HEP Summer School 2024
Tutorial Problems - The Standard Model

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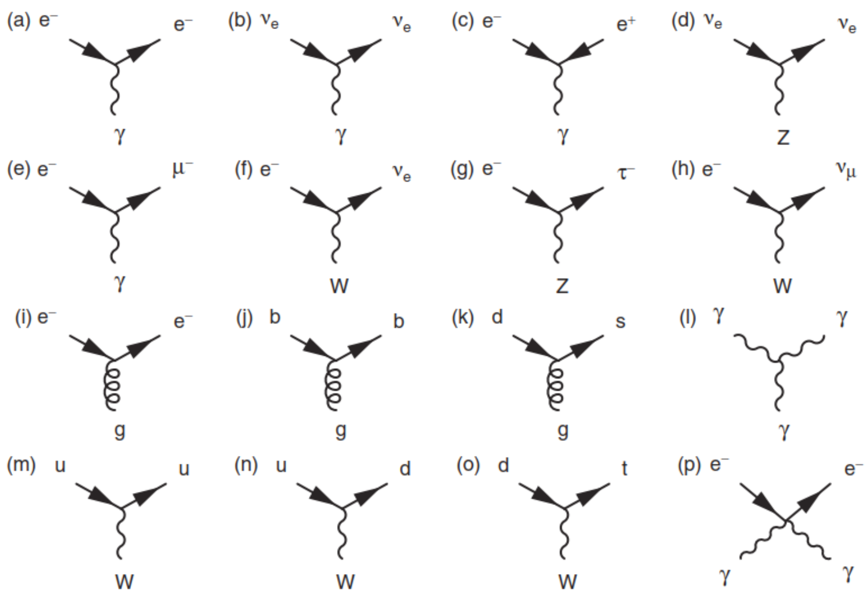
Problem 1

(From Thompson) Explaining your reasoning, state whether each of the sixteen diagrams below represents a valid Standard Model vertex.



Problem 2

Draw the Feynman diagram for the process $\tau^- \to \pi^- \nu_\tau$ (the π^- is the lightest $d\bar{u}$ meson).

Problem 3

Show that for matrices A and B it is valid that:

$$e^A\ e^B\ = \exp\left(A + B + \frac{1}{2}[A, B] + \cdots\right)\ , \tag{1}$$

where the notation $\cdots$ denotes higher order commutators.

Hint: Use the exponential expansion formula:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}\ ,$$

to show that the first term in the expansion is the same between the left and right hand side of Eq. (2).

Problem 4

(i) Show that elements of the group $SU(N)$ (i.e. special unitary $N \times N$ matrices, U) each have $N^2 - 1$ parameters.

(ii) The group elements, U, can be expressed in terms of the generators, t^a ($a = 1, \ldots, N^2 - 1$), as

$$U = e^{i\theta^a t^a}\ .$$

Show that U being unitary implies that the matrices t^a are Hermitian. Show that U having determinant 1 implies that the matrices t^a are traceless (Hint: prove first the identity $\det e^A = e^{TrA}$).

(iii) Consider the generators of $SU(2)$ in the fundamental representation:

$$t_1 = \frac{\sigma_1}{2} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ ,\ \ t_2 = \frac{\sigma_2}{2} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ ,\ \ t_3 = \frac{\sigma_3}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ ,$$

and find the structure constants of the group, f_{abc}.

(iv) Then use them to find the generators of the adjoint representation using that:

$$\left(t_a^{\text{adjoint}}\right)_{bc} = -if_{abc}$$

Hint: use the property of the Pauli matrices:

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k\ .$$

Problem 5

Use the Lorentz transformation properties of spinors that you studied in the QED/QCD course to show that the product:

$$\bar{\psi}_L \psi_R \ ,$$

where the subscripts L and R denote the left and right handed chirality, respectively, is Lorentz invariant.

Why are direct mass terms like $\bar{e}_L e_R$ not allowed in the Standard Model Lagrangian, hence the need for Yukawa couplings with the Higgs to give masses to the fermions?

Hint: First show that for $\gamma^5 \equiv \frac{i}{4!}\epsilon_{\mu\nu\kappa\lambda}\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda$:

$$\gamma^5 S = S\gamma^5 \ .$$

where $\psi(x) \to S(\Lambda)\psi(\Lambda^{-1}x)$ under a Lorentz transformation and $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$.

Problem 6

Consider the case of QCD and do the following algebra to fill in the gaps in the lecture notes.

i) Show that the commutator of the covariant derivative:

$$D_\mu = \partial_\mu + igt^a A^a_\mu \ , \tag{2}$$

is proportional to the field strength tensor field strength tensor, $F^a_{\mu\nu}t^a = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^a{}_{bc}A^b_\mu A^c_\nu)t^a$, i.e.

$$-\frac{i}{g}[D_\mu, D_\nu] = F^a_{\mu\nu}t^a \ .$$

ii) Then show how $F^a_{\mu\nu}$ transforms under SU(3). To show this in an efficient way, consider that the covariant derivative acting on a spinor field transforms like the spinor field itself, i.e.

$$D_\mu\psi \to D'_\mu\psi' = UD_\mu\psi \ , \tag{3}$$

where

$$\psi' = U\psi \ ,$$

as a step to show that:

$$F'^a_{\mu\nu}t^a = UF^a_{\mu\nu}t^aU^{-1} \ .$$

iii) Consider Eq. 3 to show that:

$$A'^a_\mu t^a = A^a_\mu U t^a U^\dagger + \frac{i}{g}(\partial_\mu U)U^\dagger$$

iv) Use the transformation

$$U = \exp(-ig\theta^a t^a)$$

to show that for small g (i.e. for an infinitessimal transformation) it holds that:

$$A'^c_\mu = A^c_\mu + g f^{bac}\theta^b A^a_\mu + \partial_\mu \theta^c \ ,$$

that is, for a global transformation, A^a_μ transforms as the adjoint representation:

$$A'^c_\mu = A^c_\mu + ig(t^{\text{adjoint}})^{bac}\theta^b A^a_\mu \, .$$

Problem 7

The CKM matrix, V_{AB}, is a unitary matrix that relates the quark weak eigenstates, d'_{LA}, to the quark mass eigenstates, d_{LA}, as $d'_{LA} = V_{AB} d_{LB}$, such that the weak charged current interaction for the quark sector can be written as:

$$\mathcal{L}_{cc} = -\frac{g}{\sqrt{2}} \sum_{A,B} \bar{u}_{LA}\gamma^\mu W^+_\mu V_{AB} d_{LB} + \bar{d}_{LB} V^*_{AB} \gamma^\mu W^-_\mu u_{LA} \tag{4}$$

with $A, B = 1, \ldots, n$ labelling the generation.

For the 3 generations known in the Standard Model:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} ,$$

with the (non-unique) parameterisation:

$$V = \begin{pmatrix} c_1 & -s_1 c_3 & -s_1 s_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} ,$$

where $c_a = \cos\theta_a$ and $s_a = \sin\theta_a$.

Assume n generations. Show that a unitary matrix has n^2 real parameters. Out of these, $n(n-1)/2$ parameters can be expressed as rotation angles and the remaining $n(n+1)/2$ parameters can be expressed as phases. Explain why only $\frac{1}{2}(n-1)(n-2)$ of these phases are physical.

Use the transformation under CP:

$$\text{CP} : \bar{\psi}_1 \gamma^\mu \psi_2 W^+_\mu \to \bar{\psi}_2 \gamma^\mu \psi_1 W^-_\mu$$

to show that (4) violates the CP-symmetry only for $n \geq 3$ generations.

Problem 8

List the 19 parameters of the Standard Model.