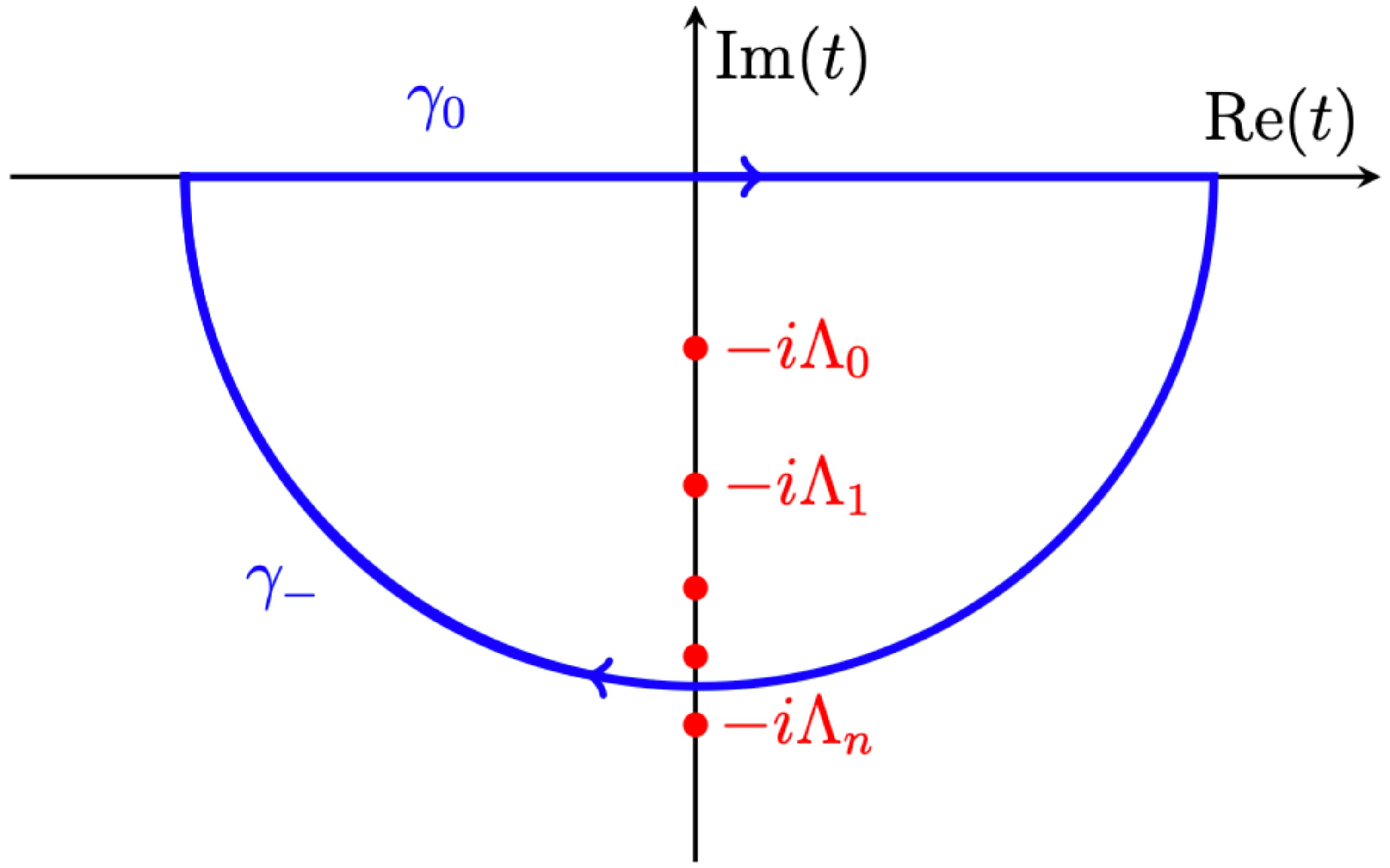
$$\chi(t,\phi) \equiv \langle e^{it\mathcal{N}} \rangle = \int_{-\infty}^{\infty} e^{it\mathcal{N}} P(\mathcal{N},\phi)\, d\mathcal{N} \longrightarrow P(\mathcal{N},\phi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-it\mathcal{N}}\chi(t,\phi)\,dt$$

$$\mathcal{L}_{FP}^{\dagger}\cdot\chi(t,\phi) = -\,i\,t\,\chi(t,\phi) \qquad \chi(t,\phi_{end}) = 1$$

- Useful trick: pole expansion

$$\chi(t,\phi) = \sum_n \frac{a_n(\phi)}{\Lambda_n - i\,t} + g(t,\phi) \longrightarrow P(\mathcal{N},\phi) = \sum_n a_n(\phi)\,e^{-\Lambda_n \mathcal{N}}$$

$$0 < \Lambda_0 < \Lambda_1 < \cdots \Lambda_n$$



[J.M. Ezquiaga, J. Garcia-Bellido, V. Vennin (2020)]

- Main task: find **poles** and **residues** of the characteristic function

Poles: zeros of the inverse characteristic function

Residues: $a_n(\phi) = -\,i\left[\frac{\partial}{\partial t}\chi^{-1}(t = -\,i\Lambda_n, \phi)\right]^{-1}$

Tail expansion: higher n terms suppressed at large $\mathcal{N}$

Tail of the PDF for ζ has an exponential fall-off behaviour

Comparison with the classical limit

Leading order in perturbation theory:
curvature perturbation ζ (and also its coarse-grained version ζ_R) features **Gaussian statistics**

variance: $\sigma_R^2 \equiv \langle \zeta_R^2 \rangle = \int_0^{a/R} \mathrm{d} \ln k \mathscr{P}_\zeta(k)$ $\quad \mathscr{P}_\zeta = 2v_0/\alpha^2$ in the titled-well model

$0 \rightarrow k_{\mathrm{IR}} = a_{\mathrm{end}} H e^{-1/d}$

1-pt probability:

$$p_M = \frac{1}{2} \mathrm{erfc}\left(\frac{\zeta_c}{\sqrt{2\sigma_R^2}} \right)$$

covariance: $\sigma_{R_1}^2, \sigma_{R_2}^2, \tau_r^2 = \langle \zeta_{R_1}(\vec{x}) \zeta_{R_2}(\vec{x} + \vec{r}) \rangle = \int_0^{a/r} \mathrm{d} \ln k \mathscr{P}_\zeta(k)$

$0 \rightarrow k_{\mathrm{IR}} = a_{\mathrm{end}} H e^{-1/d}$

2-pt probability:

$$p_{M,M}(r) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{d}x e^{-x^2/2} \mathrm{erfc}\left[\frac{\zeta_c}{\sqrt{\sigma_R^2 + \tau_r^2}} \left(1 + \sqrt{\frac{\sigma_R^2 - \tau_r^2}{2}} \frac{x}{\zeta_c} \right) \right]$$

Analytical results in the flat-well model

1-pt distribution:

$$P\left(\zeta_R\right) = -\frac{\pi \cos\left[\sqrt{3}\left(1-x_*\right)\mu\right]}{2(1-x_*)^2\mu^2}\vartheta_2'\left(\frac{\pi}{2}, e^{-\frac{\pi^2}{(1-x_*)^2\mu^2}\left\{\zeta_R+\frac{\mu}{2\sqrt{3}}(1-x_*)\tan\left[\sqrt{3}\mu(1-x_*)\right]\right\}}\right) \times e^{3\left\{\zeta_R+\frac{\mu}{2\sqrt{3}}(1-x_*)\tan\left[\sqrt{3}\mu(1-x_*)\right]\right\}}$$

2-pt distribution:

$$P(\zeta_{R_1},\zeta_{R_2}) = -\frac{\pi^3}{8\mu^6(1-x_*)^2(1-x_1)^2(1-x_2)^2}\frac{\cos\left[\sqrt{3}\mu(1-x_1)\right]\cos\left[\sqrt{3}\mu(1-x_2)\right]}{\cos\left[\sqrt{3}\mu(1-x_*)\right]}$$

$$\int \mathrm{d}\mathcal{N}_{\phi_0\to\phi_*}\vartheta_2'\left(\frac{\pi}{2}, e^{-\frac{\pi^2}{\mu^2(1-x_*)^2}\mathcal{N}_{\phi_0\to\phi_*}}\right)$$

$$\vartheta_2'\left(\frac{\pi}{2}\frac{x_*-x_1}{1-x_1}, e^{-\frac{\pi^2}{\mu^2(1-x_1)^2}\left(\zeta_{R_1}-\mathcal{N}_{\phi_0\to\phi_*}+\langle\mathcal{N}_{\phi_0}\rangle_{\mathrm{V}}-\langle\mathcal{N}_{\phi_1}\rangle_{\mathrm{V}}\right)}\right)$$

$$\vartheta_2'\left(\frac{\pi}{2}\frac{x_*-x_2}{1-x_2}, e^{-\frac{\pi^2}{\mu^2(1-x_2)^2}\left(\zeta_{R_2}-\mathcal{N}_{\phi_0\to\phi_*}+\langle\mathcal{N}_{\phi_0}\rangle_{\mathrm{V}}-\langle\mathcal{N}_{\phi_2}\rangle_{\mathrm{V}}\right)}\right)$$

$$e^{3\left(\zeta_{R_1}+\zeta_{R_2}-\mathcal{N}_{\phi_0\to\phi_*}+2\langle\mathcal{N}_{\phi_0}\rangle_{\mathrm{V}}-\langle\mathcal{N}_{\phi_1}\rangle_{\mathrm{V}}-\langle\mathcal{N}_{\phi_2}\rangle_{\mathrm{V}}\right)}$$

Volume-averaged number of e-folds:

$$\langle\mathcal{N}\rangle_{\mathrm{V}} = \frac{\mu}{2\sqrt{3}}\left\{\tan\left(\sqrt{3}\mu\right)-(1-x)\tan\left[\sqrt{3}\mu\left(1-x\right)\right]\right\}$$

Field values - coarse-graining size relation:

$$x_*(R) = 1-\frac{1}{\sqrt{3}\mu}\arccos\left[(\sigma RH)^3\cos\left(\sqrt{3}\mu\right)\right]$$

Analytical results in the tilted-well model

Characteristic function: $$\chi_{\mathcal{N}}(t,\phi) = e^{\frac{d\mu^2 x}{2}} \frac{\sqrt{4it-d^2\mu^2}\cos\left(\frac{x-1}{2}\sqrt{4it-d^2\mu^2}\,\mu\right) - d\mu\sin\left(\frac{x-1}{2}\sqrt{4it-d^2\mu^2}\,\mu\right)}{\sqrt{4it-d^2\mu^2}\cos\left(\frac{1}{2}\sqrt{4it-d^2\mu^2}\,\mu\right) + d\mu\sin\left(\frac{1}{2}\sqrt{4it-d^2\mu^2}\,\mu\right)}$$

FPT distribution: $$P_{\mathrm{FPT},\phi}(\mathcal{N}) = -\frac{\pi}{2\mu^2} e^{\mu^2 d\frac{x}{2} - \frac{\mu^2 d^2}{4}\mathcal{N}} \vartheta_3'\left(\frac{\pi}{2}x, e^{-\frac{\pi^2}{\mu^2}\mathcal{N}}\right)$$

Mean volume: $$\left\langle e^{3\mathcal{N}_\phi}\right\rangle = e^{\frac{d\mu^2 x}{2}} \frac{\sqrt{12-d^2\mu^2}\cos\left(\frac{x-1}{2}\sqrt{12-d^2\mu^2}\,\mu\right) - d\mu\sin\left(\frac{x-1}{2}\sqrt{12-d^2\mu^2}\,\mu\right)}{\sqrt{12-d^2\mu^2}\cos\left(\frac{\mu}{2}\sqrt{12-d^2\mu^2}\right) + d\mu\sin\left(\frac{\mu}{2}\sqrt{12-d^2\mu^2}\right)}$$

Mean number of e-folds: $$\langle\mathcal{N}_\phi\rangle = \frac{x}{d} + e^{-d\mu^2}\frac{1-e^{d\mu^2 x}}{d^2\mu^2}$$

Volume-averaged number of e-folds: $$\langle\mathcal{N}_\phi\rangle_{\mathrm{V}} = \left\{x\left(d^2\mu^2-6\right)\sin\left[\frac{\mu}{2}(x-2)\sqrt{12-d^2\mu^2}\right] + 2(d-3x+6)\sin\left(\frac{\mu}{2}x\sqrt{12-d^2\mu^2}\right)\right.$$
$$\left. - d^2\mu^2 x\sqrt{\frac{12}{d^2\mu^2}-1}\cos\left[\frac{\mu}{2}(x-2)\sqrt{12-d^2\mu^2}\right]\right\}$$
$$\left(d^2\mu^2\sqrt{12-d^2\mu^2}\left[\sin\left(\frac{\mu}{2}\sqrt{12-d^2\mu^2}\right) + \sqrt{\frac{12}{d^2\mu^2}-1}\cos\left(\frac{\mu}{2}\sqrt{12-d^2\mu^2}\right)\right]\right.$$
$$\left.\left\{\sqrt{\frac{12}{d^2\mu^2}-1}\cos\left[\frac{\mu}{2}(x-1)\sqrt{12-d^2\mu^2}\right] - \sin\left[\frac{\mu}{2}(x-1)\sqrt{12-d^2\mu^2}\right]\right\}\right)^{-1}$$

"Eternal " inflation

For $P_{\mathrm{FPT},\Phi_0}(\mathcal{N}) \propto e^{-\Lambda\mathcal{N}}$ and $\Lambda \leq 3$ the volume-weighted distribution is not well-defined

Flat well

Mean volume well defined only for $\mu < \mu_c \equiv \pi/(2\sqrt{3})$

$$\langle e^{3\mathcal{N}_\phi}\rangle = \frac{\cos\left[\sqrt{3}\mu(1-x)\right]}{\cos\left(\sqrt{3}\mu\right)}$$

$$P_{\mathrm{FPT},\phi}(\mathcal{N}) = -\frac{\pi}{2\mu^2}\vartheta_2'\left(\frac{\pi}{2}x, e^{-\frac{\pi^2}{\mu^2}\mathcal{N}}\right)$$

If $\mu \ll \mu_c$ the mean volume is order 1 (in σ-Hubble units): large-volume approximation does not apply

→ Need to work at values of μ **close to (but smaller than)** μ_c

Consequence:
for small x_* the tails of the 1-pt distributions $P(\zeta_R)$ are almost flat and $P(\zeta_R)$ peaks at rather large, negative values of ζ_R.
In the large-volume approx. $R_1, R_2 \ll r \to x_1, x_2 \ll 1$, also the 2-pt distribution peaks at large negative values of ζ_{R_1}, ζ_{R_2}

Tilted well

$P_{\mathrm{FPT},\phi}(\mathcal{N}) \propto e^{-(\pi^2/\mu^2 + \mu^2 d^2/4)\mathcal{N}}$ for large $\mathcal{N}$

Convergence conditions: $\alpha^2 > 12v_0$ or $\alpha^2 < 12v_0$ and $\mu < \pi/\sqrt{3-\alpha^2/(4v_0)}$