Compaction function profiles from stochastic inflation

New Horizons in Primordial Black Hole physics
Edinburgh, 17 May 2024

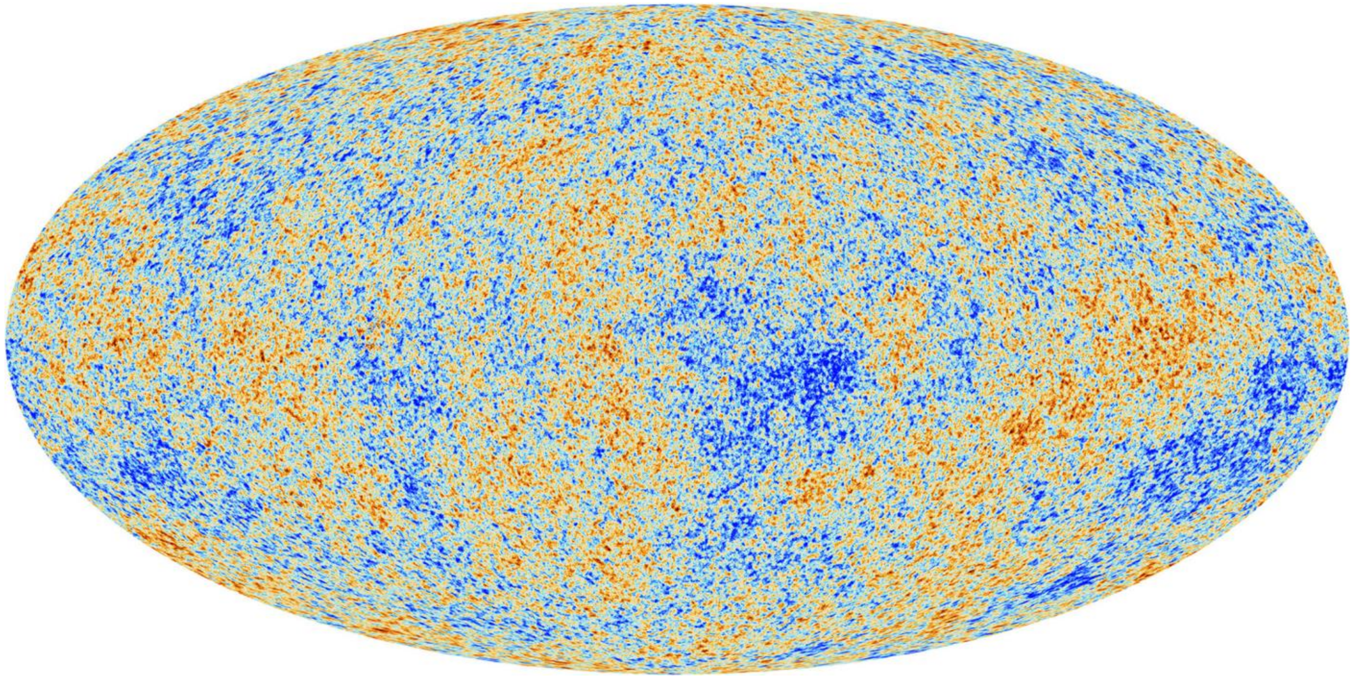
Eemeli Tomberg, Lancaster University

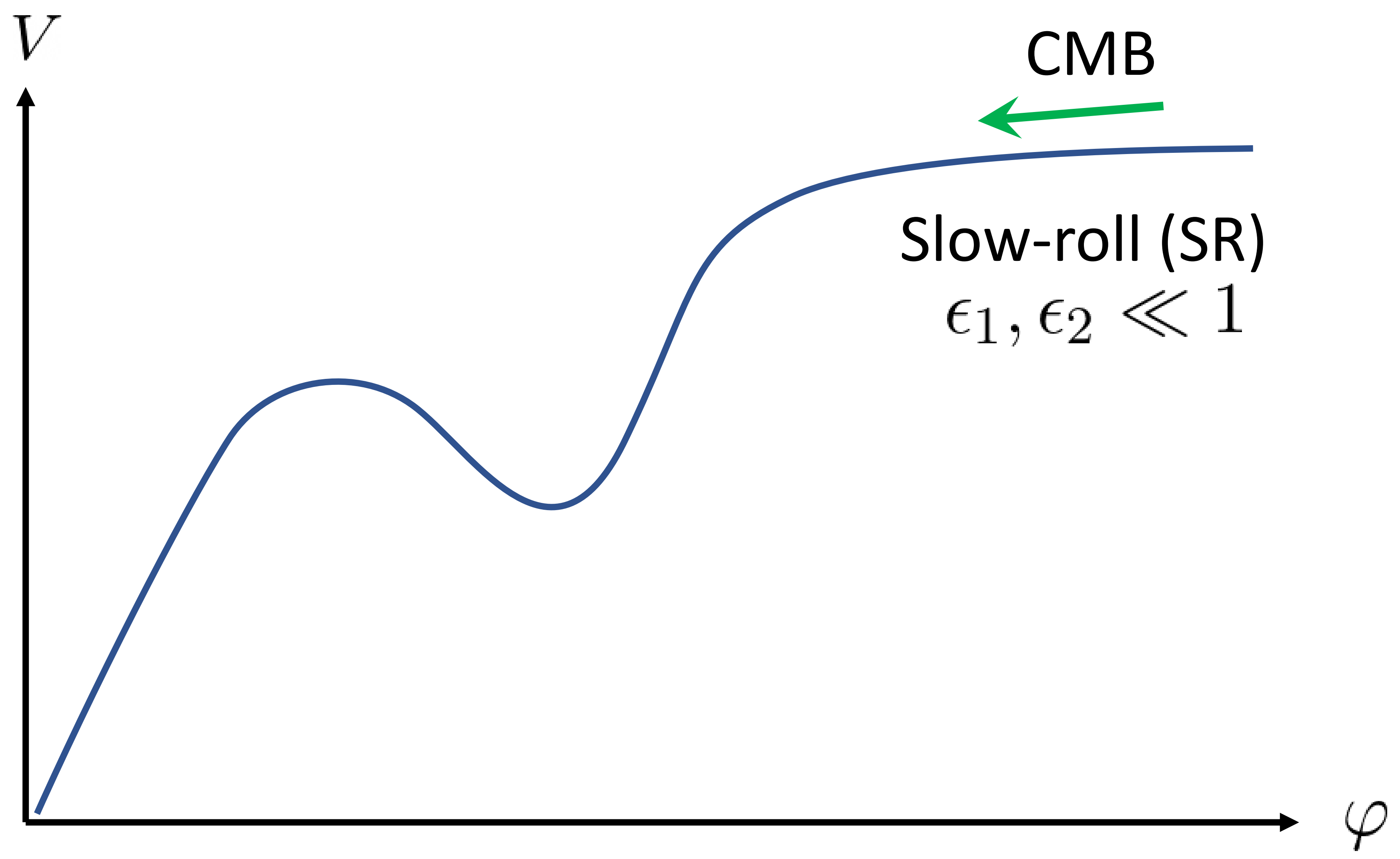
Based on 2012.06551, 2111.07437, 2210.17441, 2304.10903, 2312.12911
in collaboration with D. Figueroa, S. Raatikainen, S. Räsänen

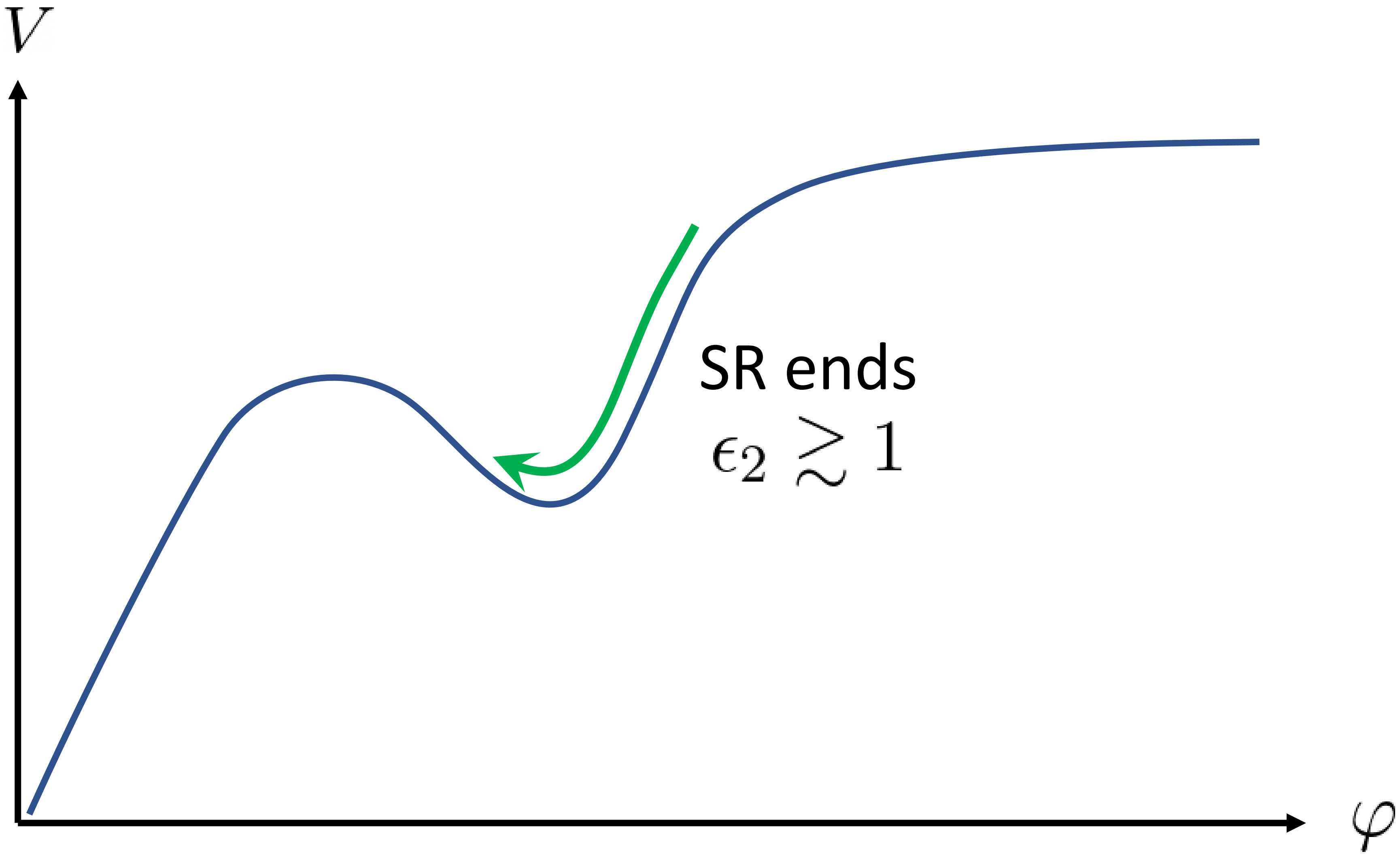
Black holes from primordial perturbations

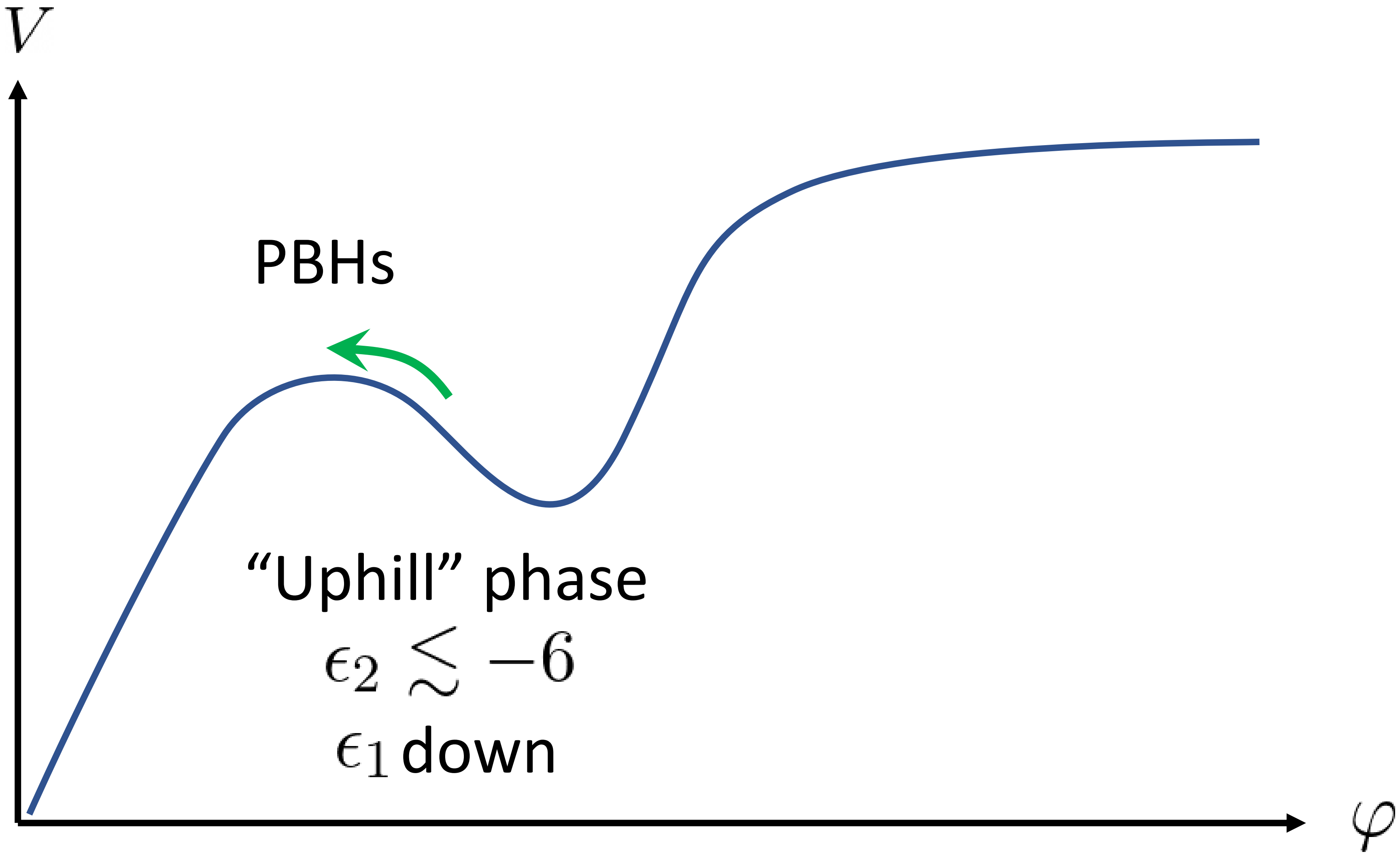
Cosmic inflation: quantum fluctuations

Later: strongest collapse into black holes

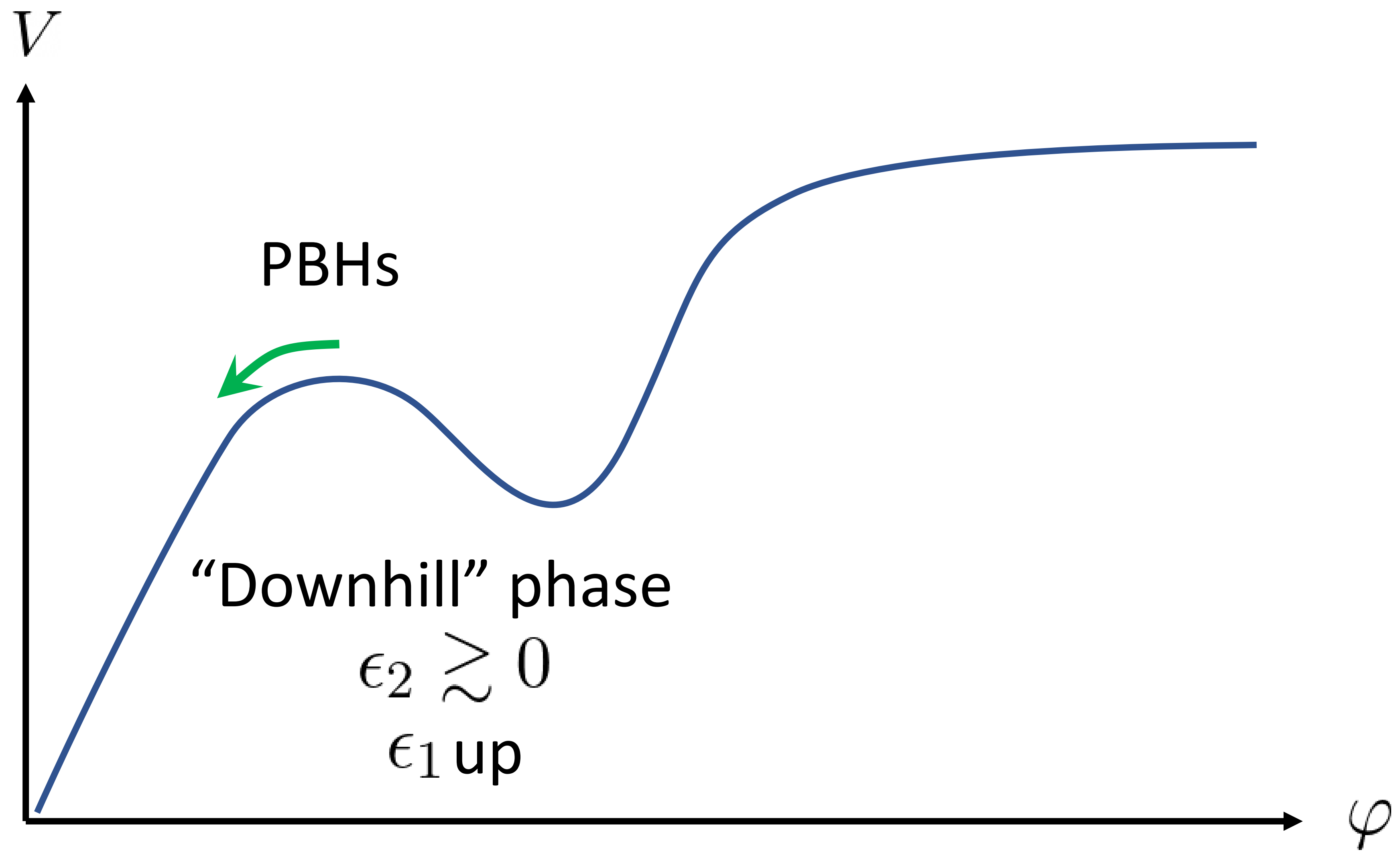



V
CMB
Slow-roll (SR)
$\epsilon_1, \epsilon_2 \ll 1$
φ


V
SR ends
$\epsilon_2 \gtrsim 1$
φ

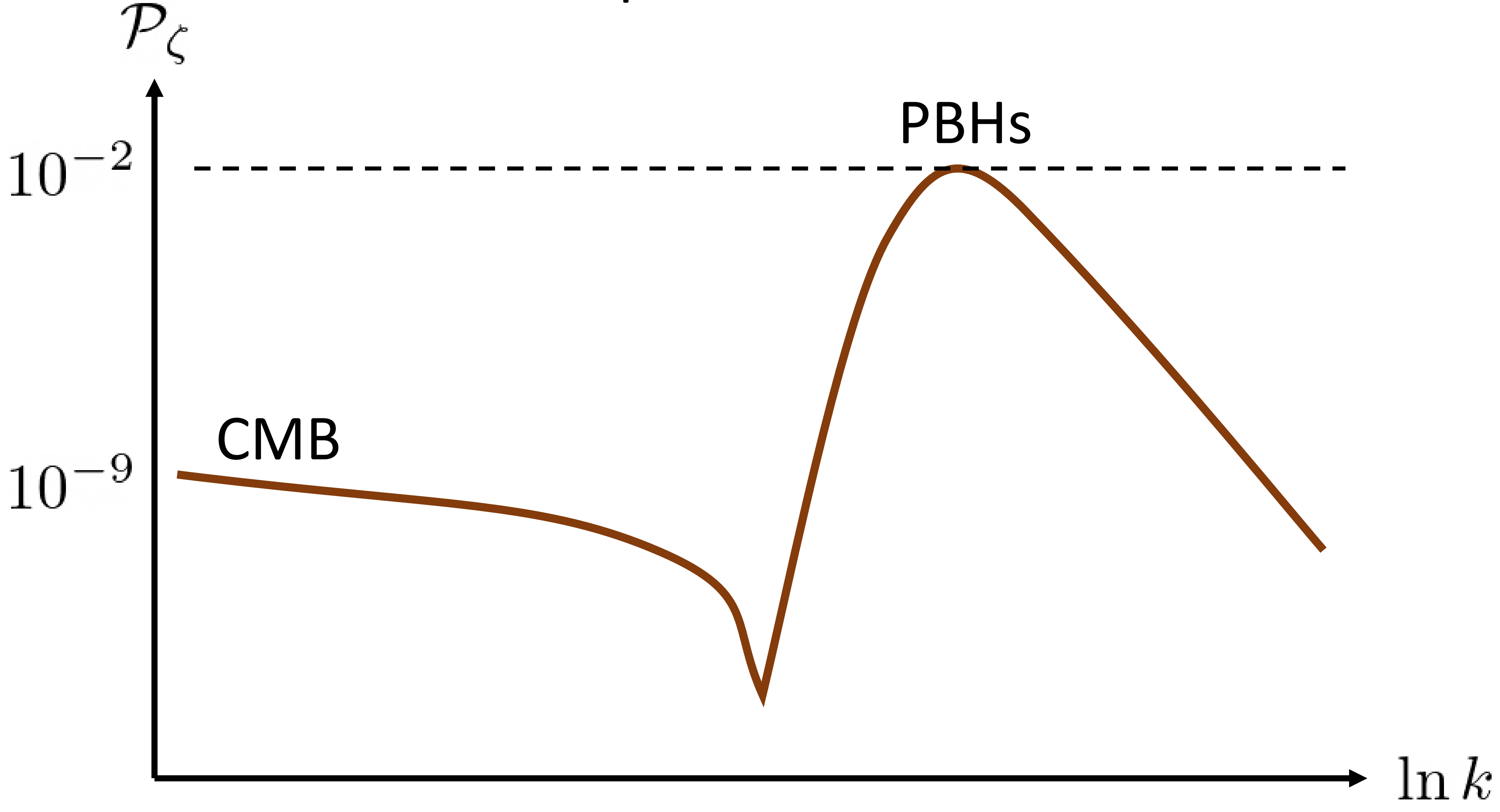


V
PBHs
"Uphill" phase
$\epsilon_2 \lesssim -6$
ϵ_1 down
φ

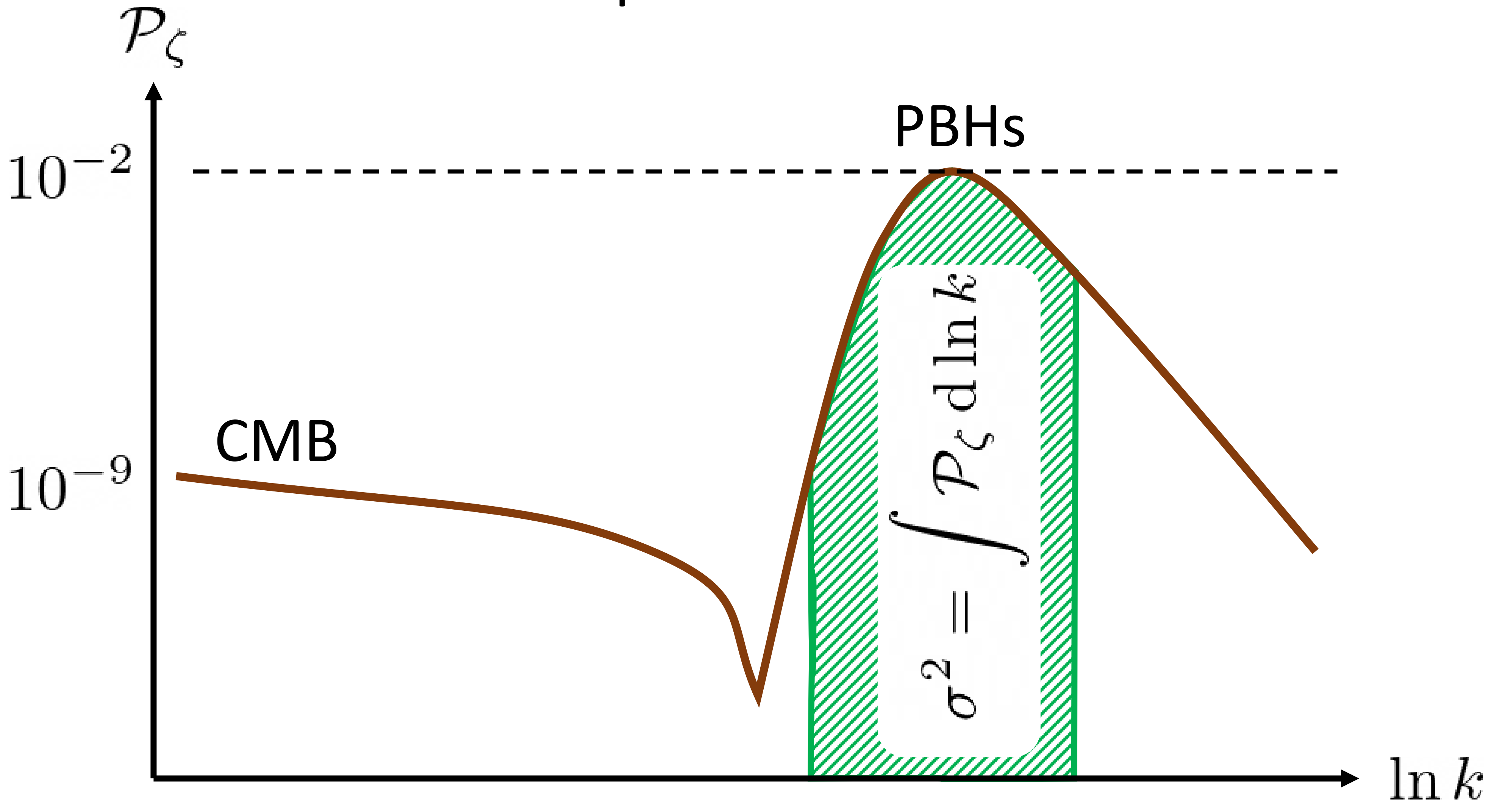


V
PBHs
"Downhill" phase
$\epsilon_2 \gtrsim 0$
ϵ_1 up
φ

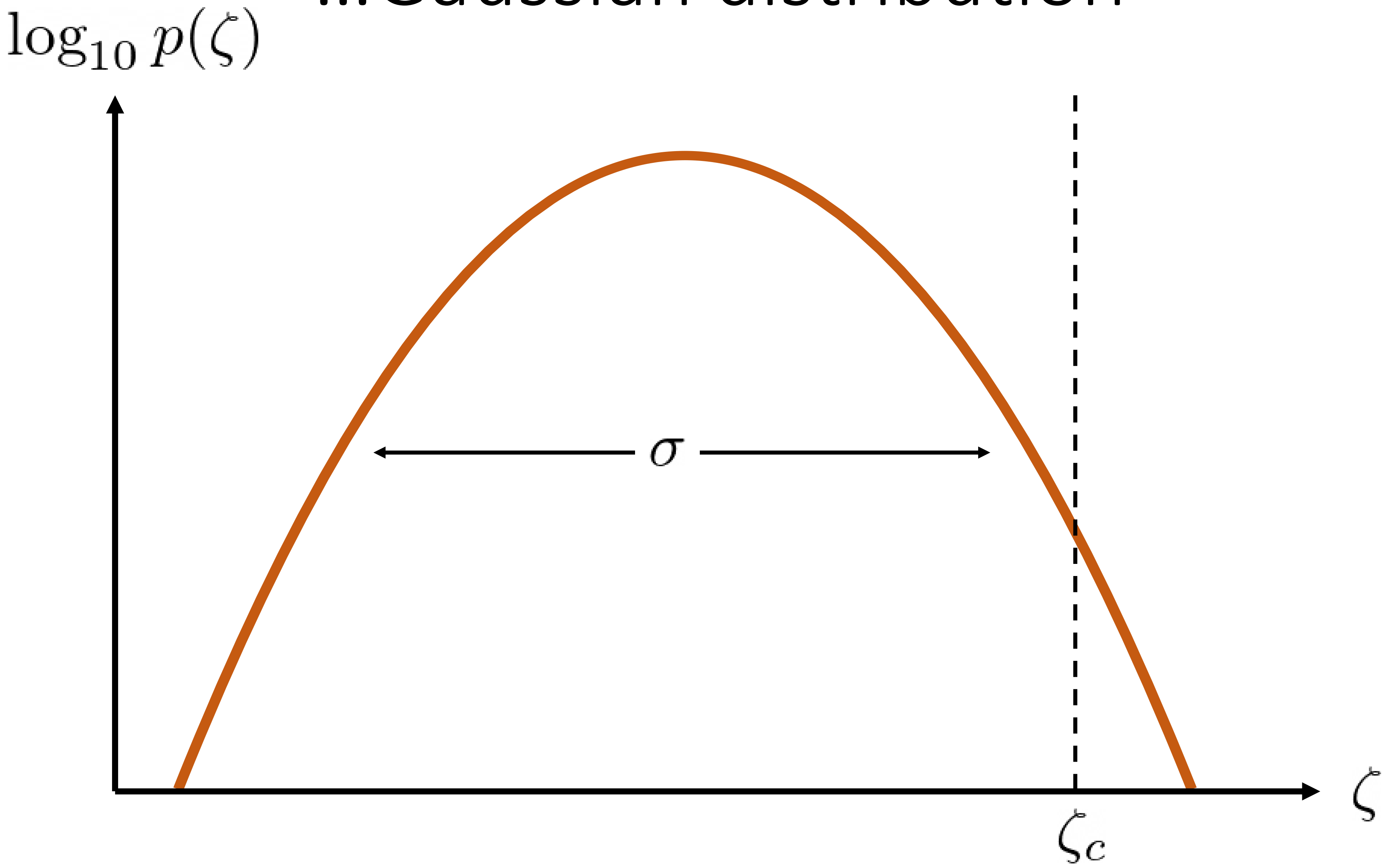
Linear perturbations...



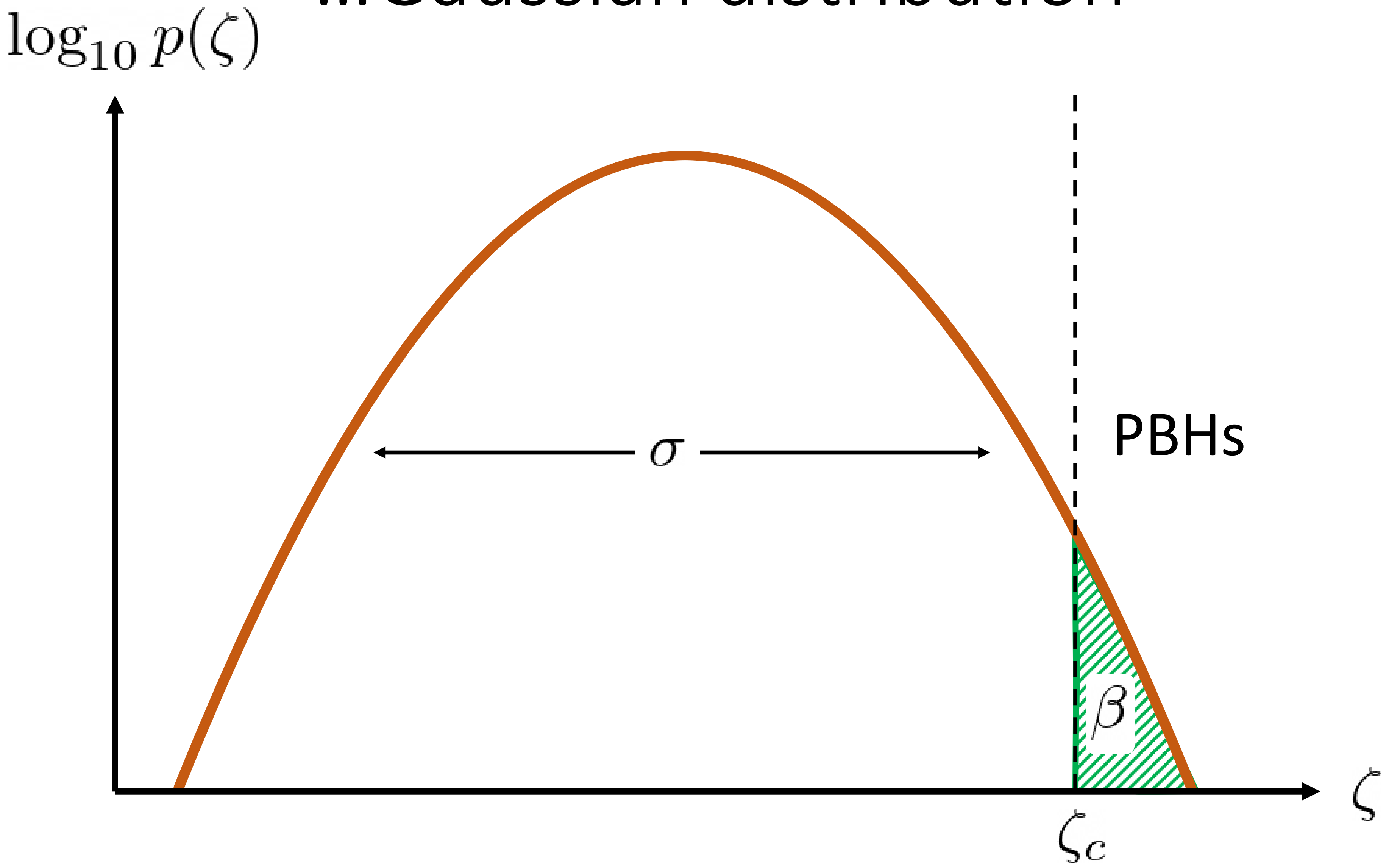
Linear perturbations…



…Gaussian distribution



…Gaussian distribution



Why this is wrong

Why this is wrong

Perturbations in the tail are not Gaussian

Why this is wrong

Perturbations in the tail are not Gaussian

Instead of curvature perturbation: need compaction function

Compaction function

$$\mathcal{C} \equiv 2\frac{M_{\rm MS} - M_{\rm bg}}{R}$$

Collapse: $\mathcal{C}_{\rm max} > \mathcal{C}_c \approx 0.4$

Compaction function

$$\mathcal{C} \equiv 2\frac{M_{\rm MS} - M_{\rm bg}}{R}$$

Collapse: $\mathcal{C}_{\rm max} > \mathcal{C}_c \approx 0.4$

In inflationary variables:

$$\mathcal{C}(r) = \frac{2}{3}(1 - [1 + r\zeta'(r)]^2)$$

Radial profile of non-Gaussian perturbations?

Stochastic inflation

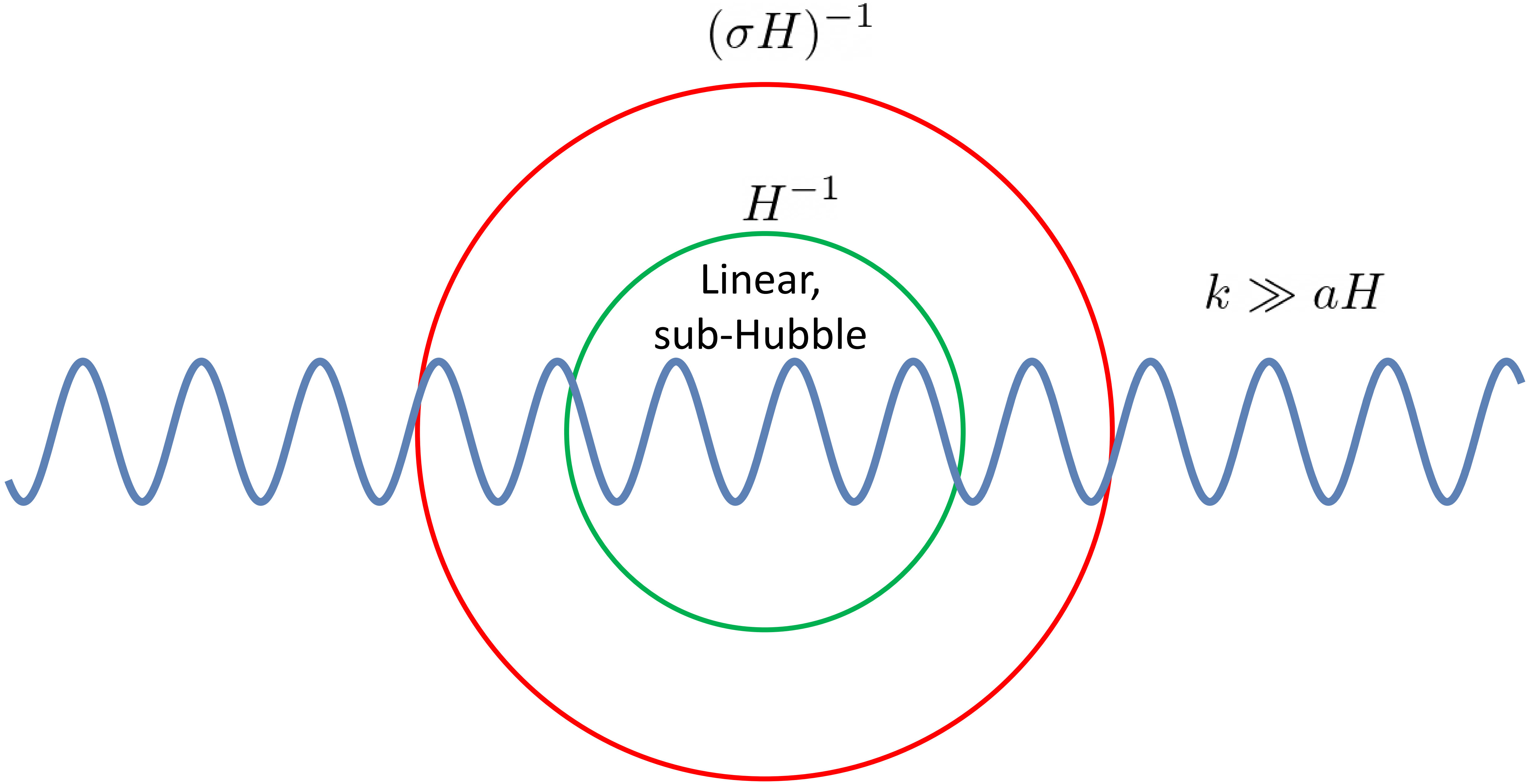
Inflaton field: $\varphi = \phi + \delta\phi$

Coarse-grained:
FLRW

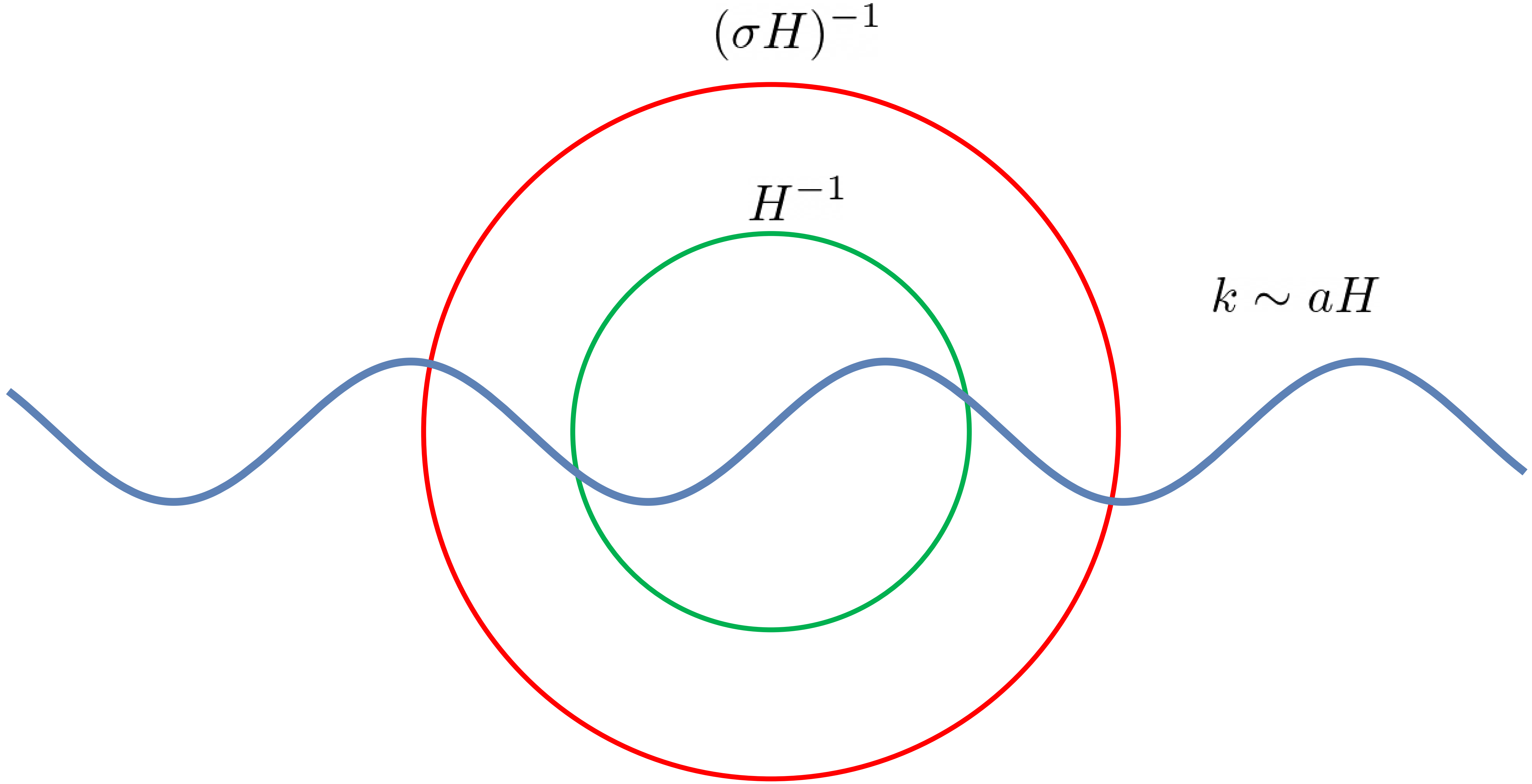
Short-wavelength:
linear perturbation theory

$$\phi \equiv \int_{k<k_\sigma} \frac{\mathrm{d}^3 k}{(2\pi)^{2/3}} \varphi_k(N) e^{-i\vec{k}\cdot\vec{x}} \qquad \delta\phi \equiv \int_{k>k_\sigma} \frac{\mathrm{d}^3 k}{(2\pi)^{2/3}} \varphi_k(N) e^{-i\vec{k}\cdot\vec{x}}$$

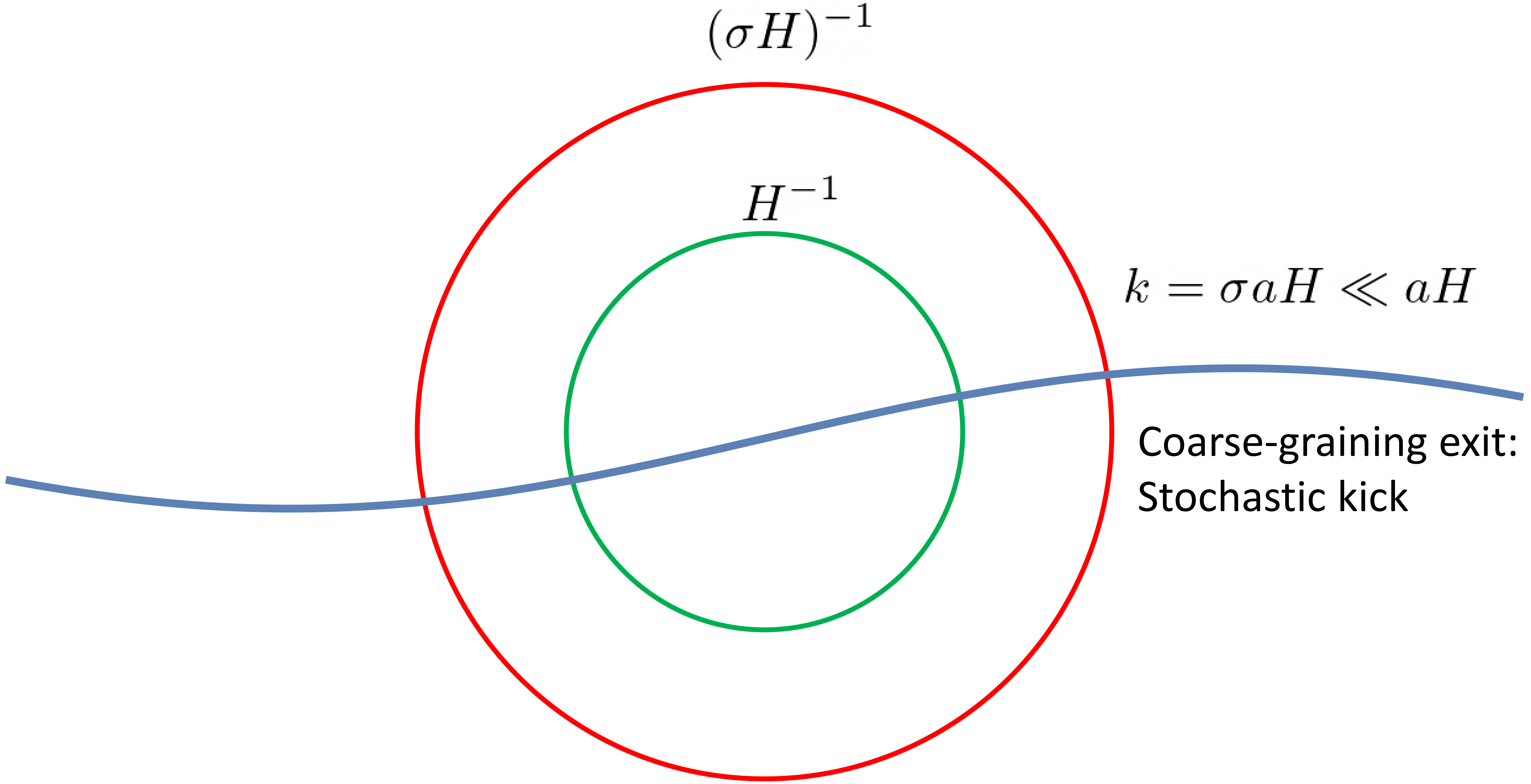
Patched together at the coarse-graining scale $k = k_\sigma \equiv \sigma a H$



$(\sigma H)^{-1}$
H^{-1}
Linear,
sub-Hubble
$k \gg aH$



$(\sigma H)^{-1}$
H^{-1}
$k \sim aH$


$(\sigma H)^{-1}$
H^{-1}
$k = \sigma a H \ll aH$
Coarse-graining exit:
Stochastic kick

Stochastic inflation

$$\phi' = \pi + \xi_\phi\,, \quad \pi' = -\left(3 - \frac{1}{2}\pi^2\right)\pi - \frac{V'(\phi)}{H^2} + \xi_\pi\,, \quad H^2 = \frac{V(\phi)}{3 - \frac{1}{2}\pi^2}$$

$$\delta\phi_k'' = -(3 - \frac{1}{2}\pi^2)\delta\phi_k' - \left[\frac{k^2}{a^2H^2} + \pi^2(3 - \frac{1}{2}\pi^2) + 2\pi\frac{V'(\phi)}{H^2} + \frac{V''(\phi)}{H^2}\right]\delta\phi_k$$

$$\langle\xi_\phi(N)\xi_\phi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}|\delta\phi_{k_\sigma}(N)|^2\delta(N - N')$$

$$\langle\xi_\pi(N)\xi_\pi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}|\delta\phi'_{k_\sigma}(N)|^2\delta(N - N')$$

$$\langle\xi_\phi(N)\xi_\pi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}\delta\phi_{k_\sigma}(N)\delta\phi'^{*}_{k_\sigma}(N)\delta(N - N')$$

$$\zeta_{<k} = \Delta N = N - \bar{N}$$

Stochastic inflation

$$\phi' = \pi + \xi_\phi\,, \quad \pi' = -\left(3 - \frac{1}{2}\pi^2\right)\pi - \frac{V'(\phi)}{H^2} + \xi_\pi\,, \quad H^2 = \frac{V(\phi)}{3 - \frac{1}{2}\pi^2}$$

$$\delta\phi_k'' = -(3 - \frac{1}{2}\pi^2)\delta\phi_k' - \left[\frac{k^2}{a^2H^2} + \pi^2(3 - \frac{1}{2}\pi^2) + 2\pi\frac{V'(\phi)}{H^2} + \frac{V''(\phi)}{H^2}\right]\delta\phi_k$$

$$\langle\xi_\phi(N)\xi_\phi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}|\delta\phi_{k_\sigma}(N)|^2\delta(N-N')$$

$$\langle\xi_\pi(N)\xi_\pi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}|\delta\phi'_{k_\sigma}(N)|^2\delta(N-N')$$

$$\langle\xi_\phi(N)\xi_\pi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}\delta\phi_{k_\sigma}(N)\delta\phi'^*_{k_\sigma}(N)\delta(N-N')$$

$$\zeta_{<k} = \Delta N = N - \bar{N}$$

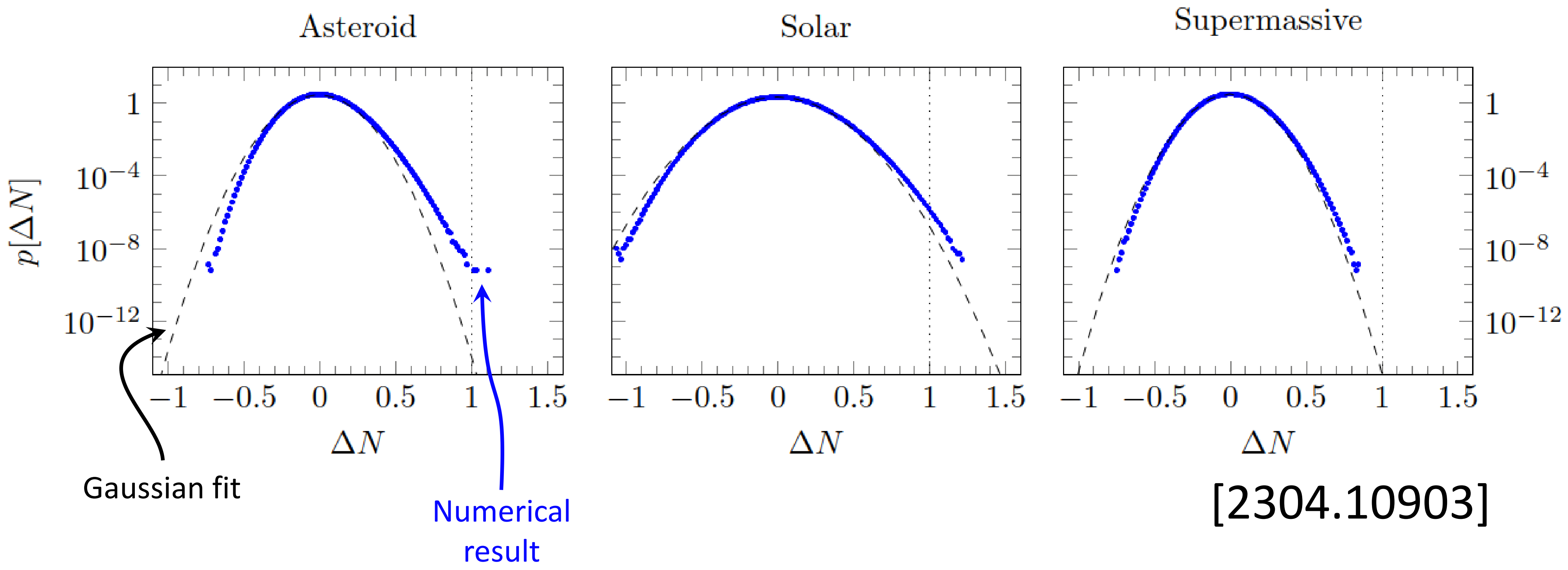
← Curvature perturbations coarse-grained to k-scale

Constant-roll inflation: analytical approximation

$$\zeta_{<k} = -\frac{2}{\epsilon_2} \ln\left(1 - \frac{\epsilon_2}{2} \underbrace{\sum_{\tilde{k}=k_{\min}}^{k} \sqrt{\mathcal{P}_\zeta(\tilde{k})\,\mathrm{d}\ln k}\,\hat{\xi}_{\tilde{k}}}_{\equiv X_{<k}}\right)$$

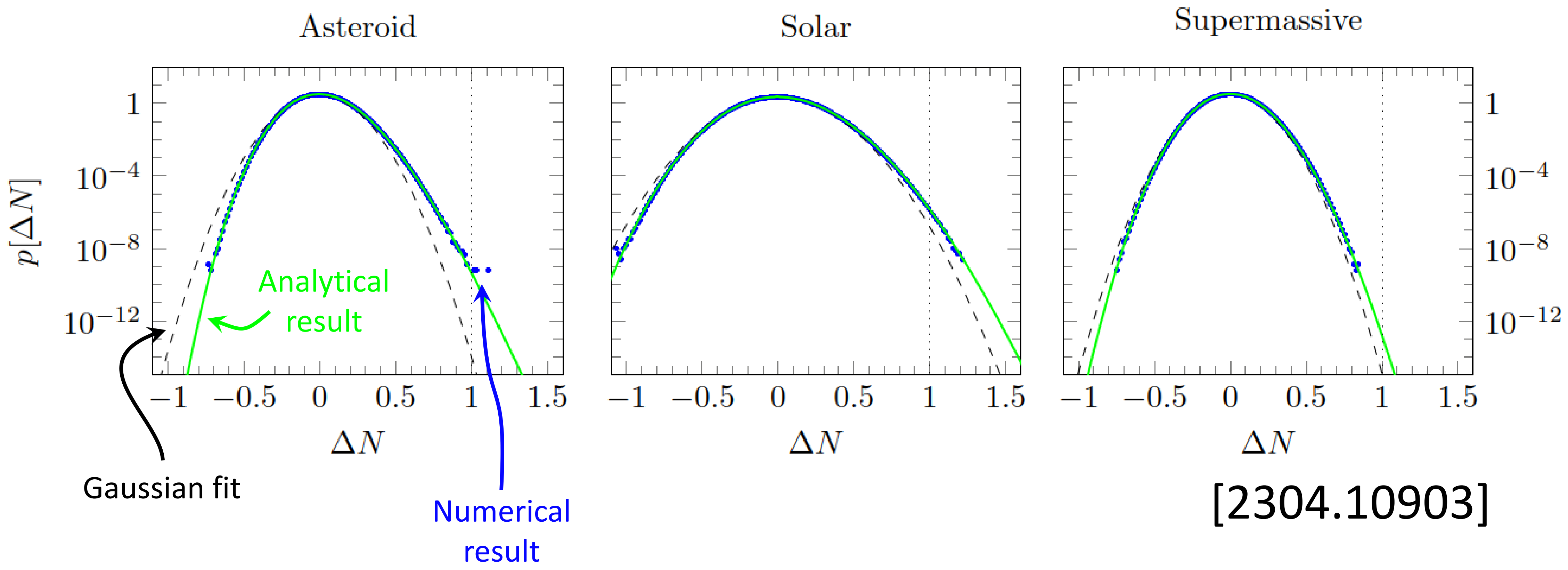
$$p(\zeta_{<k}) = \frac{1}{\sqrt{2\pi}\sigma_k} \exp\left[-\frac{2}{\sigma_k^2\epsilon_2^2}\left(1 - e^{-\frac{\epsilon_2}{2}\zeta_{<k}}\right)^2 - \frac{\epsilon_2}{2}\zeta_{<k}\right]$$

Comparison to numerics



[2304.10903]

Comparison to numerics



[2304.10903]

Recall:

$$\mathcal{C}(r) = \frac{2}{3}(1 - [1 + r\zeta'(r)]^2)$$

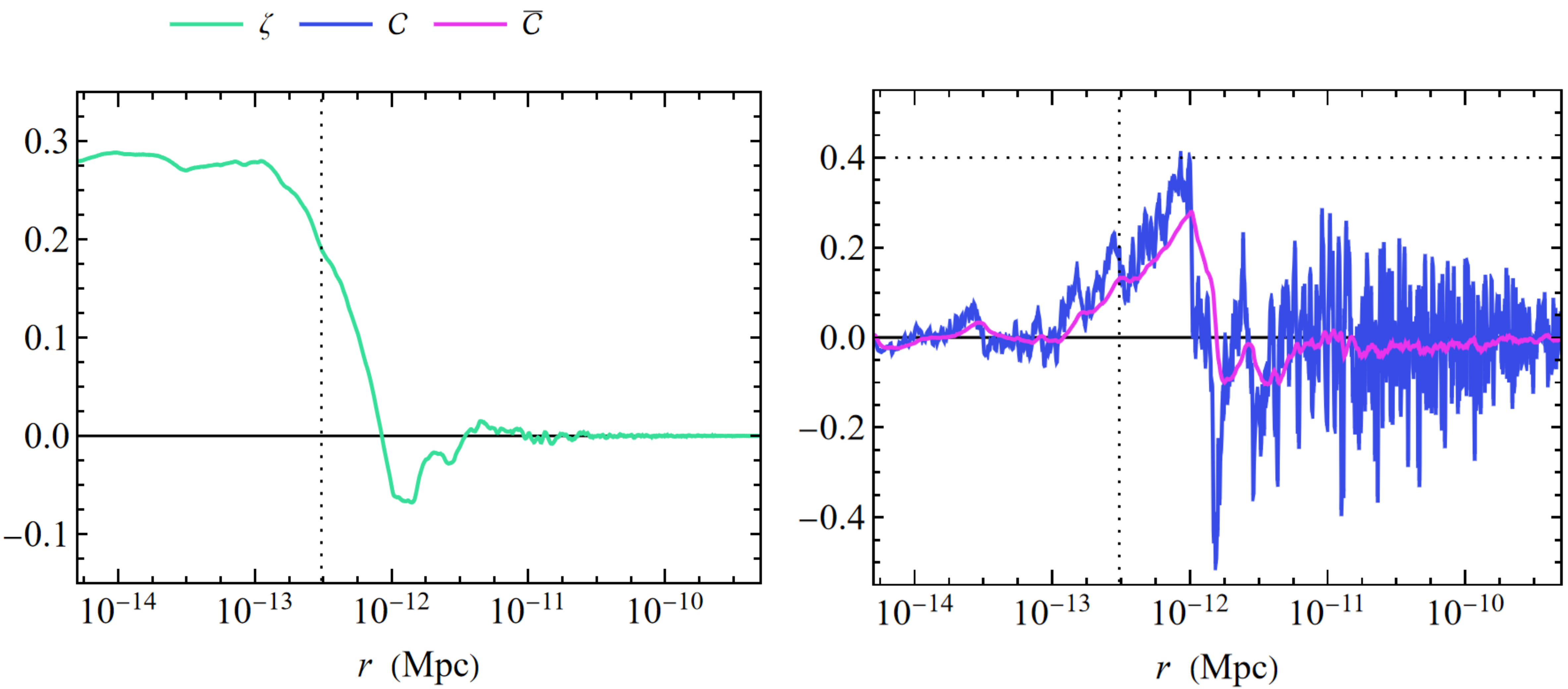
Assuming spherical symmetry…

$$\zeta_k = \frac{\sqrt{2\pi}}{2k^3}\frac{\mathrm{d}\zeta_{<k}}{\mathrm{d}\ln k}$$

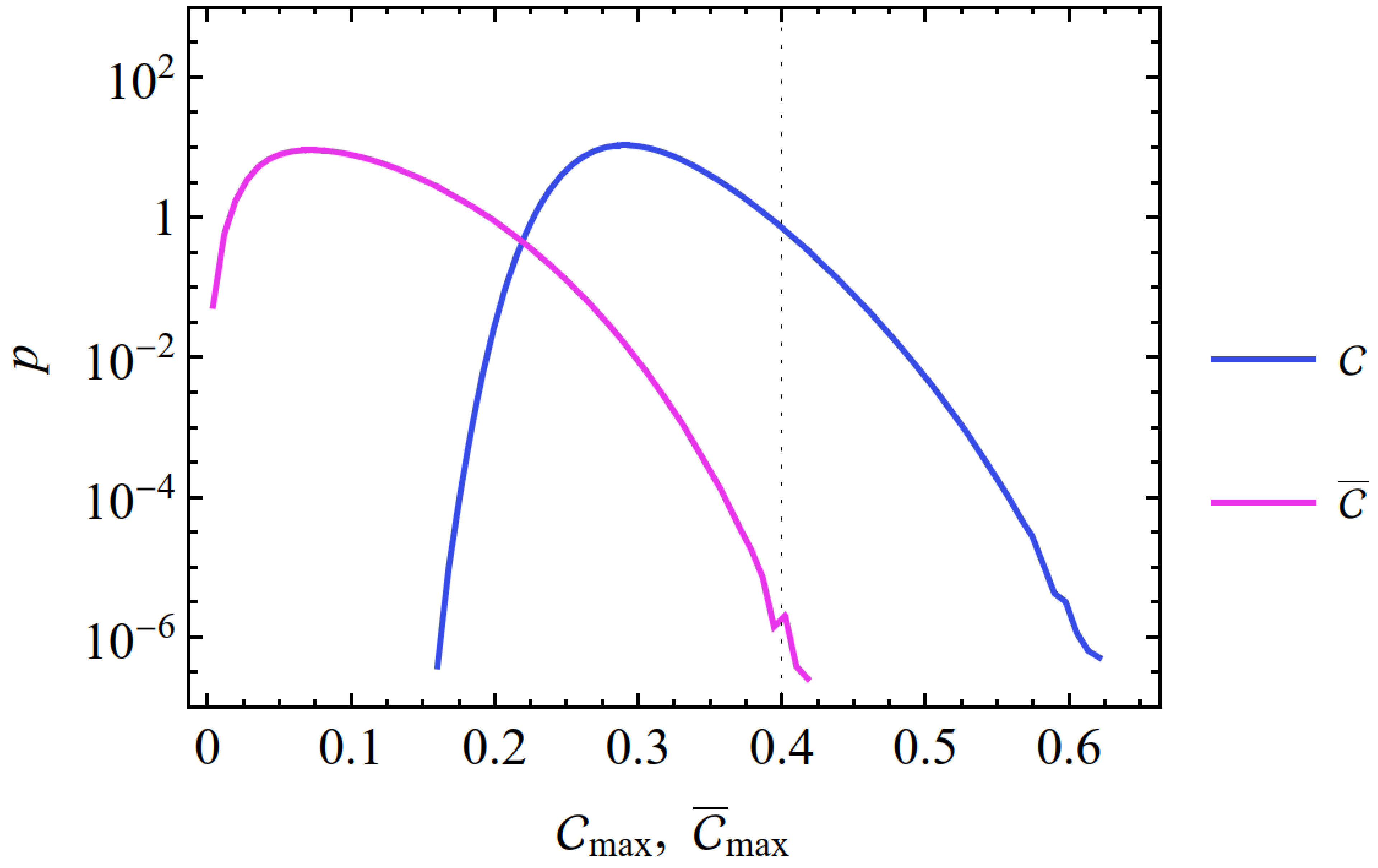
$$r\zeta'(r) = \sum_k \frac{2k^2\,\mathrm{d}k}{\sqrt{2\pi}}\,\zeta_k\left[\cos(kr) - \frac{\sin(kr)}{kr}\right]$$

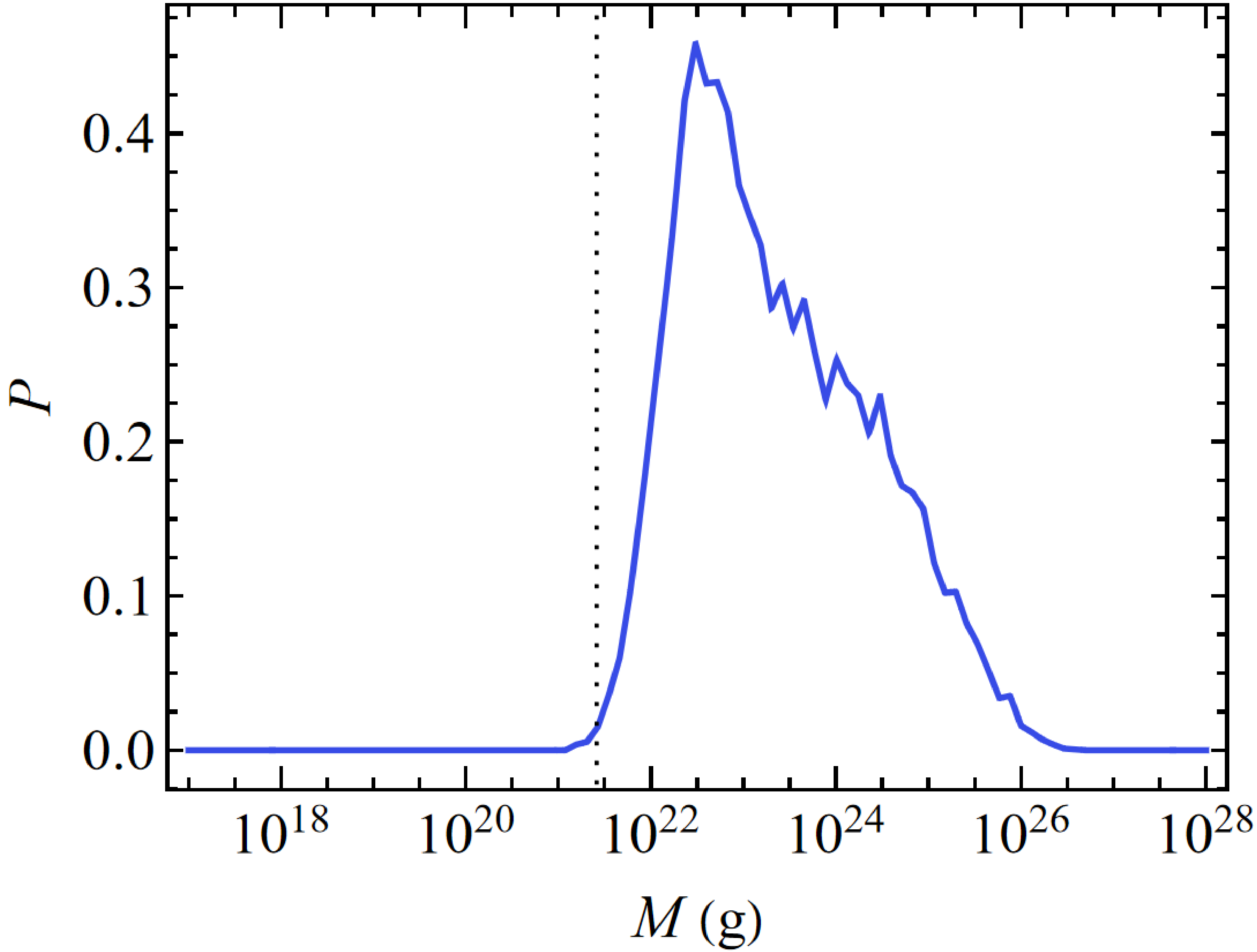
…get master formula

$$r\zeta'(r) = \sum_k \Bigg[-\frac{\hat{\xi}_k}{1 - \frac{\epsilon_2}{2} X(k)} \sqrt{\mathcal{P}_\zeta(k)\, \mathrm{d}\ln k} + \frac{\epsilon_2}{4\left[1 - \frac{\epsilon_2}{2} X(k)\right]^2} \mathcal{P}_\zeta(k)\, \mathrm{d}\ln k \Bigg] \times \left[\cos(kr) - \frac{\sin(kr)}{kr} \right]$$

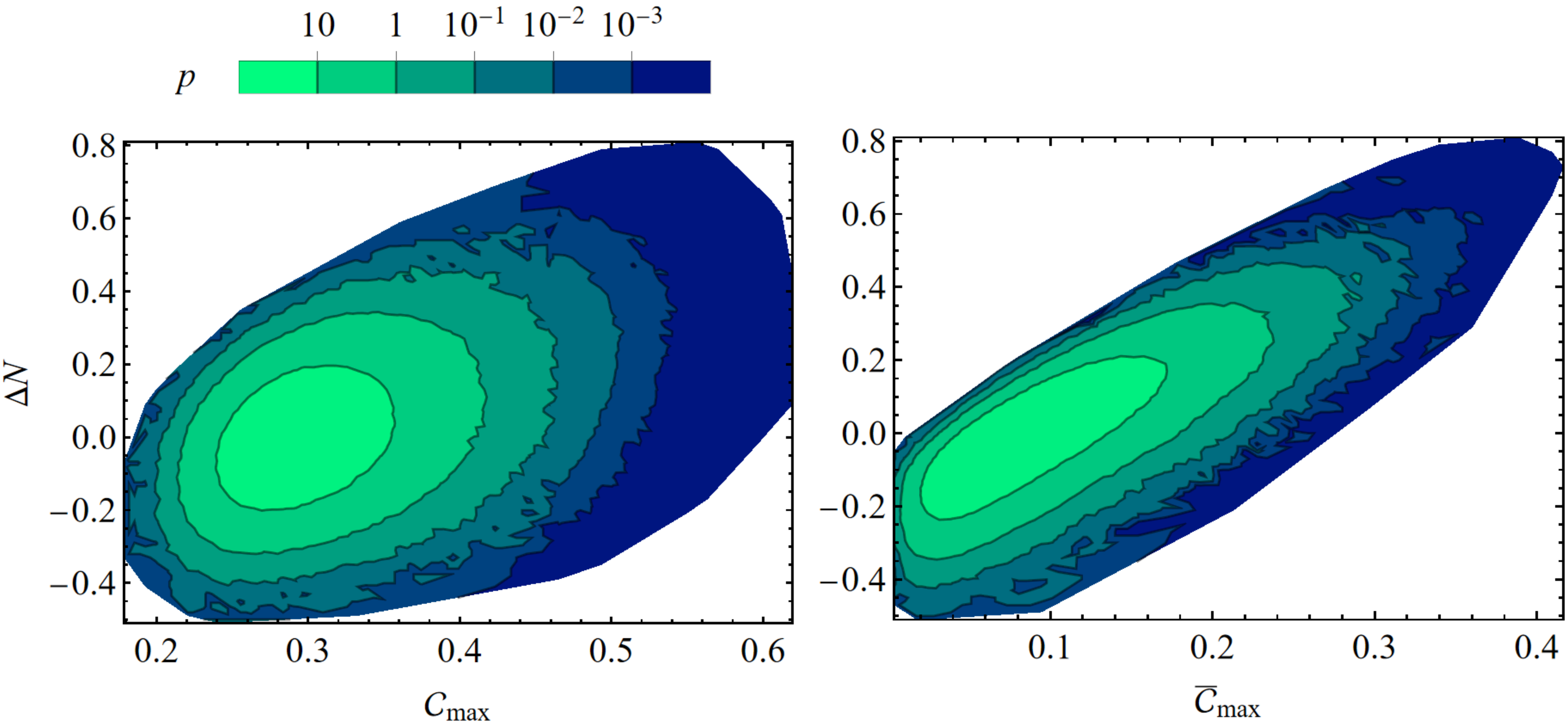


ζ
C
C̄
0.3
0.2
0.1
0.0
−0.1
10^{-14} 10^{-13} 10^{-12} 10^{-11} 10^{-10}
r (Mpc)
0.4
0.2
0.0
−0.2
−0.4
10^{-14} 10^{-13} 10^{-12} 10^{-11} 10^{-10}
r (Mpc)


10^2
1
10^{-2}
10^{-4}
10^{-6}
p
0
0.1
0.2
0.3
0.4
0.5
0.6
$C_{\max}$, $\overline{C}_{\max}$
C
$\overline{C}$



0.4
0.3
0.2
0.1
0.0
P
10^{18}
10^{20}
10^{22}
10^{24}
10^{26}
10^{28}
M (g)



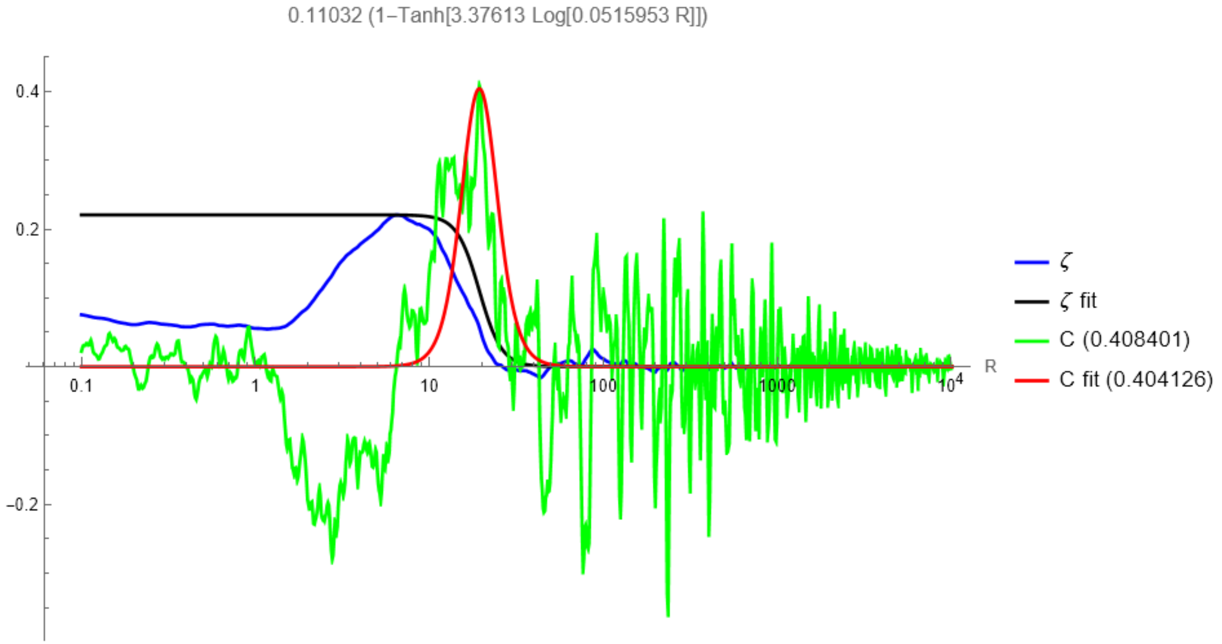
p
10
1
10^{-1}
10^{-2}
10^{-3}
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
ΔN
0.2
0.3
0.4
0.5
0.6
$C_{\max}$
0.1
0.2
0.3
0.4
$\overline{C}_{\max}$

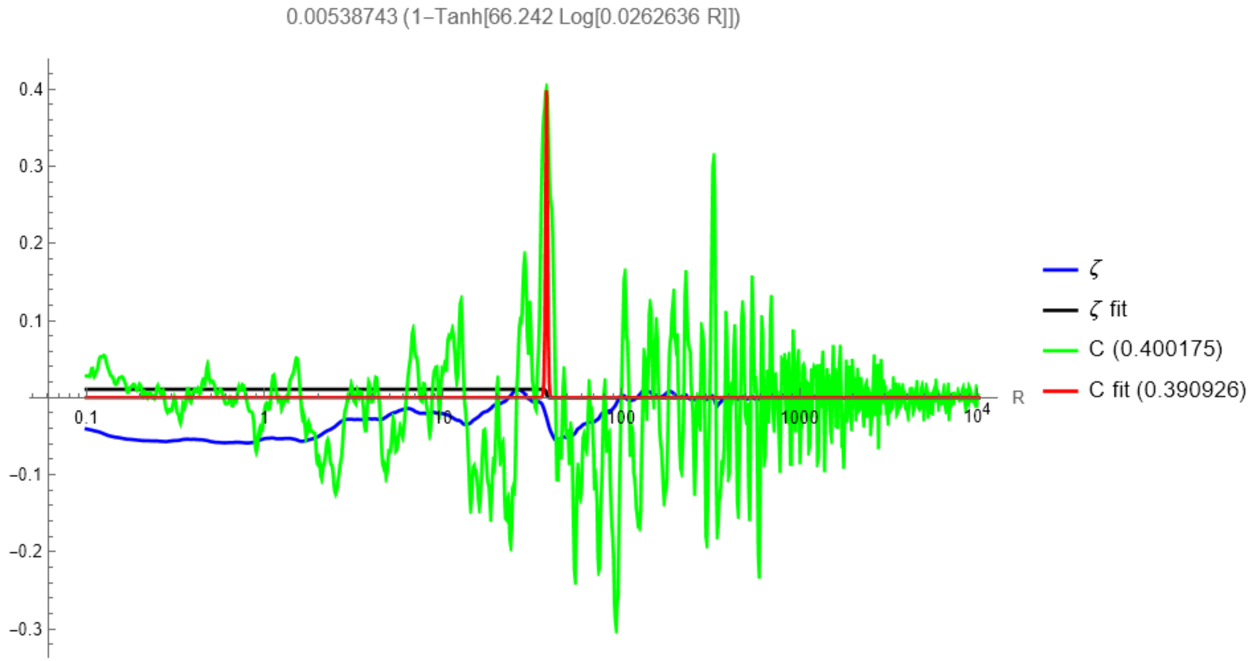
Conclusions

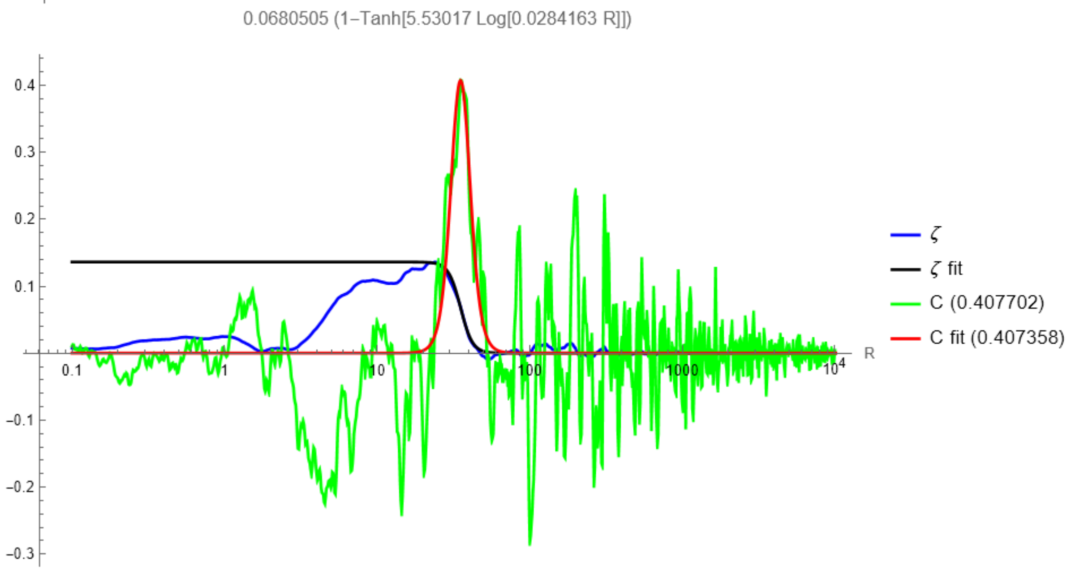
Compaction function formalism needed for accurate PBH predictions

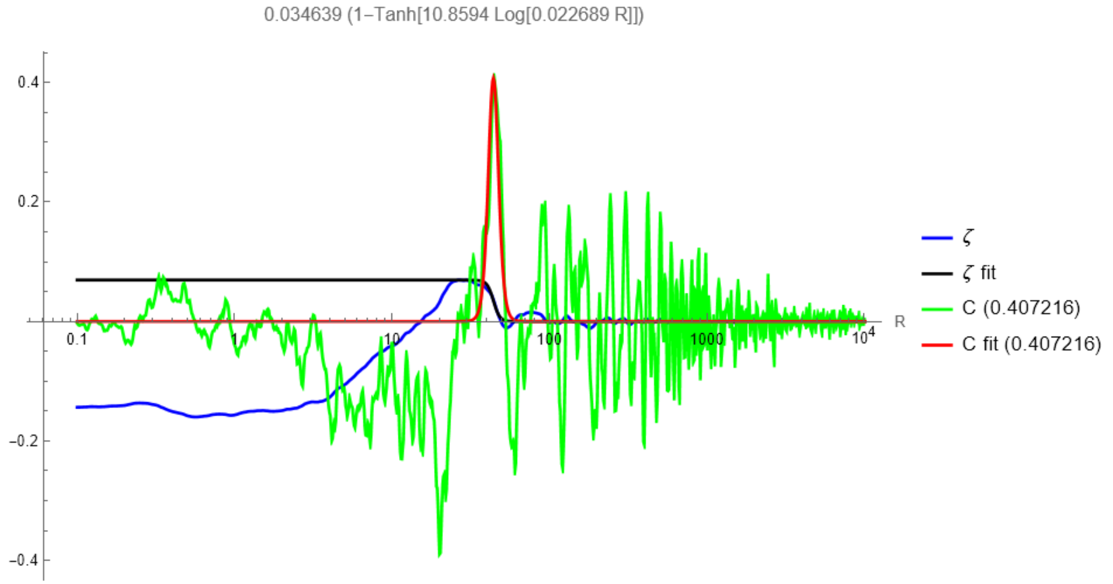
Stochastic inflation gives compaction function profiles including non-Gaussianity

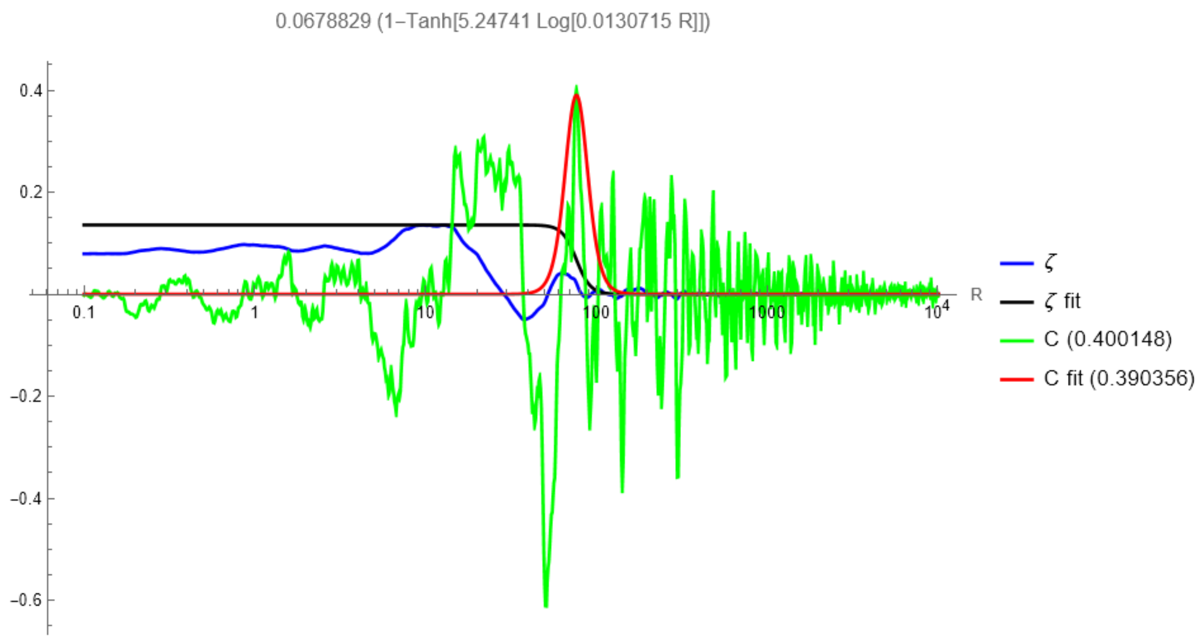
Spiked radial profiles: what to do?


0.11032 (1−Tanh[3.37613 Log[0.0515953 R]])
ζ
ζ fit
C (0.408401)
C fit (0.404126)
R

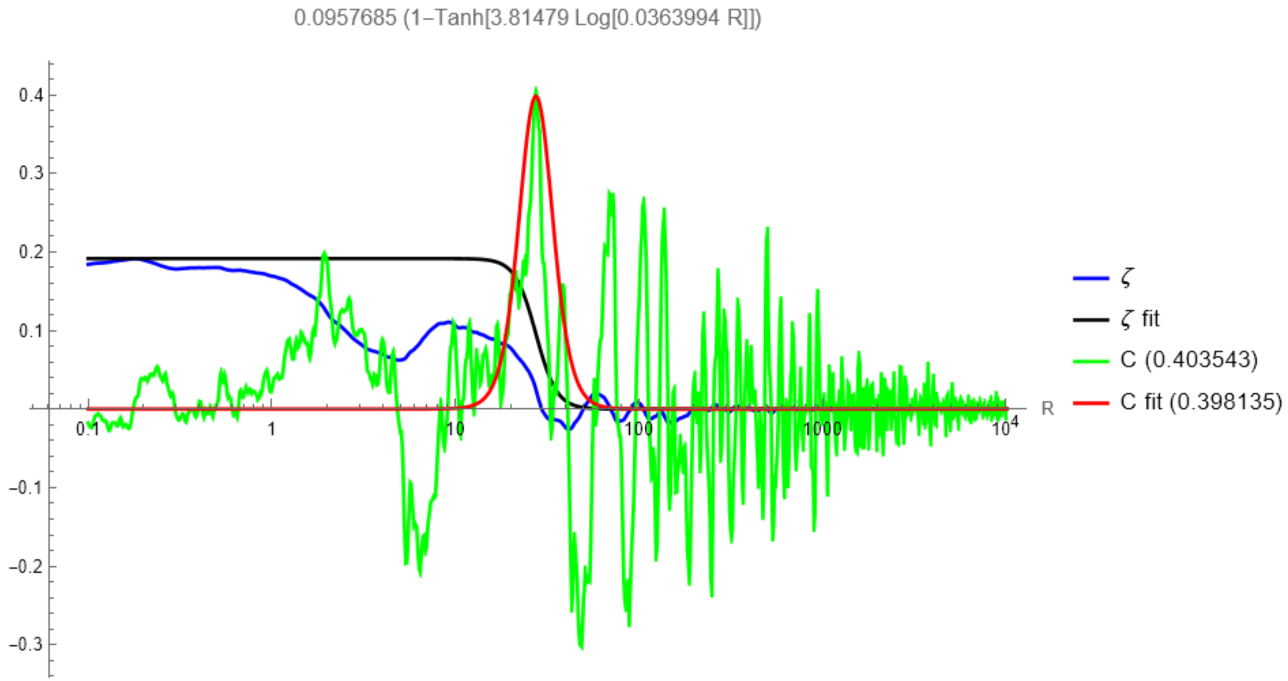

0.00538743 (1−Tanh[66.242 Log[0.0262636 R]])
ζ
ζ fit
C (0.400175)
C fit (0.390926)
R


0.0680505 (1−Tanh[5.53017 Log[0.0284163 R]])
ζ
ζ fit
C (0.407702)
C fit (0.407358)
R

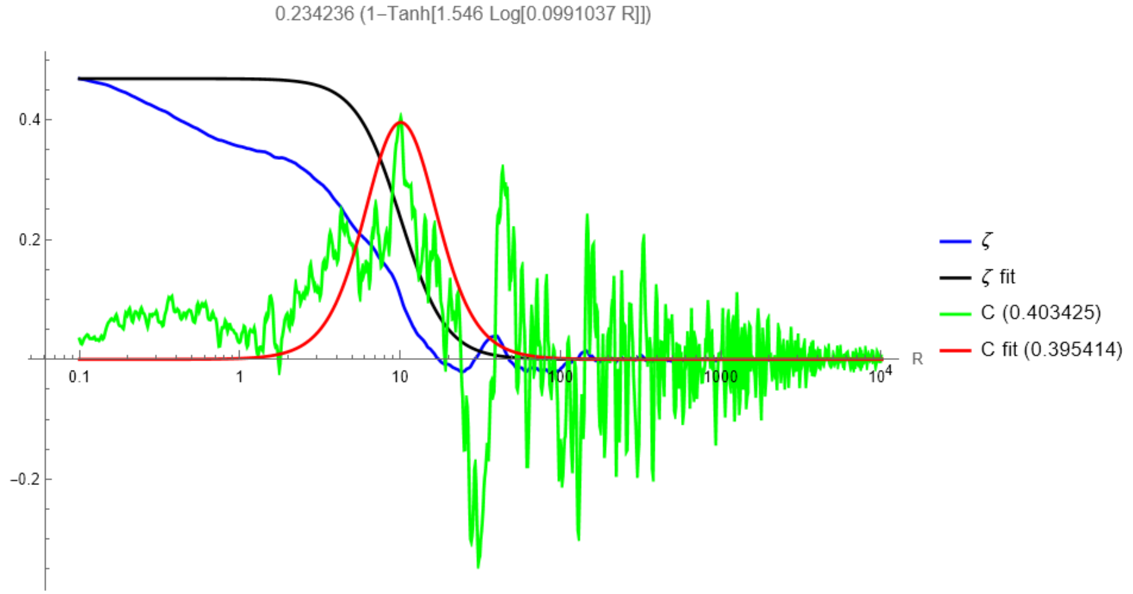

0.034639 (1−Tanh[10.8594 Log[0.022689 R]])
ζ
ζ fit
C (0.407216)
C fit (0.407216)
R



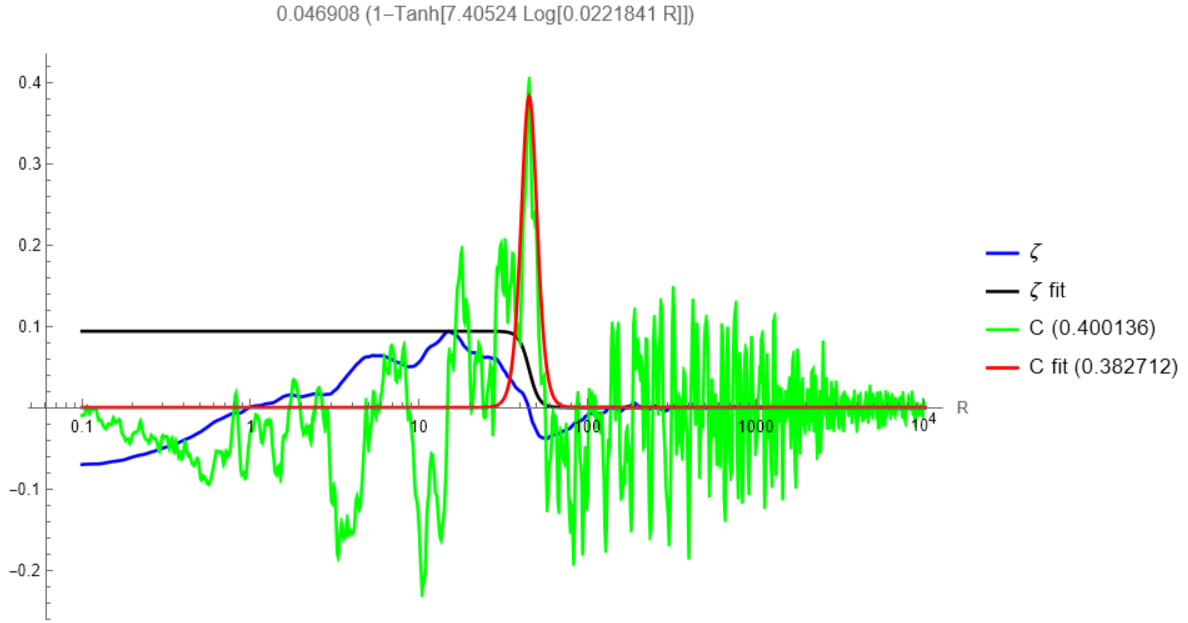
0.0678829 (1−Tanh[5.24741 Log[0.0130715 R]])
ζ
ζ fit
C (0.400148)
C fit (0.390356)
R



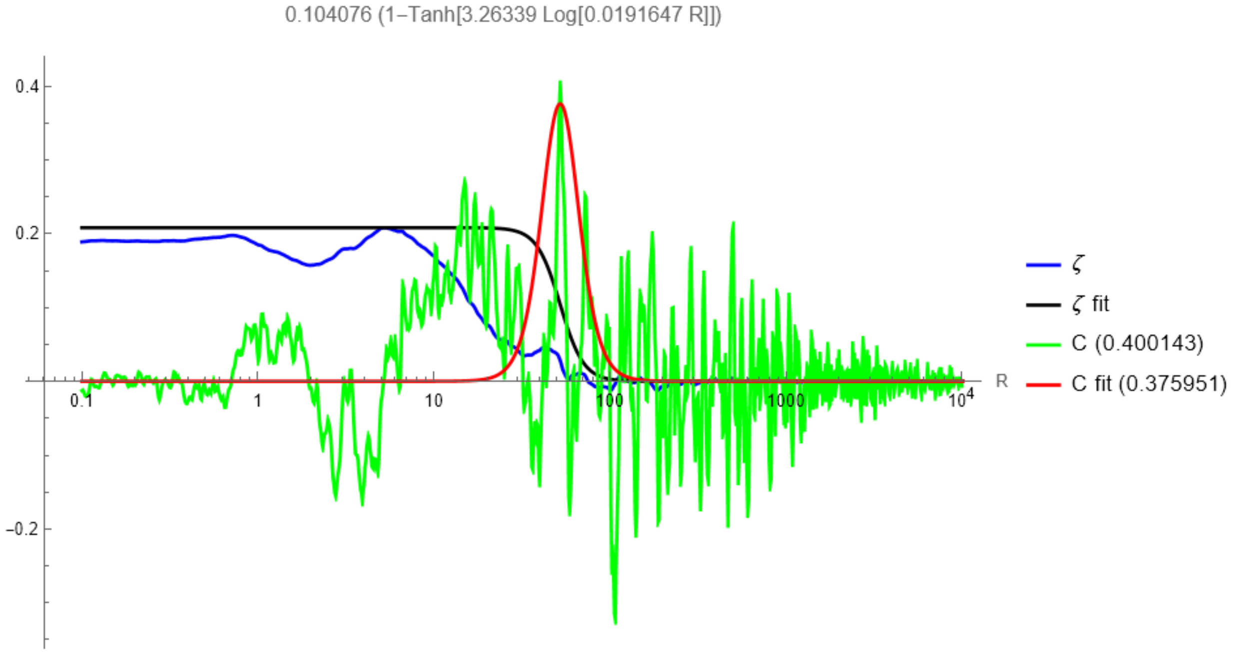
0.0957685 (1−Tanh[3.81479 Log[0.0363994 R]])
ζ
ζ fit
C (0.403543)
C fit (0.398135)
R

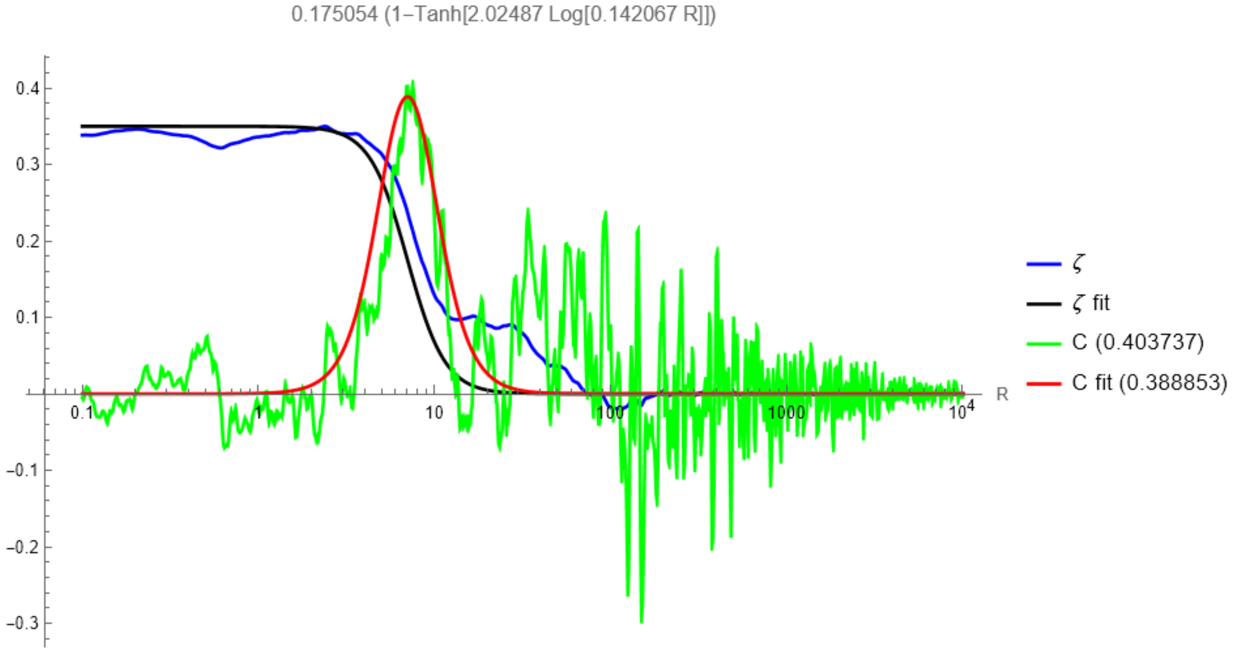


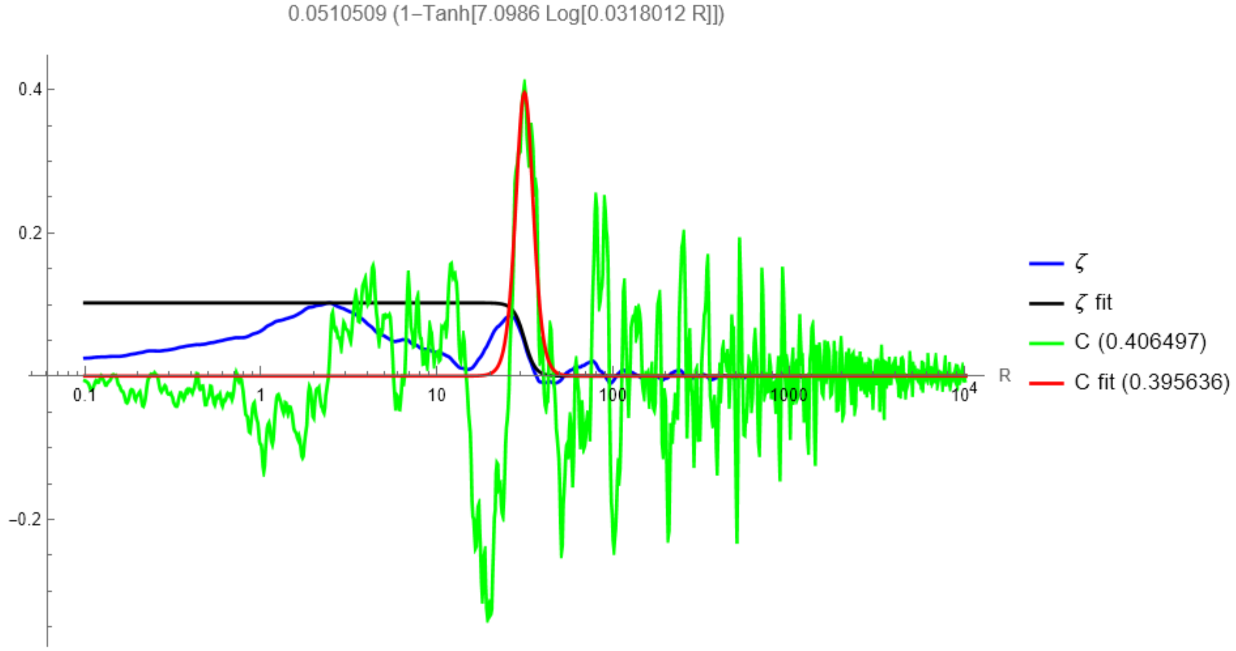
0.234236 (1−Tanh[1.546 Log[0.0991037 R]])
ζ
ζ fit
C (0.403425)
C fit (0.395414)
R

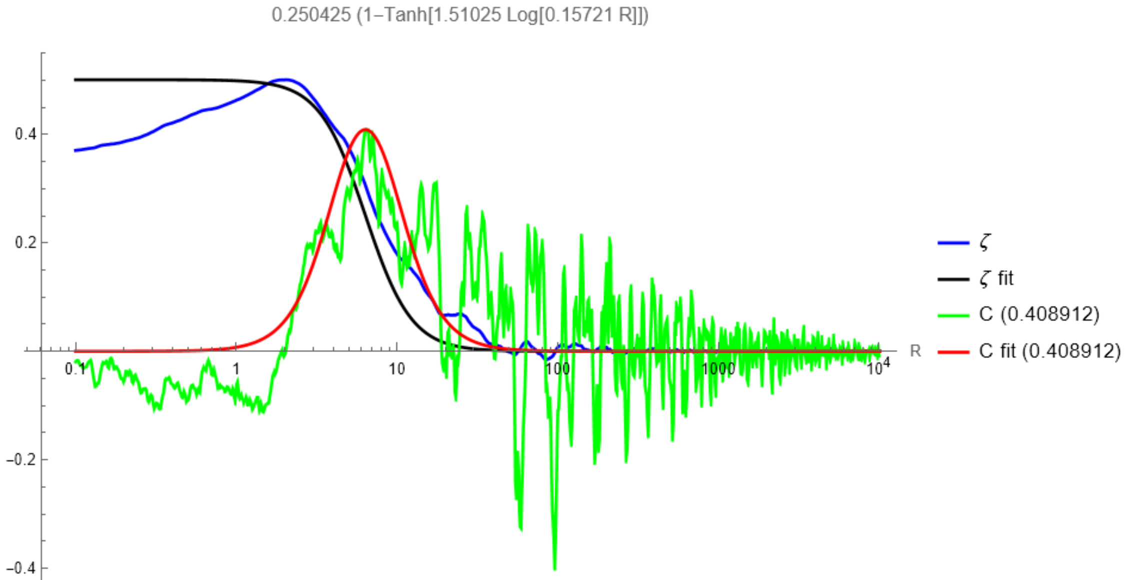


0.046908 (1−Tanh[7.40524 Log[0.0221841 R]])
ζ
ζ fit
C (0.400136)
C fit (0.382712)
R

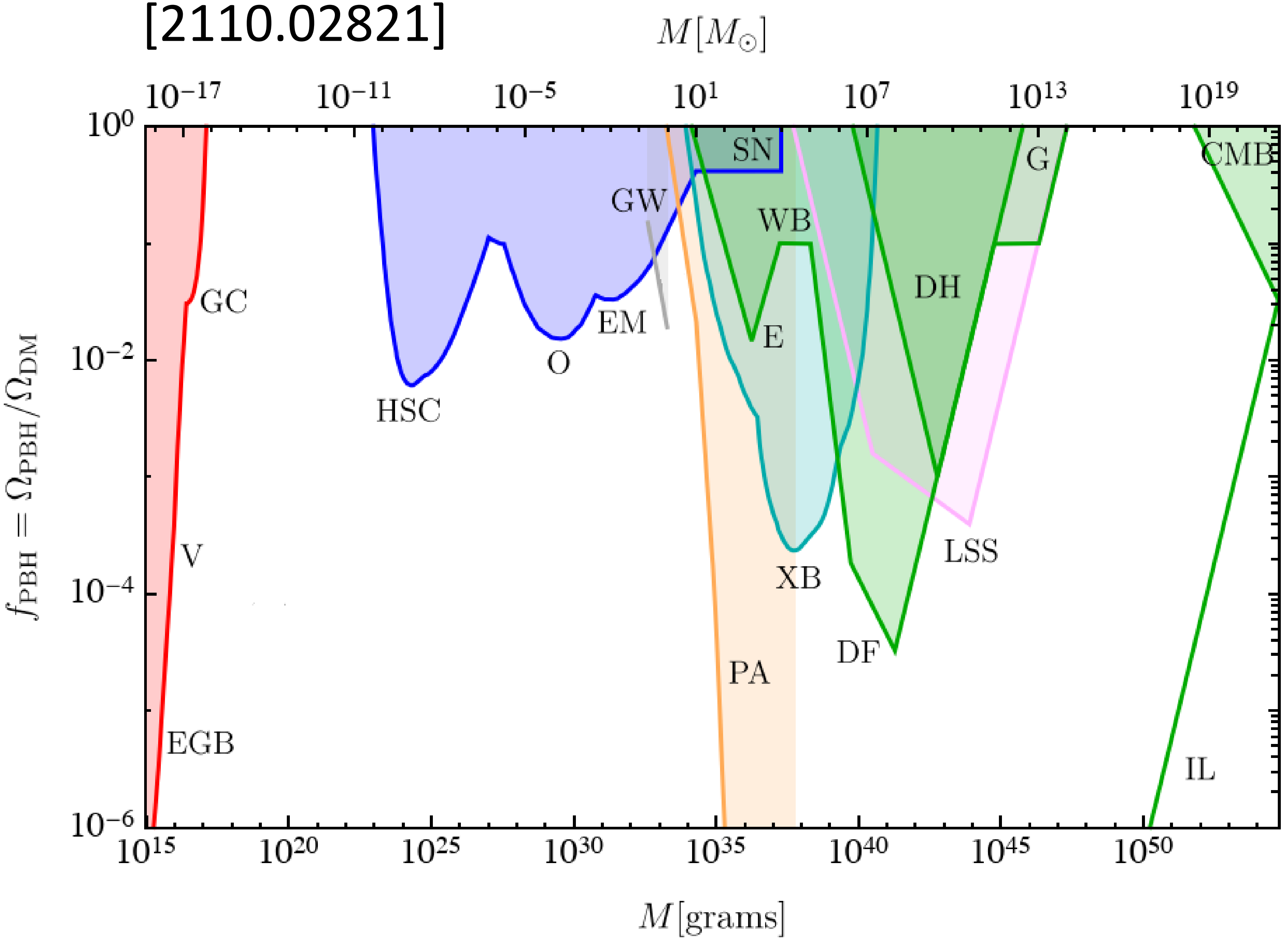

0.104076 (1–Tanh[3.26339 Log[0.0191647 R]])
0.4
0.2
-0.2
0.1
1
10
100
1000
10^4
R
ζ
ζ fit
C (0.400143)
C fit (0.375951)


0.175054 (1–Tanh[2.02487 Log[0.142067 R]])
0.4
0.3
0.2
0.1
-0.1
-0.2
-0.3
0.1
1
10
100
1000
10^4
R
ζ
ζ fit
C (0.403737)
C fit (0.388853)


0.0510509 (1–Tanh[7.0986 Log[0.0318012 R]])
0.4
0.2
-0.2
0.1
1
10
100
1000
10^4
R
ζ
ζ fit
C (0.406497)
C fit (0.395636)


0.250425 (1–Tanh[1.51025 Log[0.15721 R]])
0.4
0.2
-0.2
-0.4
0.1
1
10
100
1000
10^4
R
ζ
ζ fit
C (0.408912)
C fit (0.408912)

[2110.02821]



$M[M_\odot]$
10^{-17}
10^{-11}
10^{-5}
10^{1}
10^{7}
10^{13}
10^{19}
10^{0}
10^{-2}
10^{-4}
10^{-6}
$f_{\rm PBH} = \Omega_{\rm PBH}/\Omega_{\rm DM}$
SN
G
CMB
GW
WB
GC
DH
EM
E
O
HSC
V
LSS
XB
DF
PA
EGB
IL
10^{15}
10^{20}
10^{25}
10^{30}
10^{35}
10^{40}
10^{45}
10^{50}
$M[\rm grams]$

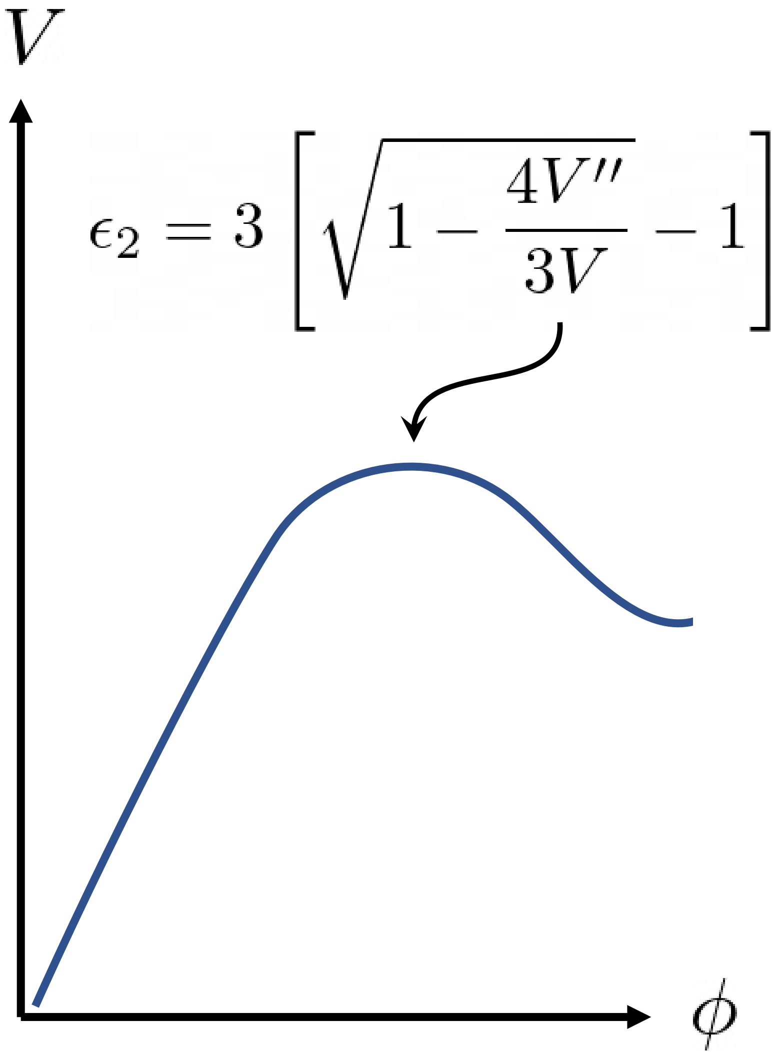
Initial PBH fractions

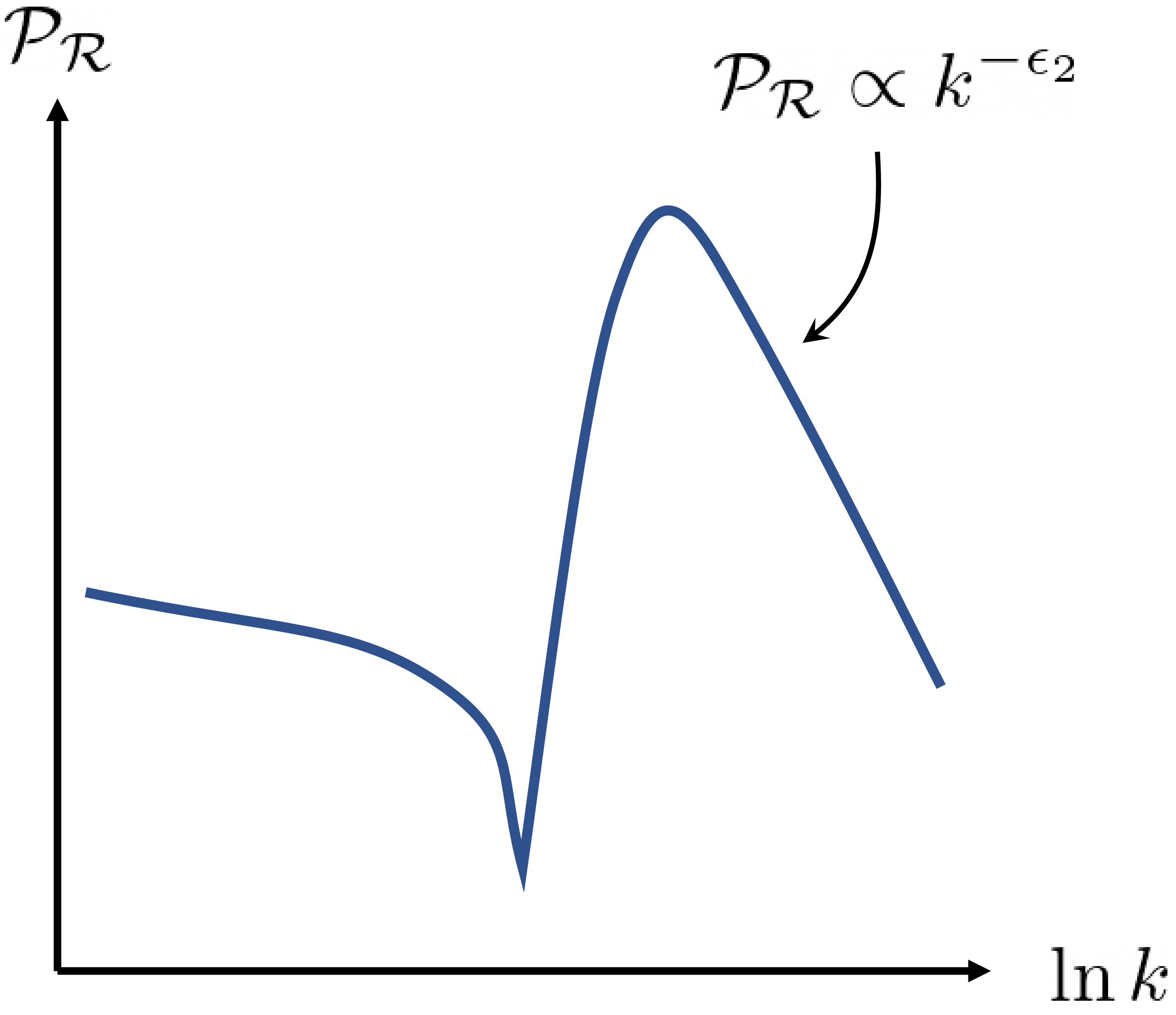
Gaussian approximation, $\mathcal{R}_{<k} > 1$, fixed k: $\beta \approx 5 \times 10^{-16}$

Non-Gaussian statistics, $\mathcal{R}_{<k} > 1$, fixed k: $\beta \approx 2.2 \times 10^{-11}$

$\bar{\mathcal{C}}_{\max} > 0.4 : \quad \beta \approx 1.4 \times 10^{-8}$

$\mathcal{C}_{\max} > 0.4 : \quad \beta \approx 0.016$

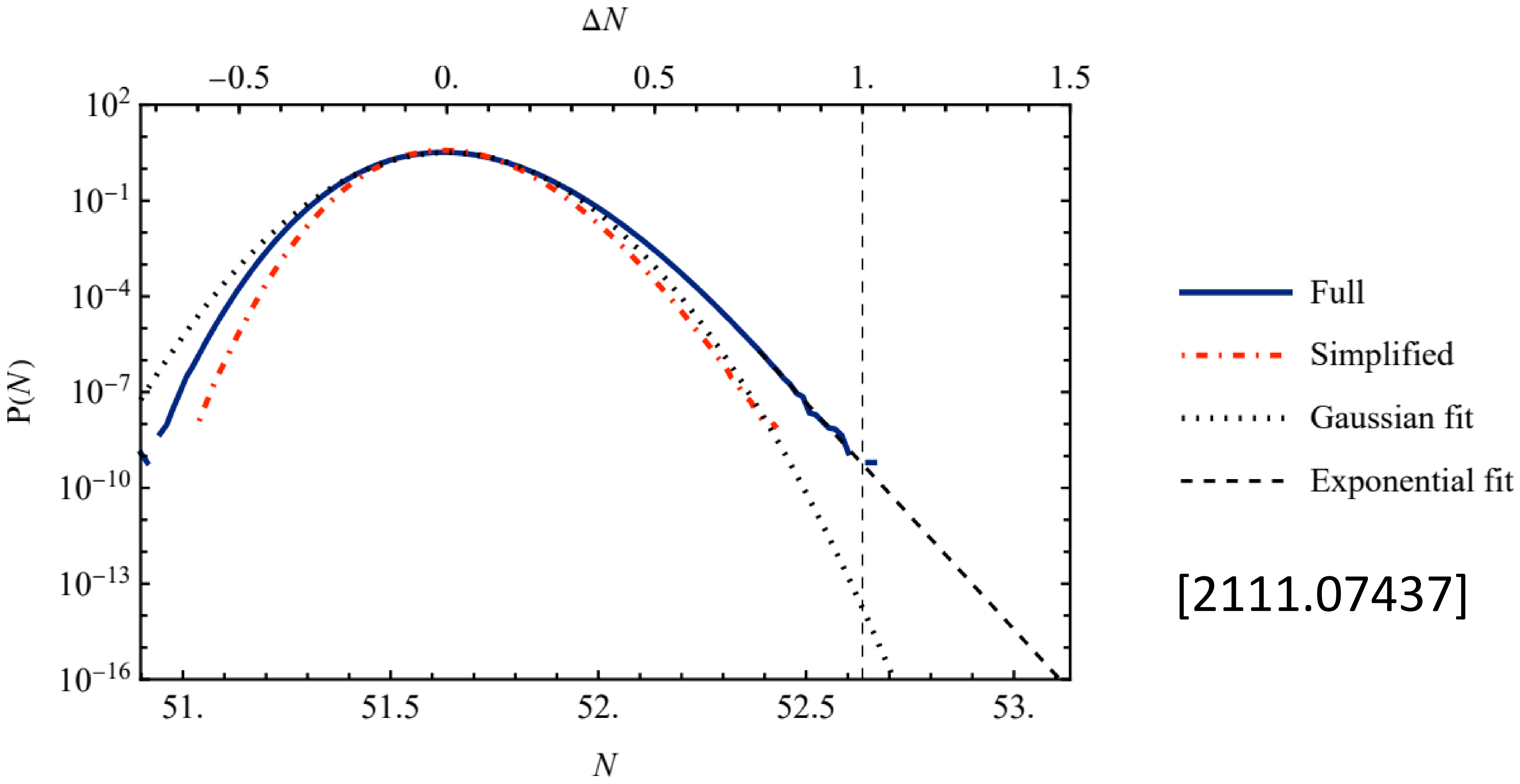

V
$\epsilon_2 = 3\left[\sqrt{1 - \frac{4V''}{3V}} - 1\right]$
ϕ


$\mathcal{P}_{\mathcal{R}}$
$\mathcal{P}_{\mathcal{R}} \propto k^{-\epsilon_2}$
$\ln k$

[2205.13540]

Full numerical computations

One million CPU hours

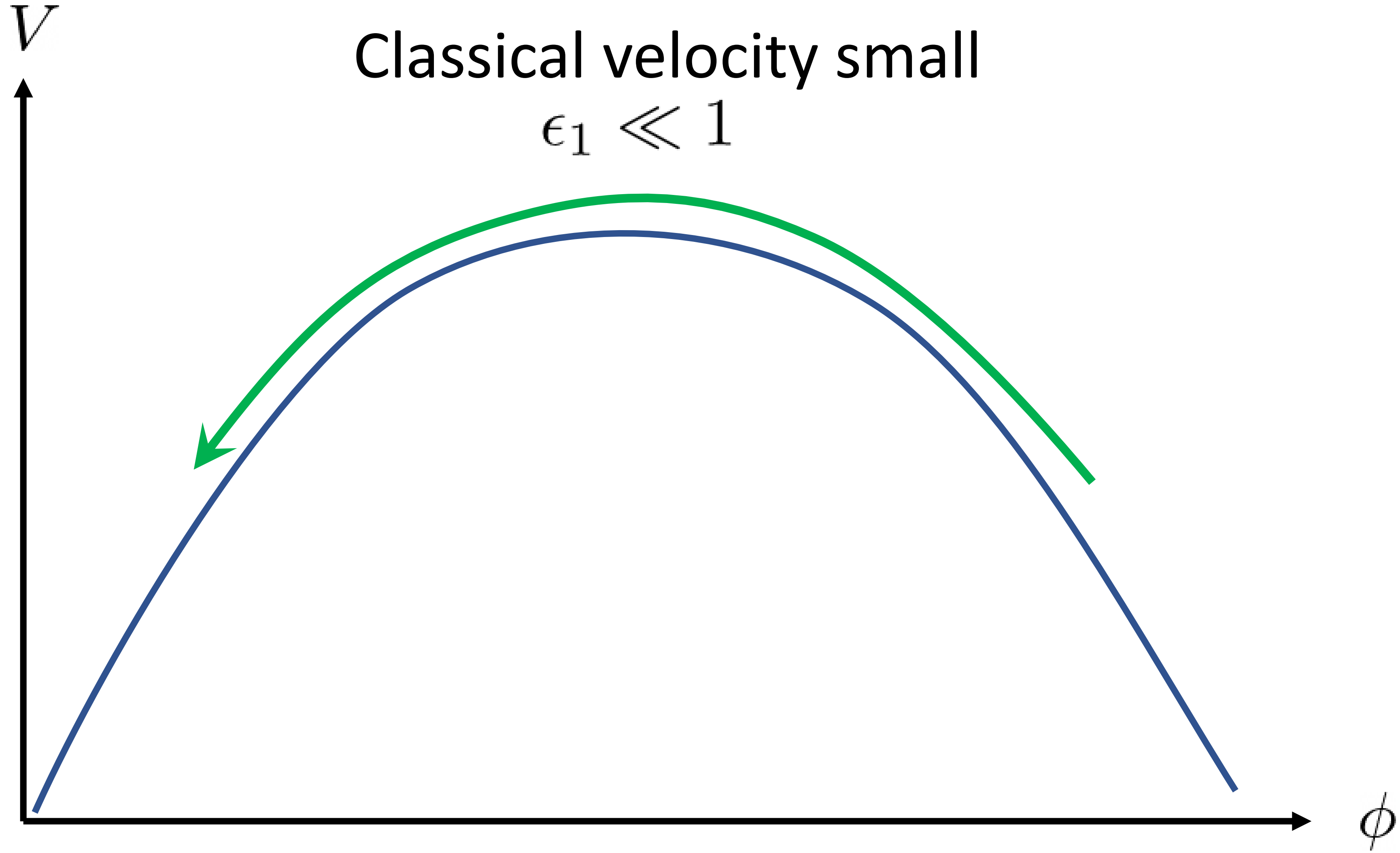


[2111.07437]

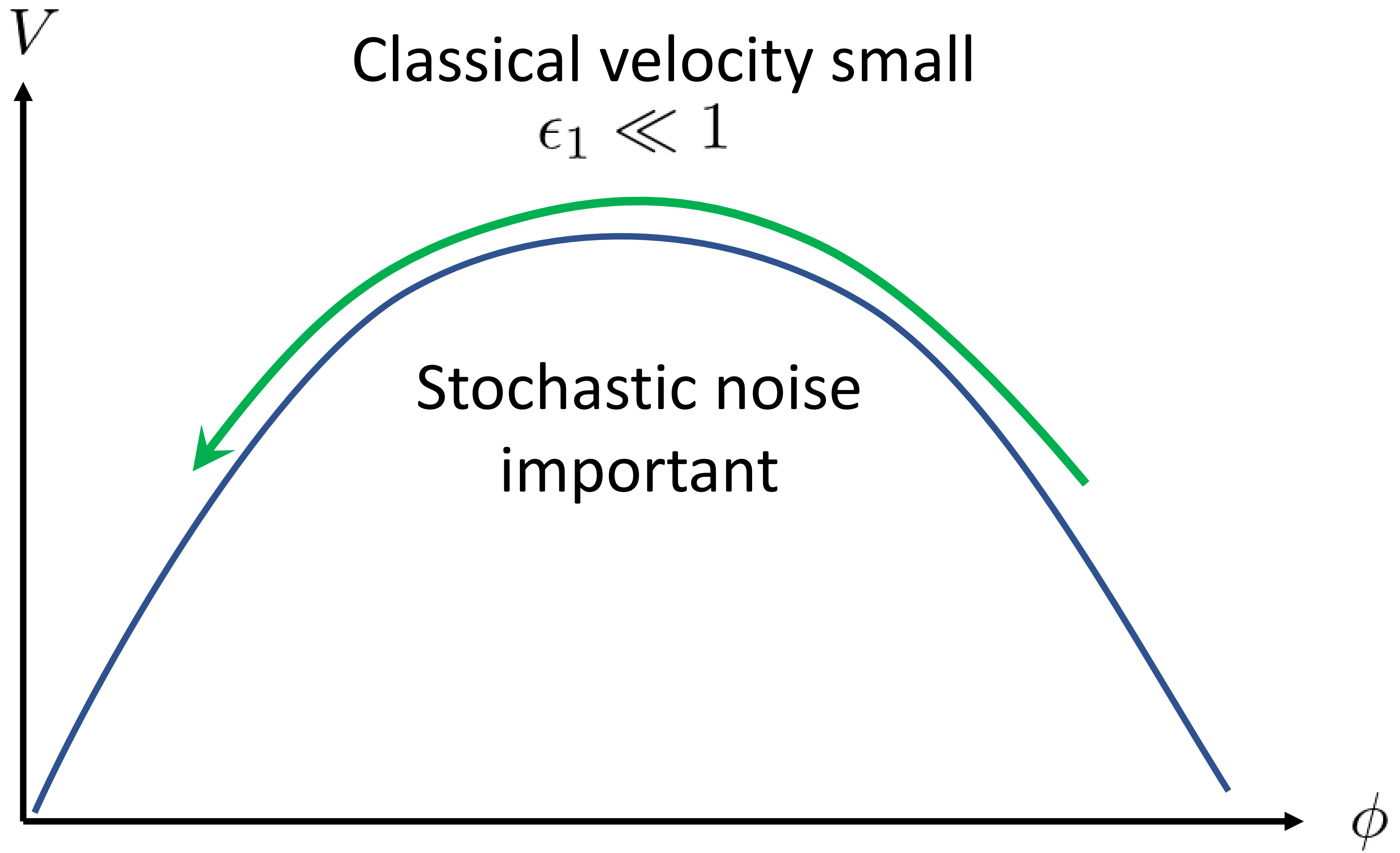
Zoom into the hilltop

Classical velocity small

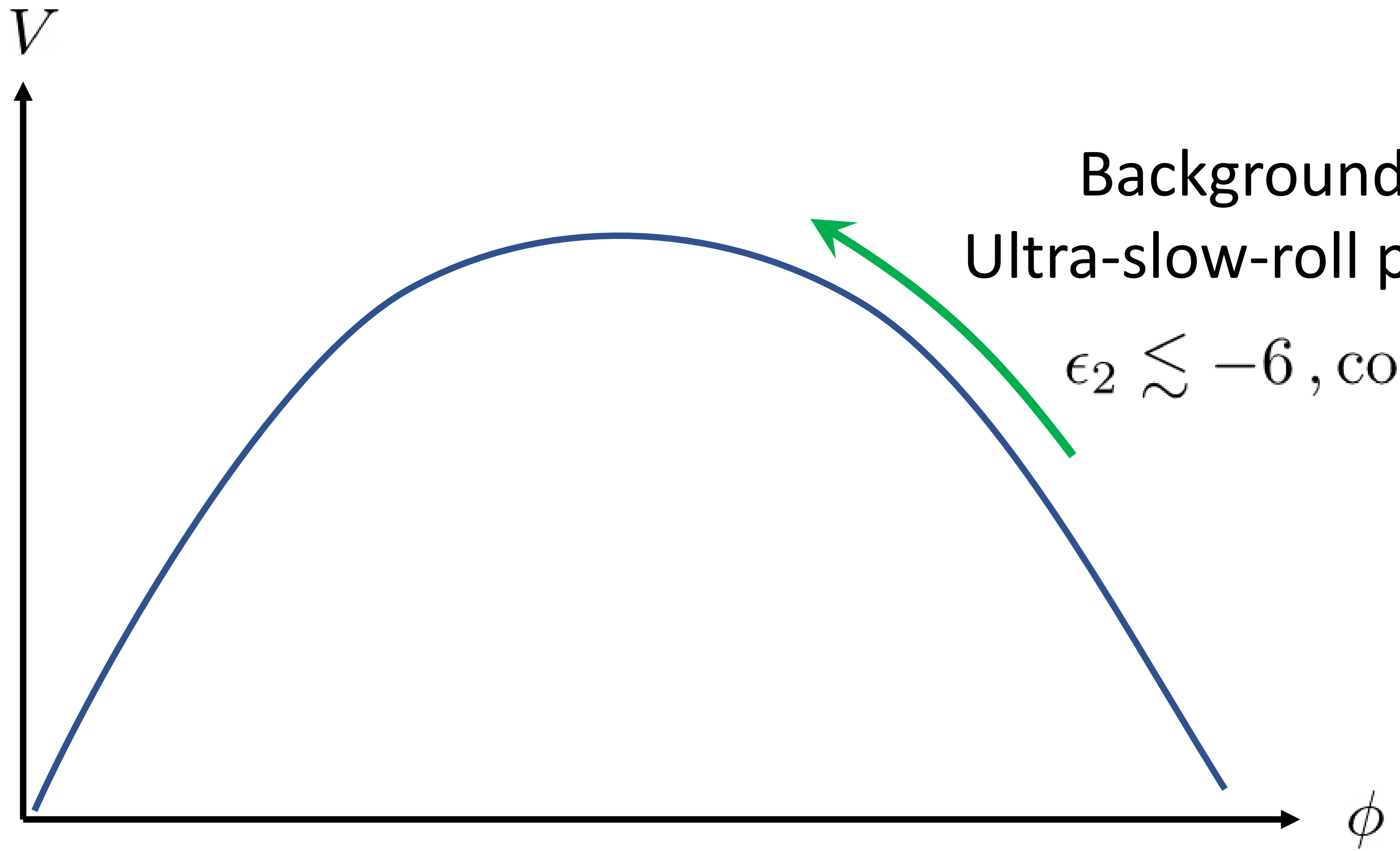
$\epsilon_1 \ll 1$



Zoom into the hilltop



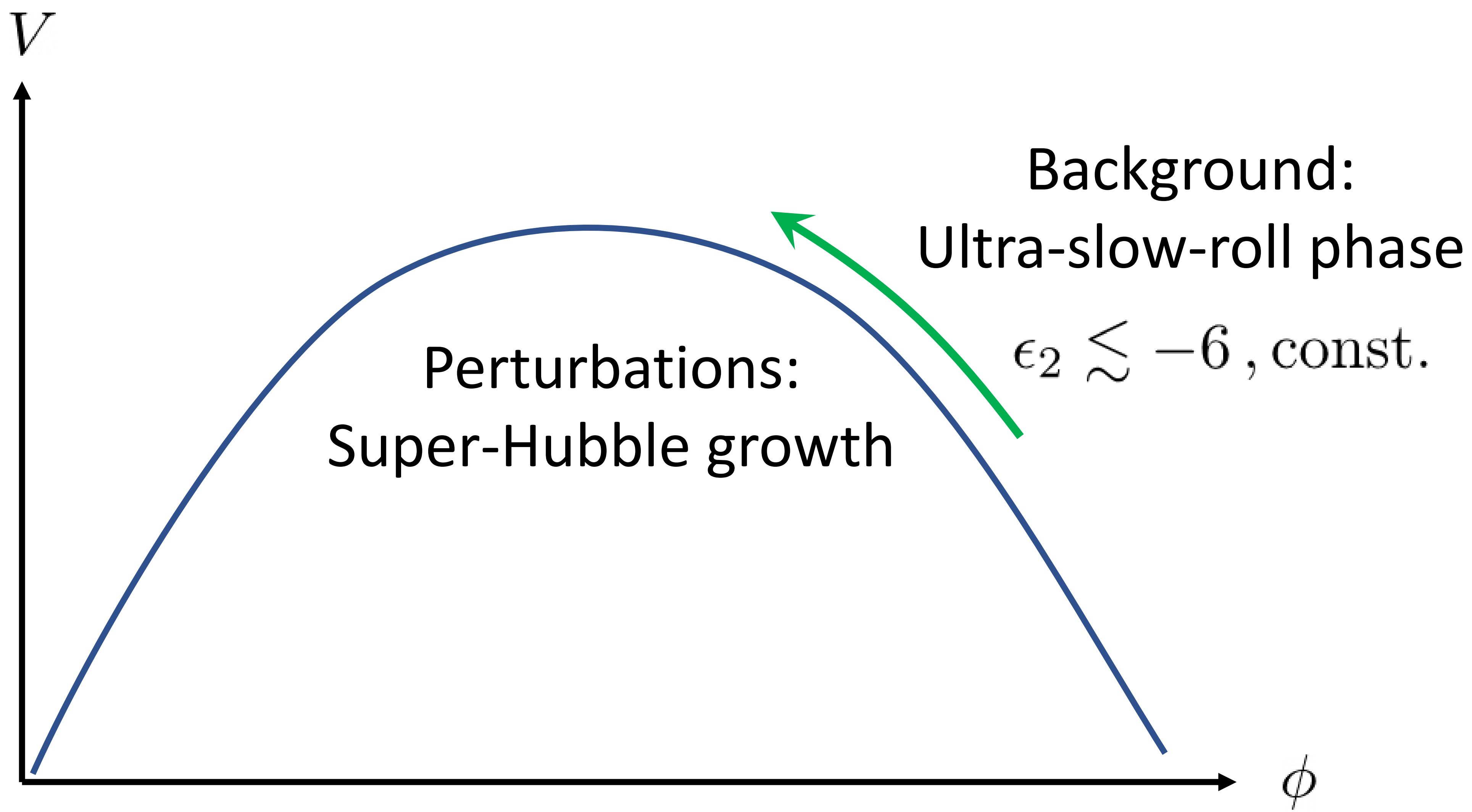
Zoom into the hilltop



Background:
Ultra-slow-roll phase

$\epsilon_2 \lesssim -6\,,\mathrm{const.}$

Zoom into the hilltop

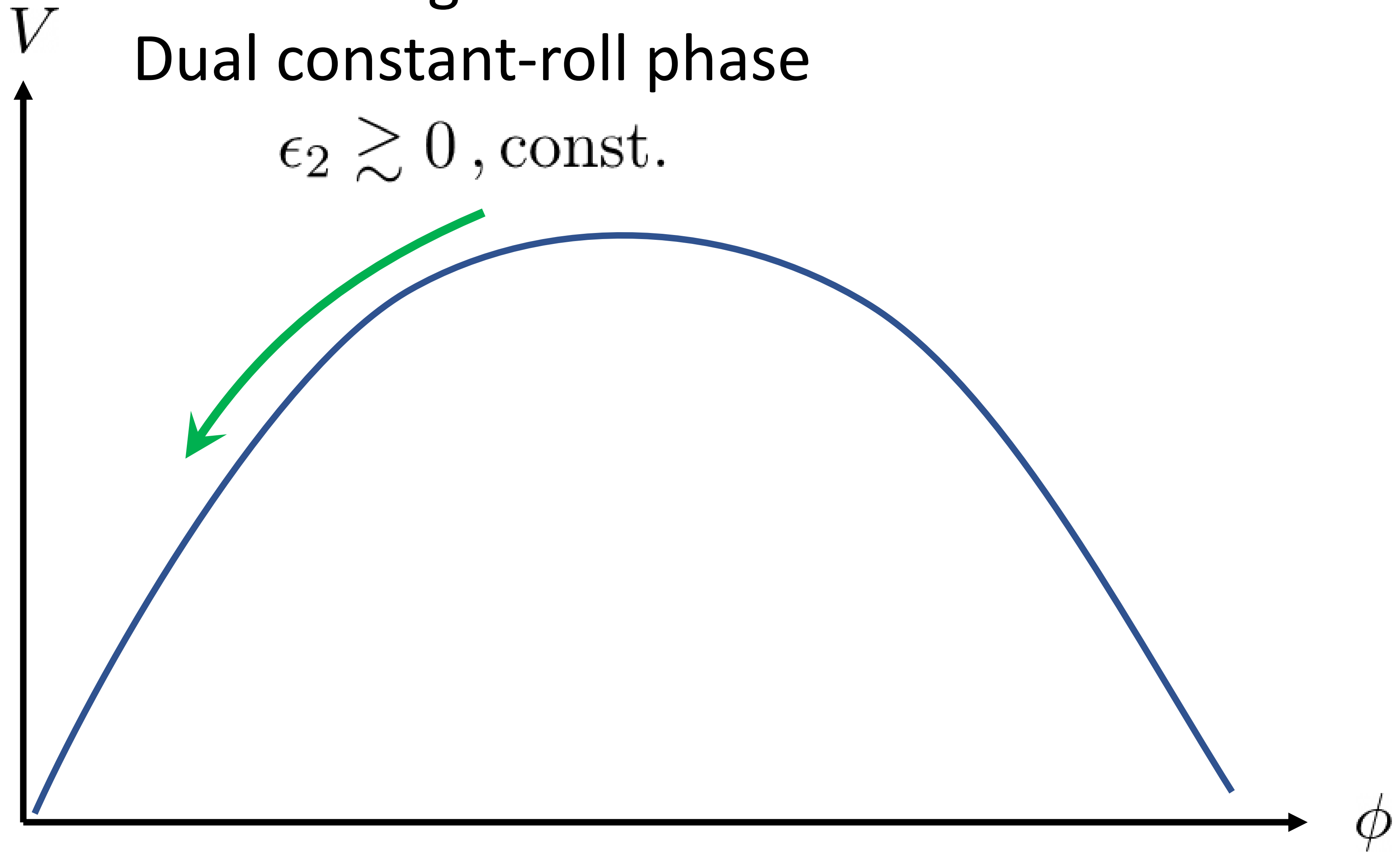


Zoom into the hilltop

Background:

Dual constant-roll phase

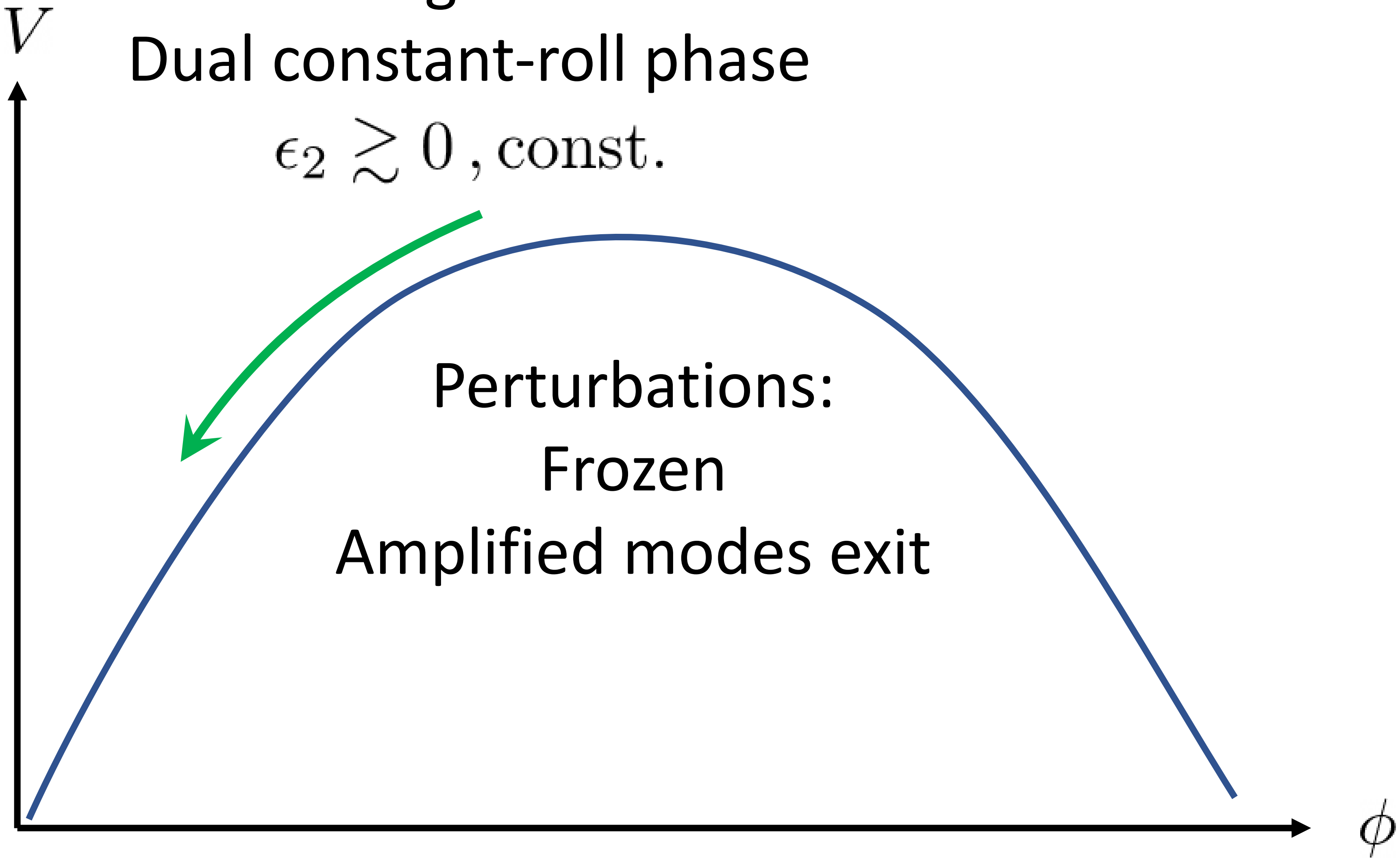
$\epsilon_2 \gtrsim 0\,,\mathrm{const.}$



Zoom into the hilltop

Background:
Dual constant-roll phase
$\epsilon_2 \gtrsim 0\,, \mathrm{const.}$

Perturbations:
Frozen
Amplified modes exit



Simplified stochastic equation:

$$\mathrm{d}\phi = \frac{\epsilon_2}{2}(\phi - \phi_0)\mathrm{d}N + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N}\sqrt{\mathcal{P}_{\mathcal{R}}(k_\sigma)\mathrm{d}N}\,\hat{\xi}_N$$

$$\langle \hat{\xi}_N \hat{\xi}_{N'} \rangle = \delta_{NN'}$$

Simplified stochastic equation:

$$\mathrm{d}\phi = \frac{\epsilon_2}{2}(\phi - \phi_0)\mathrm{d}N + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N}\sqrt{\mathcal{P}_{\mathcal{R}}(k_\sigma)\mathrm{d}N}\,\hat{\xi}_N$$

$$\langle \hat{\xi}_N \hat{\xi}_{N'} \rangle = \delta_{NN'}$$

$$\phi(N) = \phi_0\left(1 - e^{\frac{\epsilon_2}{2}N}\right) + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N} X_{<k_\sigma}$$

$$X_{<k} \equiv \sum_{\tilde{k}=k_{\mathrm{ini}}}^{k} \sqrt{\mathcal{P}_{\mathcal{R}}(\tilde{k})\,\mathrm{d}\ln k}\,\hat{\xi}_{\tilde{k}}$$

ΔN distribution

$$p(X_{<k}) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{X_{<k}^2}{2\sigma_k^2}}, \quad \sigma_k^2 \equiv \int_{k_{\rm ini}}^{k} \mathcal{P}_{\mathcal{R}}(\tilde{k})\, \mathrm{d}\ln\tilde{k}$$

ΔN distribution

$$p(X_{<k}) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{X_{<k}^2}{2\sigma_k^2}}\,, \qquad \sigma_k^2 \equiv \int_{k_{\rm ini}}^{k} \mathcal{P}_{\mathcal{R}}(\tilde{k})\,\mathrm{d}\ln\tilde{k}$$

$$X_{<k} = \frac{2}{\epsilon_2}\left(1 - e^{-\frac{\epsilon_2}{2}\Delta N_{<k}}\right)$$

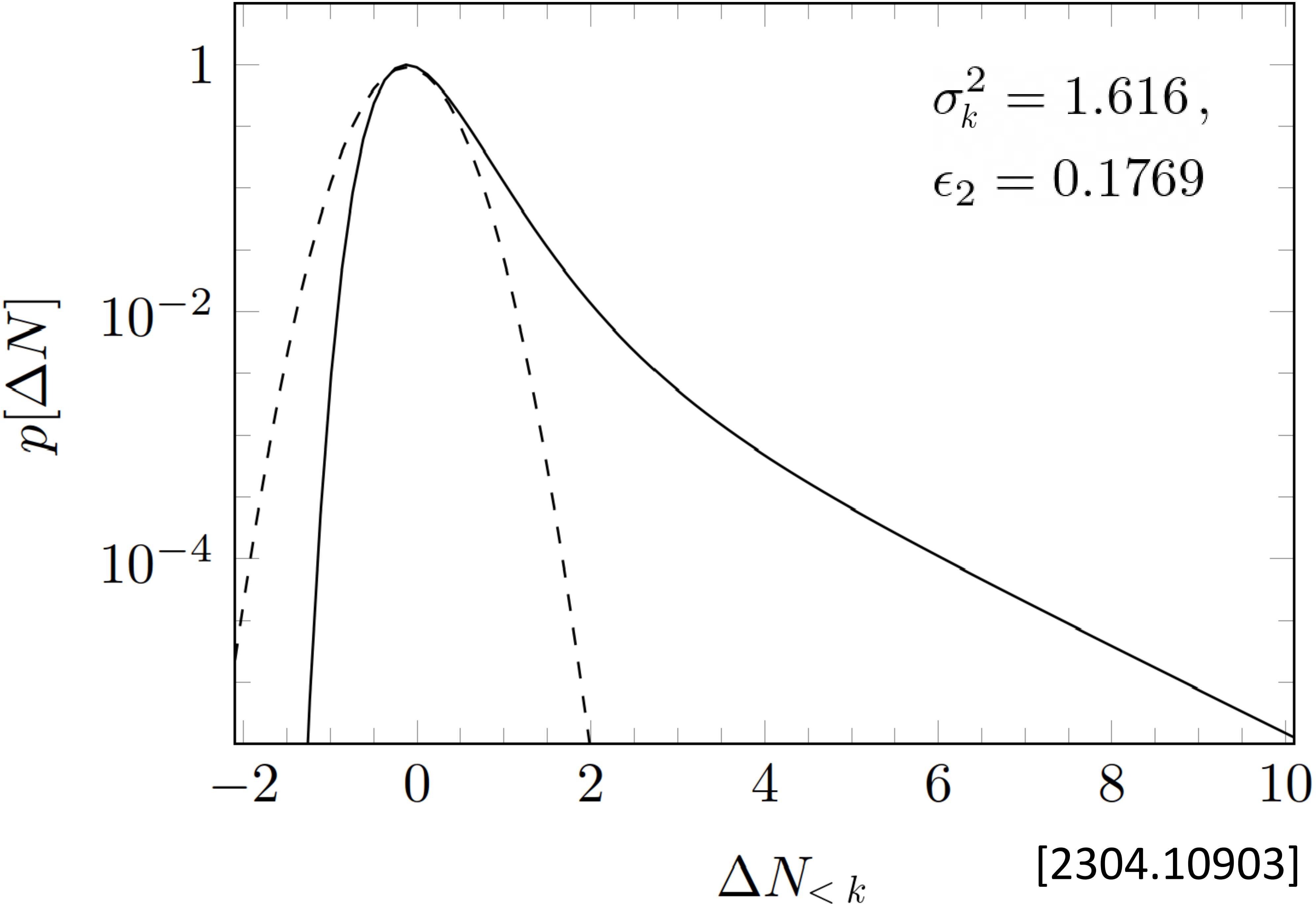
ΔN distribution

$$p(X_{<k}) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{X_{<k}^2}{2\sigma_k^2}}, \quad \sigma_k^2 \equiv \int_{k_{\rm ini}}^{k} \mathcal{P}_{\mathcal{R}}(\tilde{k})\,\mathrm{d}\ln\tilde{k}$$

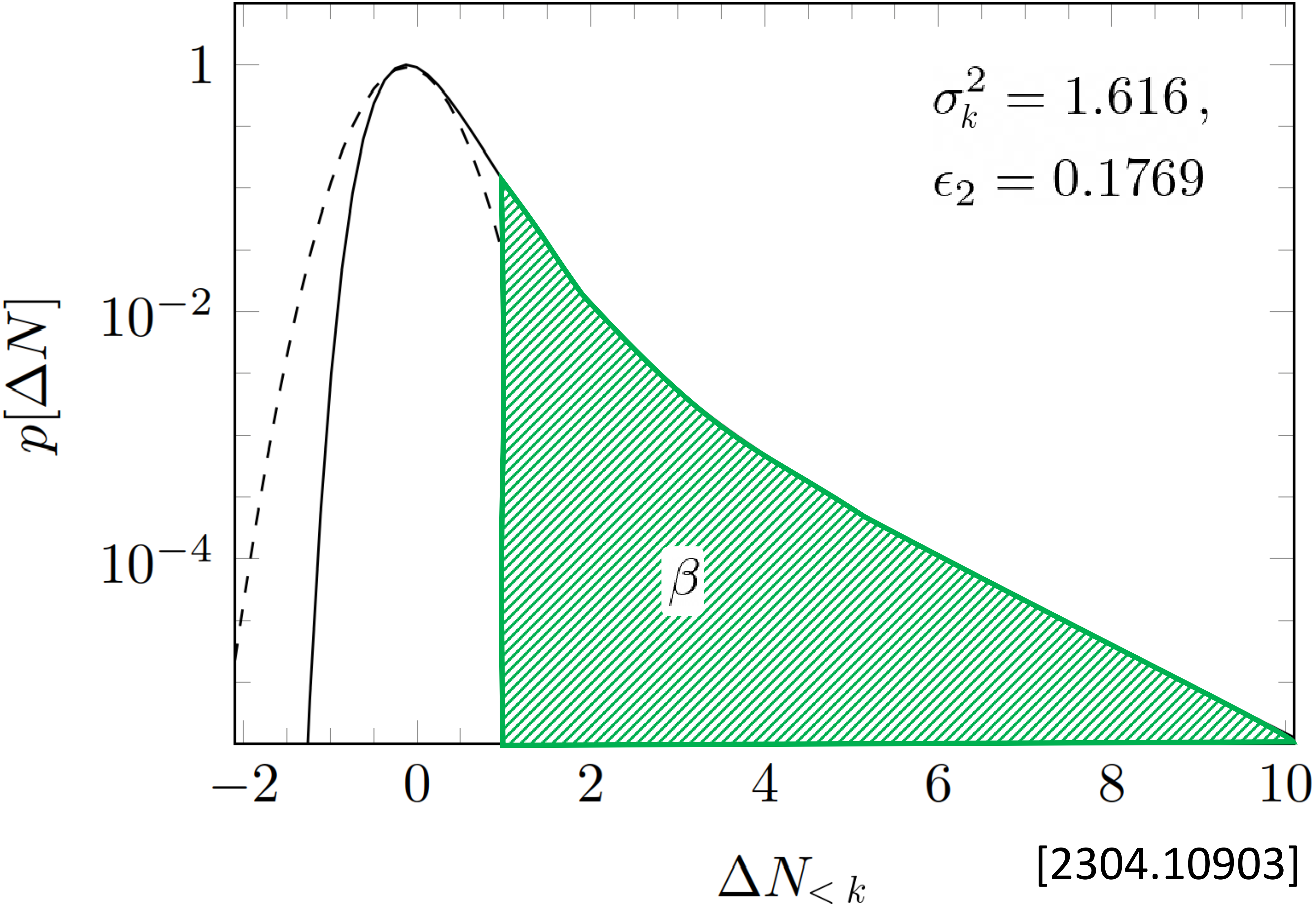
$$X_{<k} = \frac{2}{\epsilon_2}\left(1 - e^{-\frac{\epsilon_2}{2}\Delta N_{<k}}\right)$$

$$p(\Delta N_{<k}) = \frac{1}{\sqrt{2\pi}\sigma_k} \exp\left[-\frac{2}{\sigma_k^2 \epsilon_2^2}\left(1 - e^{-\frac{\epsilon_2}{2}\Delta N_{<k}}\right)^2 - \frac{\epsilon_2}{2}\Delta N_{<k}\right]$$

$$\Delta N_{<k} = \mathcal{R}_{<k}$$


$\sigma_k^2 = 1.616\,,$
$\epsilon_2 = 0.1769$
1
10^{-2}
10^{-4}
$p[\Delta N]$
−2
0
2
4
6
8
10
$\Delta N_{<\,k}$

[2304.10903]



1
10^{-2}
10^{-4}
$p[\Delta N]$
$\sigma_k^2 = 1.616\,,$
$\epsilon_2 = 0.1769$
β
−2
0
2
4
6
8
10
$\Delta N_{<k}$

[2304.10903]