STFC HEP Summer School Lectures- The Standard Model

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1 Introduction

Plan for course:

1. Overview of SM - QFT and symmetries
2. Some group theory and representations- symmetries and fields.
3. Lagrangians and Feynman rules
4. Some historical motivation- Fermi theory for 4-fermion interactions
5. More symmetries- global and gauged
6. Spontaneous symmetry breaking
7. Building SM Lagrangian
   - Gauge and Higgs bosons
   - Leptons and quarks
   - CKM matrix and CP violation
8. Some features of SM- more symmetries
9. Problems with/in + beyond the SM- hints towards new physics

Refs:

- 2019 SM lectures notes
- Palash B. Pal, Kane, Thomson, Burgess

2 What is the SM?

- A particular QFT defined by its symmetries and particle content encapsulated in $\mathcal{L}_{SM}$ $\rightarrow$ precise understanding of **all** interactions of fundamental particles (except gravity) $\rightarrow$ decay rates, scattering cross-sections.
- Developed in the 1970s $\rightarrow$ final piece Higgs boson (2012). No conflicts with experiment.
- Based on symmetries:
  - POINCARÉ INVARIANCE (Lorentz and translations)
  - $SU(3) \times SU(2)_L \times U(1)_Y$.
    (SU(3) associated with the strong force, gluons with $m_g = 0$). $SU(2)_L$ the weak force, and $W^{\pm}, Z$ bosons. $U(1)_Y$ associated with weak hypercharge.)
  - With SPONTANEOUS SYMMETRY BREAKING $SU(2)_L \times U(1)_Y \rightarrow U(1)_{EM}$
    (Leading to a Higgs boson with m_H 125 GeV, massless W, Z bosons and a photon with $m_\gamma = 0$ associated with the unbroken $U(1)_{EM}$.)
  - and 3 generations of quarks and leptons.

					Strong	EM	Weak
QUARKS	UP-TYPE	u	c	t	✓	✓	✓
6 FLAVOURS	DOWN-TYPE	d	s	b			
LEPTONS	CHARGED	e^-	μ^-	τ^-		✓	✓
6 FLAVOURS	NEUTRINOS	ν_e	ν_μ	ν_τ			✓

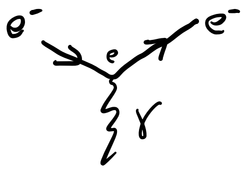
$m_e \sim 0.5$ MeV to $m_t \sim 173$ GeV, $\sum m_\nu < 25$ eV

  – plus antiparticles (same mass and spin, opposite charges).

- The symmetries restrict the possible interactions.

SM interaction vertices:

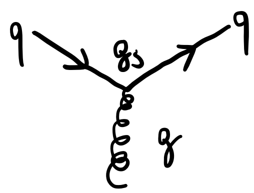
EM



all charged particles
never changes flavour

INTERACTION STRENGTH $\alpha = \frac{1}{137}$

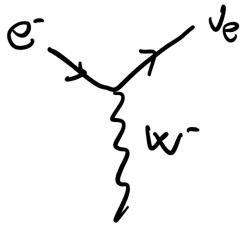
STRONG



only quarks
never changes flavour

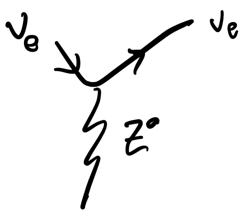
$\alpha_s \simeq 1$

WEAK



CHARGED CURRENT
all fermions
always changes flavour

$\alpha_{W/Z} \hat{\sim} \frac{1}{30}$



NEUTRAL CURRENT
all fermions
never changes flavour

Note that the intrinsic strength of the weak interaction > that of QED but the large masses of the $W^{\pm}, Z$ bosons lead to suppression → weak interaction is weakest force.

Additional notes:

- 3 generations: ≥ 3 needed for CP violation and breaking symmetry between matter and antimatter.
- Known universe is composed of u,d,e,ν_e, gauge bosons → others are short-lived and only relevant in the early universe.

- SM has 19 free parameters (masses and couplings) that are fixed by experiments.

Successes of SM:

- RENORMALISABILITY
- UNITARITY
- UNIFICATION of EM and weak forces (GUTs?)
- ALL ASPECTS (e.g. W and Z boson mass relations) have impressive agreement w/ experimental data.

Natural units:

$\hbar = c = 1 \rightarrow$ energy (mc^2), momentum (mc), mass (m) all appear as mass $\simeq$ energy. 1 GeV $\sim$ mass of proton.

Conversion factors

- $\hbar = 6.6 \times 10^{-25}\text{GeV}\, s$
- $c = 3 \times 10^8 m/s$

e.g. $1 = \hbar\, c = 10^{-17}\text{GeV}\, m \rightarrow$ length appears as inverse energy $[\text{GeV}]^{-1}$

The action $S = \int d^4x\mathcal{L}$ needs to be dimensionless so $\mathcal{L}$ has mass dimensions 4. $\mathcal{L} \sim m^4$.

The modern viewpoint of SM is it is likely to be an **EFFECTIVE FIELD THEORY-** the low energy limit of some more fundamental physics- new physics encapsulated in irrelevant operators

$$\mathcal{L} = \mathcal{L}_{SM} + \sum_{d>4} \frac{c_i \mathcal{O}^{(d)}}{M^{d-4}}$$

where d refers to the dimension of the operator in the SM fields and M the scale of the new physics. $\mathcal{L}_{SM}$ is the renormalisable part of the Lagrangian density.

In the EFT approach we include all known degrees of freedom in $\mathcal{L}$ and all possible interactions allowed by the symmetry. The sizes of the c_i are determined by experiments.

3 Group theory and symmetries

QFT provides a set of rules which- given the $\mathcal{L}$ of a system, allow us to calculate the rates of physical processes, i.e. Γ, σ.

In the SM, symmetries give structures to $\mathcal{L}$- restrict allowed interactions and bound states.

GROUP THEORY is the mathematical structure to deal with symmetries.

Group: a set of elements with a binary composition rule $\circ$ between all elemments such that:

- $\circ$ closes: $a \circ b \in \mathcal{G}$, $\forall\ a, b \in \mathcal{G}$
- $\circ$ is associative:
$$a \circ (b \circ c) = (a \circ b) \circ c, \ \ \forall a, b, c \in \mathcal{G}$$
- $\exists$ identity element I where $a \circ I = I \circ a = a$, $\forall a \in \mathcal{G}$
- Each element has an inverse:
$$\forall a \in \mathcal{G}, \exists a^{-1} \in \mathcal{G}$$
such that $a \circ a^{-1} = a^{-1} \circ a = I$

Symmetries: consider a set whose elements are "symmetry operations" i.e. that leave a system (e.g. physical object, mathematical equation, $\mathcal{L}$) unchanged.

Applying one operation after the other gives a composition rule $\rightarrow$ all symmetry operastions form a group.

Some examples of groups

* the group $\mathbb{Z}_2$: elements $\{a, b\}$ w/ composition rule

	a	b
a	a	b
b	b	a

a is the identity element

$a^{-1} = a$, $b^{-1} = b$

eg $\{-1, 1\}$

$a \circ b = b \circ a$ $\Rightarrow$ ABELIAN GROUP

- **Parity:** space inversion $\mathbf{x} \to -\mathbf{x}$. Scalars are invariant under parity (i.e. kinetic energy). Vectors transform as $\mathbf{p} \to -\mathbf{p}$. Axial vectors $\mathbf{L} = \mathbf{x} \times \mathbf{p} \to \mathbf{L}$ (i.e. invariant). Pseudo-scalars e.g. helicity (spin projected onto the direction of momentum) $h \propto \mathbf{S}.\mathbf{p} \to -h$
- The group $U(1)$ has elements $z = e^{-i\theta}$ where θ is a real, continuous parameter, with a composition rule given by ordinary multiplication, so it is abelian.
- For unitary groups $U(N)$ the elements are $N \times N$ unitary matrices with unit determinant ($U^\dagger U = I$). The composition law is matrix multiplication. $|\det(U^\dagger U)|^2 = 1 \to \det(U) = e^{i\alpha} \neq 0 \to$ inverse exists ($U^{-1} = U^\dagger$)
- SU(N): elements set of $N \times N$ unitary matrices w/ unit determinant, composition rule matrix multiplication.

We can think of a group $SU(N)$ in terms of some invariance. Suppose an element U acts on a column matrix ψ with N elements

$$\psi' = U\psi$$

where ψ' is also a N-dimensional column matrix. Examples are:

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} \nu_e \\ e \end{pmatrix} \quad \text{for } SU(2)_L$$

$$\psi = \begin{pmatrix} r \\ g \\ b \end{pmatrix} \text{for } SU(3)$$

then $\psi^\dagger\psi \to \psi^{\dagger\prime}\psi' = \psi^\dagger\psi$ so SU(N) transformations keep the norm of a state invariant.

- Lorentz group SO(3,1)- leaves invariant the line element $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. This is the "cousin" of SO(4) which leaves $dt^2 + dx^2 + dy^2 + dz^2$ invariant.
- Poincaré group- Lorentz transformations and translations.

4 Properties of group

- Discrete vs continuous: a group is continuous if one element can be changed to another element without going outside the group. e.g. $U(1)$ with elements $e^{-i\theta}$ (real θ) is continuous, as changing θ continuously gives different elements of the group. e.g. Z_2 with elements {+1,-1} are discrete, as you can't go from +1 to −1 without leaving the group.
- Spacetime vs internal: a group is spacetime if its elements inflict change on spacetime coordinates e.g. parity $\mathbf{x} \to -\mathbf{x}$, or Poincaré group. A group is internal if its elements transform quantities defined at the same point in spacetime. e.g. changing phase of the wave-function or changing one kind of particle to another.

- Local vs global: transformations are characterised by a parameter e.g. θ for $U(1)$ with $g = e^{i\theta}$. If the parameter is independent if the spacetime coordinate then the symmetry is global. If the parameter depends on the spacetime coordinates the symmetry is local. We also call local symmetries "gauge symmetries".

5 Generators, algebra, representations

For infinite groups (with infinite numbers of elements) we characterise the group by a finite number of parameters (e.g. θ for U(1))

We can then write a group element g as:

$$g = \exp -iT_\alpha \theta_\alpha$$

where T_α are the generators of the group and θ_α are the parameters. The generators can be defined through the equation:

$$T_a = i\frac{\partial g}{\partial \theta_a}|_{\theta_a=0}$$

Note that for each parameter we have 1 generator.

e.g. SU(N): an SU(N) matrix has N^2-1 independent elements so we need N^2-1 independent parameters and generators. The generators are traceless, Hermitian operators. SU(2) has 3 generators whilst SU(3) has 8.

Algebra: now $g_a = \exp -iT_a\theta_a$ and we need $g_a \circ g_b = g_c,\ g_c \in g$

Use Baker-Campbell-Hausdorff formula $e^A e^B = \exp A + B + \frac{1}{2}[A,B] + ...$ then

$$\exp -iT_a\theta_a \exp -iT_b\theta_b = \exp -iT_c\theta_c$$

$$[T_a, T_b] = if_{abc}T_c$$

where f_{abc} are the structure constants. The equation above is called the "algebra" of the group.

Representations A way of visualising group elements by considering what they do as operations on a particular choice of states.

Assign each group element an operator R_g such that:

$$R_{g1}R_{g2} = R_{g1g2}$$

We saw above how SU(N) matrices form a matrix representation of the group SU(N) and how the matrix operators act on the states ψ, N dimensional column vectors

$$\psi \to \psi' = U\psi$$

(leaving $\psi^\dagger\psi$ invariant)

The operations themselves form the representation of the group- we often also say that the states themselves form a representation.

Examples

- For any group $\mathcal{G}$ we can define the trivial or "singlet" representation

$$R_g = I \forall \text{ elements } g \in \mathcal{G}$$

- For N-dimensional column vectors, form the "defining" or "fundemental" representation of SU(N). For SU(2) the generators in the fundamental representation are the Pauli matrices $\frac{1}{2}\sigma_a$
- The structure constants also define a representation $(t_a^{adj})_{bc} = -if_{abc}$ called the "adjoint" representation $[t_a, t_b] = f_{abc}t_c$
- Conjugate representation: if $R_{g1}R_{g2} = R_{g1g2}$ then $R^*_{g1}R^*_{g2} = R^*_{g1g2}$ so $\{\bar{R}_g\}$=$\{R_g^{ast}\}$ also form a representation.

e.g. denote fundamental representation of SU(3) as $\underline{3}$ and the conjugate representation as $\underline{\bar{3}}$ or $\underline{3}^*$

6 Fields

Aim: construct an action functional involving fields, from which all dynamics can be obtained. Action should enjoy spacetime Poincaré invariance and internal $SU(3) \times SU(2)_L \times U(1)_Y$ invariance. Need fields to have definite transformation properties. We start with representations of the Lorentz group and corresponding quantum fields/operators.

- Scalar fields ("spin-0"): scalar wavefunction should solve the Klein Gordon equation:

$$(\Box^2 + m^2)\phi(x) = 0$$

$$\phi(x) = \int \frac{d^3\mathbf{p}}{\sqrt{(2\pi)^2 2E_p}}(\hat{a}(\mathbf{p})\mathbf{e^{-p.x}} + \mathbf{\hat{a}^\dagger(p)e^{ip.x}})$$

where $\hat{a}(\mathbf{p})$ annihilates a particle with 3-momentum $\mathbf{p}$ and $\hat{a}^\dagger(\mathbf{p})$ creates a particle with 3-momentum $\mathbf{p}$.

- Vector fields ("spin-1"): one example is the photon field $A_\mu(x)$, a 4-vector, defined through $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The source-free Maxwell's equations: $\partial_\mu F^{\mu\nu}$ are satisfied by:

$$A^\mu(x) = \sum_{r=0}^{3} \int \frac{d^3\mathbf{p}}{\sqrt{(2\pi)^2 2E_p}} (\hat{a}_r(\mathbf{p})\epsilon^\mu_\mathbf{r} \mathbf{e}^{-\mathbf{p}.\mathbf{x}} + \hat{\mathbf{a}}^\dagger_\mathbf{r} \epsilon^{\mu\,*}_\mathbf{r}(\mathbf{p})\mathbf{e}^{\mathbf{ip}.\mathbf{x}})$$

  where ϵ^μ_r is the polarisation vector. As the photon only carries 2 degrees of freedom, there are only two independent polarisation vectors.

- A massive vector field: satisfies equation $\partial_\mu F^{\mu\nu} + M^2 A^\mu = 0$ and has 3 independent polarisation states.

- Dirac field ("spin-1/2"): spin-$\frac{1}{2}$ fermion wavefunction satisfies the Dirac Equation: $i\gamma^\mu \partial_\mu \psi - m\psi = 0$ where $\gamma^\mu, \gamma^{nu} = 2\eta^{\mu\nu}$ e.g.:

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}$$

$$\psi(x) = \sum_{s=1,2} \int \frac{d^3\mathbf{p}}{\sqrt{(2\pi)^2 2E_p}} (\hat{d}_s(\mathbf{p})\mathbf{u_s}(\mathbf{p})\mathbf{e}^{-\mathbf{ip}.\mathbf{x}} + \hat{\mathbf{d}}^\dagger_\mathbf{s}(\mathbf{p})\mathbf{v_s}(\mathbf{p})\mathbf{e}^{\mathbf{ip}.\mathbf{x}})$$

  where $\hat{d}_s$ annihilates a particle of momentum $\mathbf{p}$ and $\hat{d}^\dagger_s$ creates a particle of momentum $\mathbf{p}$. $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ are 4-component column vectors. Note that Dirac spinors can be decomposed into Weyl spinors via the projections:

$$L = \frac{1}{2}(1-\gamma_5), \quad R = \frac{1}{2}(1+\gamma_5) \to \psi = (L+R)\psi = L\psi + R\psi = \psi_L + \psi_R$$

  Chirality is a LI property conserved for massless partiles $\to$ can write $\mathcal{L}$ in terms of ψ_L, ψ_R

Lagrangian: aim is to construct an action functional involving the fields:

$$\mathcal{L} = \int d^4x \mathcal{L}(\Phi^A, \partial_\mu \Phi^A)$$

where Φ^A are some general fields.

Classically: action principle $\to$ classical solutions are extrema of action and solve:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(partial_\mu \Phi^A)} \right) - \frac{\partial \mathcal{L}}{\partial \Phi^A}$$

Properties of Lagrangian:

- $\mathcal{L}$ should be Hermitian (+h.c. if needed).
- The dimensions of $\mathcal{L}$ should be 4 in natural units.
- Terms of order > 2 in fields are interaction terms.
- $\mathcal{L}$ is Poincaré invariant (translation $\to$ no explicit dependence on x^μ. Lorentz invariant $\to$ only Lorentz invariant combinations of fields included.

e.g. $\phi^n, \partial_\mu\phi\partial^\mu\phi, V_\mu V^\mu, (\partial_\mu V_\nu)(\partial^\mu V^\nu), (\partial_\mu V_\nu)(\partial^\nu V^\mu)$ Fermion fields must occur in even numbers (as they cary angulasr momentum $\frac{1}{2}$ and total angular momentum must be 0.

Define $\bar{\psi} = \psi^\dagger \gamma^0$ and use bilinears to construct Lorentz invariants e.g. $\bar{\psi}\psi$ (scalar), $\bar{\psi}\gamma_5\psi$ (pseudoscalar), $\bar{\psi}\gamma_\mu\psi$ (vector), $\bar{\psi}\gamma_\mu\gamma_5\psi$ (axial-vector), $\bar{\psi}\gamma_{\mu\nu}\psi$ where $\gamma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$ (tensor)

$\mathcal{L}$ will also have internal symmetries (anomaly free)- every internal symmetry of the action (through Noether's theorem) gives rise to conservation laws.

For an intfinitesimal transformation:

$$\Psi^A(x) \to \Psi'^A(x) = \Psi^A + \delta\Psi^A$$

e.g. for U(1) $\psi \to \psi' = e^{-i\theta}\psi \approx 1 - i\theta\phi$ for $\theta << 1$.

Then

$$\partial_\mu J^\mu_r = 0$$

where

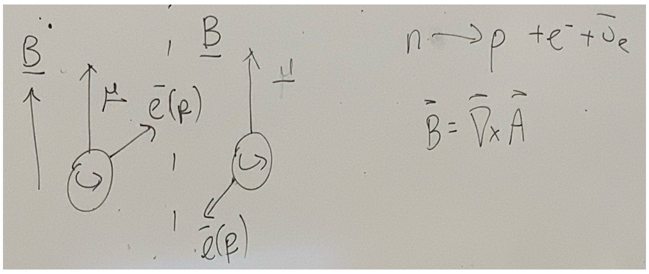
$$J^\mu_r = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta\phi}{\delta\theta_r} \text{ and } \frac{\partial Q_r}{\partial t} = 0 \text{ where} Q_r = \int J^0_r d^3\mathbf{x}$$

i.e. a "conserved current" and a "conserved charge".

7 Fermi theory of weak interactions

Although the weak interactions are very weak compared to the strong and EM interactions, they give rise to many important physical phenomena e.g. flavour changing and parity violation.

Consider the $\beta-$decay of ^{60}Co:



Now consider:

$$n \to p + e^- + \bar{\nu}_e$$

Fermi supposed 4 fermions interact at a point, via a current-current interaction

$$\mathcal{L}_{int} = -G_F \left[\bar{\psi}(p)\gamma^\mu\psi(n)\right]\left[\bar{\psi}(e)\gamma^\mu\psi(\nu_e)\right]$$

where the two terms are vector bi-linears being contracted to give a scalar. Going left to write, the $\bar{\psi}(p)$ creates a proton, the $\psi(n)$ annihilates a neutron, the $\bar{\psi}(e)$ creates an electron and the $\psi(\nu_e)$ corresponds to annihilating a ν_e or creating a $\bar{\nu}_e$.

Many other phenomena can be described by such 4-fermion current-current interactions e.g. $n+\nu_e \to p+e^-$, $\mu^- \to e^- + \nu_\mu + \bar{\nu}_e$, $e^- + \nu_e \to e^- + \nu_e$

In fact, these processes can involve parity violation. Parity violation cannot arise from [VV] type interactions, nor from [AA] interactions, need a [VA] or [AV] type interaction.

$$\mathcal{L}_{int} = \frac{G_F}{\sqrt{2}}\left[\bar{\psi}_1\gamma^\mu(c_V \pm c_A\gamma_5)\psi_2\right]\left[\bar{\psi}_3\gamma_\mu(1 \pm \gamma_5)\psi_4\right]$$

c_V and c_A are constants fixed experimentally- 4-fermi interactions turn out to involve V-A currents.

Note: we can use Fierz identifies to express $\mathcal{L}_{int}$ with different couplings.

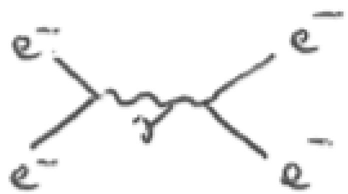
Problems of Fermi Theory

Fermi theory is very successful at describing low-energy weak interactions but

- Recall that $\mathcal{L}_{kin} = \bar{\psi}\gamma^\mu\partial_\mu psi$ and since $[\partial_\mu] = 1$, and $[\mathcal{L}_{kin}] = 4$ then $[\psi] = \frac{3}{2}$. This means that Fermi constant G_F has mass dimension -2 and the theory is non-renormalisable.
- Even sticking to tree-level diagrams, cross-sections $\sigma \propto G_F^2 s$ where s is the COM available for the process $\to$ cross-section grows linearly with s which gives a breakdown of unitarity at energies $\gtrsim G_F^{-\frac{1}{2}}$
- Better to see Fermi theory as a low energy approximation to some more complete theory which should be renormalisable and unitary.

8 Intermediate gauge bosons

Like in QED where interaction proceeds via a photon:



$e^- - e^-$ or $e^- - e^+$ elastic scattering also involves 4-fermions via a basic interaction vertex

Rather than current-current interaction, consider current-vector boson - current interaction that can be described by the 4-fermion vertex at low energies:

Write the vector boson interaction with the fermions as

$$\mathcal{L}_{int} = -gJ^{\mu}W^{\dagger}_{\mu} + h.c$$

where g is the coupling constant, J^{μ} is a superposition of Fermion bilinears and W^{+}_{μ} describes the (positively charged) vector boson. The amplitude for the annihilation diagram above is:

$$i\mathcal{M} = (-ig)^2 J^{\mu\dagger}_{i} D_{\mu\nu} J^{\nu}$$

where $D_{\mu\nu}$ is the propagator for the intermediate vector boson. For a massive vector field:

$$D_{\mu\nu}(q) = \frac{1}{q^2 - M_W^2}\left(-g_{\mu\nu} + \frac{q_{\mu}q_{\nu}}{M_W^2}\right)$$

In the limit of low energies $q^{\mu} << M_W$

$$D_{\mu\nu} \to \frac{g_{\mu\nu}}{M_W^2} \to i\mathcal{L} \sim \frac{g^2}{M_W^2} J^{\mu\dagger} J_{\mu}$$

exactly as in Fermi ineraction but with $G_F \sim \frac{g^2}{M_W^2}$. So Fermi theory emerges as an approximation to current-vector boson-current interaction for momentum transfers small compared to the vector boson mass.

But rules for renormalisibility:

1. No coupling constant with negative mass dimension.
2. $D_{boson}\ \frac{1}{q^2}$ and $D_{fermion}\ \frac{1}{q}$ for large q.

The first criteria is satisfied, but our propagator for massive vector bosons:

$$D_{\mu\nu}(q) = \frac{1}{q^2 - M_W^2}\left(-g_{\mu\nu} + \frac{q_{\mu}q_{\nu}}{M_W^2}\right) \to \frac{1}{q^2}\frac{q_{\mu}q_{\nu}}{M_W^2}$$

as $q >> M_W$ so it does not fall off as required → UV divergences.

We need to revisit the theory for massive vector bosons (need mass for Fermi theory to arise as an EFT) - this leads us to gauge symmetries (and their breakings).

9 More symmetries! Global and gauged; abelian and non-abelian

Global abelian U(1):
Consider e.g. a complex scalar field transforming under a global U(1) symmetry:

$$\phi(x) \to \phi'(x) = e^{i\theta}\phi(x)$$

with an invariant Lagrangian desnity (up to dim 4):

$$\mathcal{L}[\phi, \partial_{\mu}\phi] = \partial_{\mu}\phi^* \partial^{\mu}\phi - m^2\phi^*\phi - \frac{\lambda}{2}(\phi^*\phi)^2$$

Symmetry → masses of $\phi_{1,2}$ where $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ are equal ($\mathcal{L} \in -m^2\phi^*\phi = -m^2((\phi_1)^2 + (\phi_2)^2)$) i.e. $m_1 = m_2 = m$

Symmetry → conserved current

$$j^{\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\frac{\delta\phi}{\delta\theta} \to j_{\mu} = -i\phi^*\partial_{\mu}\phi + i\phi\partial_{\mu}\phi^*$$

with conserved charge $Q = \int d^3\mathbf{x}\mathbf{j_0}$.

Gauged (local) U(1):
Consier a local symmetry transformation $\phi(x) \to \phi'(x) = e^{i\theta(x)}\phi(x)$.
$\mathcal{L}_{kin}$ is no longer invariant

$$\partial_{\mu}\phi^*\partial^{\mu}\phi \to \partial_{\mu}(e^{-i\theta(x)}\phi^*(x))\partial^{\mu}(e^{i\theta(x)}\phi(x))$$

Promote $\partial_\mu = \frac{\partial}{\partial x^\mu}$ to a covariant derivative

$$D_\mu \phi = (\partial_\mu - ieA_\mu)\phi$$

that transforms covariantly $D_\mu\phi \to e^{i\theta(x)} D_\mu\phi$. In the expression e is the coupling constant and A_μ the "gauge field" (corresponding to the vector boson.

Now,

$$D_\mu\phi \to (D_\mu\phi)' = (\partial_\mu - ieA'_\mu)\phi' = (\partial_\mu - ieA'_\mu)(e^{i\theta(x)}\phi(x)) = e^{i\theta(x)}(\partial_\mu\phi + i\phi\partial_\mu\theta - ieA'_\mu\phi) = e^{i\theta(x)}(\partial_\mu\phi - ieA_\mu\phi) = e^{i\theta(x)}D_\mu\phi$$

provided that the gauge field transforms as

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta(x)$$

The kinetic term in the Lagrangian now transforms as

$$\mathcal{L}_{kin} = D_\mu\phi\, D^\mu\phi^* \to e^{i\theta(x)} D_\mu\phi e^{-i\theta(x)} D^\mu\phi^* = D_\mu\phi\, D^\mu\phi^*$$

Having now a massless vector boson, introduce its field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{i}{e}[D_\mu, D_\nu]$$

where the final term corresponds to the covariant form.

This can be combined into a invariant Lagrangian (scalar electrodynamics)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - m^2(\phi^*\phi) - V(\phi^*\phi)$$

Note that we still have $m_1 = m_2 = m$ for the scalar degrees of freedom and interaction as functions of $\phi^*\phi$, two polarisations for the massless vector field and conserved charge.

Global non-Abelian:

Consider now e.g. N scalar fields (e.g. N=3 for "scalar" QCD, and N=2 for the EW case).

$$\Phi = \begin{pmatrix} \Phi_1 \\ \ldots \\ \ldots \\ \ldots \\ \Phi_N \end{pmatrix}$$

with transformation $\phi_i(x) \to \phi'_i(x) = U_{ij}\phi_j(x)$ where $U \in SU(N)$ (i.e. $N \times N$ matrices with $UU^\dagger = I$ and $\det(U) = 1$ with components U_{ij}).

SU(N) is parameterise by $r = N^2 - 1$ parameters θ^a as $U = \exp\left(i\sum_{a=1}^{r}\theta^a t^a\right)$ e.g. $t^a = \sigma_a/2$ for SU(2).

The unitary transformation $\Phi \to \Phi' = U\Phi$ leaves $\Phi^\dagger\Phi \to (U\Phi)^\dagger(U\Phi) = \Phi^\dagger U^\dagger U\Phi = \Phi^\dagger\Phi$ invariant (as $U^\dagger U = I$).

The Lagrangian can be written:

$$\mathcal{L} = \partial_\mu\Phi^\dagger\partial^\mu\Phi - m^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2 = \partial_\mu\phi_i^*\partial^\mu\phi_i = m^2\phi_i^*\phi_i - \lambda(\phi_i^*\phi_i)^2 (*)$$

This is invariant under a global SU(N) transformation. By Symmetry, all fields ϕ_i have the same mass and coupling constant and set of conserved charges.

Gauged (local) non-Abelian

$\mathcal{L}$ in $(*)$ is no longer invariant if the SU(N) transformation is coordinate dependent i.e. $\phi_i(x) \to \phi'_i(x) = U_{ij}(x)\phi_i(x)$ where $U(x) \in SU(N)$. This is because $\partial_\mu\Phi'(x) = \partial_\mu(U(x)\Phi(x)) = U(x)\partial_\mu\Phi(x) + \partial_\mu U(x).\Phi(x)$ renders the kinetic term not invariant.

As for the Abelian case, introduce gauge fields and promote the partial derivatives to covariant derivatives

$$D_\mu\phi_i = \partial_\mu\phi_i - igA^a_\mu T^a_{ij}\phi_j$$

where $a = 1, ..., N^2 - 1$ and $i, j = 1, ..., N$. The $A_{\mu ij}$ are new vector fields c.f. photon and the T^a_{ij} are the generators of SU(N) in the representation of ϕ_j.

The $A_{\mu ij}$ transforms such that $(D_\mu\phi_i)' = U_{ij}D_{mu}\phi_j$ $(D_\mu\Phi)' = U(D_\mu\Phi)$.

$$D_\mu\Phi' = \partial_\mu\Phi' - igA'_\mu\Phi' = U\partial_\mu\Phi + \partial_\mu U.\Phi - igA'_\mu U\Phi$$

$$\to A_\mu \to A'_\mu = UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}$$

writing $A_{\mu ij} = A^a_\mu T^a_{ij}$ and $U = \exp -igT_a\theta^a$, $\partial_\mu U = (-igT_b\partial_\mu\theta^b)U$ we can show that (expand in small θ^a

$$A^{a\prime}_\mu = A^a_\mu + gf_{bca}\theta^b A^c_\mu + \partial_\mu\theta^a + ... = A^a_\mu - g\theta^b(t^{adj.}_b)_{ac}A^c_\mu$$

i.e. A^a_μ transforms in the adjoint of the symmetry group. Now

$$\mathcal{L} = (D)_\mu\Phi)^\dagger D_\mu\Phi - m^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2 - \frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$$

where $F_{\mu\nu} = \sum F^a_{\mu\nu}t^a$ and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf_{bca}A^b_\mu A^c_\nu$

$F_{\mu\nu}(x) \to F'_{\mu\nu}(x) = U(x)F_{\mu\nu}U^{-1}(x)$ so we have a theory with N^2-1 massless vector bosons that self-interact $\to$ dynamics distinct from QED.

10 Spontaneous symmetry breaking

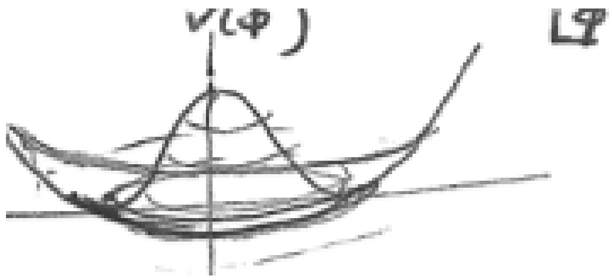
Aim: a theory of massive vector bosons.

Consider $\mathcal{L}$ invariant under some internal symmetry, with a vacuum state which is **not** invariant under the symmetry transformation $\to$ "spontaneous symmetry breaking".

Global symmetry:
e.g. Our U(1) symmetric theory of potential

$$V(\phi,\phi^*) = -\frac{\mu^2}{2}\phi\phi^* + \frac{\lambda}{4}(\phi\phi^*)^2$$

as sketched below (note $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$)



The minimum of the potential lies at any point along the curve $\phi\phi^* = \frac{1}{2}(\phi_1^2+\phi_2^2) = \frac{1}{2}V^2 = \frac{\mu^2}{\lambda}$.

The U(1) symmetry $\phi \to \phi e^{i\theta}$ takes a point on the circle and moves it to another point in the circle $\to$ the vacuum is not invariant.

Take e.g. the Vacuum $\phi_1 = V, \phi_2 = 0$ and expand the fields on top of this vacuum:

$$\phi_1(x) = v + \chi^x, \;\; \phi_2(x) = b(x)$$

where $\chi(x), b(x)$ are quantum fields expanded in creation/annihilation operators, acting on the vacuum (from right/left respectively).

The expansion of $V(\phi_1,\phi_2)$ gives:

$$V = -\frac{\mu^2}{2}\frac{1}{2}[(v+\chi)^2 + b^2] + \frac{\lambda}{4}[\frac{1}{2}((v+\chi)^2+b^2)]^2$$

up to quadratic order (and dropping constant terms)

$$\mathcal{L}^2_{\chi,\theta} = \frac{1}{2}(\partial_\mu\chi)^2 + \frac{1}{2}(\partial_\mu b)^2 - \frac{1}{2}\mu^2\chi^2$$

$\to$ one massive field χ, and one massless field b and higher order terms (interactions between χ and b). The massless field is called a "Goldstone boson" $\to$ generic consequence of spontaneously breaking a global symmetry, one massless boson appears in the spectrum of perturbations.

e.g. pions are approximate Goldstone bosons of the global SU(2) symmetry in QCD of massless quarks, broken by non-perturbative quark condensate. Because quarks are not exactly massless in fundamental theory, the SU(2) symmetry is only approximate, and pions are not exactly massless, but much lighter than the other states in the theory.

11 Gauge symmetry breaking- the Higgs mechanism

Consider the Abelian Higgs model:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - [-\mu^2\phi^*\phi + \lambda(\phi^*\phi)^2]$$

where under the gauge transformation $\phi(x) \to \phi^*(x) = e^{i\theta(x)}\phi(x)$ and the covariant derivative is $D_\mu\phi = (\partial_\mu - ieA_\mu)\phi$ where the gauge field transforms as:

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x)$$

Choose a vacuum (values of field with minimum energy):

$$A_\mu^{(v)} = 0, \quad \phi^{(v)} = \frac{1}{\sqrt{2}}v \to \text{ SSB}$$

Expand into small perturbations on top of the vacuum

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \chi(x) + ib(x))$$

so that $D_\mu\phi = (\partial_\mu\phi - ieA_\mu\phi) = \frac{1}{\sqrt{2}}(\partial_\mu\chi + i\partial_\mu b - ievA_\mu)$ to leading orders.

Plugging into $\mathcal{L}$ and expanding to 2nd order:

$$\mathcal{L}^2 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi) - \mu^2\chi^2 + \frac{e^2v^2}{2}(A_\mu - \frac{1}{ev}\partial_\mu b)^2$$

Note b and A_μ only enter in the combination $B_\mu = A_\mu - \frac{1}{ev}\partial_\mu b$. Define $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ and rewrite again

$$\mathcal{L}^{(2)} = -\frac{1}{2}B_{\mu\nu}B^{\mu\nu} + \frac{e^2v^2}{2}B_\mu B^\mu + \frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi) - \mu^2\chi^2$$

So we have a theory for a massive vector field and a massive scalar field where:

$$m_v = ev = \frac{e\mu}{\sqrt{\lambda}}, \quad m_\chi = \sqrt{2}\mu = \sqrt{2\lambda}v$$

Note that b disappears A_μ has "eaten" the Goldstone boson tobecome massive- recall that a massless vector field has 2 dofs whereas a massive one has 3 dofs.

12 Standard EW theory

Two key features- SSB and chirality.

Recall that Dirac spinors can be decomposed into chiral spinors

$$\psi_L = L\psi = \frac{1}{2}(1-\gamma_5)\psi, \quad \psi_R = R\psi = \frac{1}{2}(1+\gamma_5)\psi, \quad \psi = \psi_L + \psi_R$$

Chirality is preserved under Lorentz transforations.

Conjugate of a LH spinor is $\bar{\psi}_L = \bar{\psi}R$.

Mass term can be written as $\bar{\psi}\psi = \bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R \to$ need both chirality to write down mass terms e.g. need e_L, e_R but we only have ν_L.

No reason for internal symmetries to act in the same way on L and R chiral sectors $\to$ L and R fermions can have different interactions.

Note that ψ_R is the parity transformation of ψ_L, $\psi_L(x) \overset{P}{\to} \pm\gamma_0\psi_R(\tilde{x})$, so any symmetry under which ψ_L and ψ_R transform differently will violate parity (i.e. weak interaction).

Gauge and Higgs bosons

Consider gauge symmetries $SU(2)_L \times U(1)_Y$ where $SU(2)_L$ is the weak gauge symmetry and Y corresponds to the hypercharge. $SU(2)_L$ acts non trivially only on L chiral fermions.

Gauge fielda $W_\mu^{a=1,2,3}$ and B_μ and field strengths $F^a_{\mu\nu}(= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - f^a_{bc}W^b_\mu W^c_\nu)$ and $B_{\mu\nu}$

Add one complex scalar field $SU(2)_L$ doublet $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$, i.e. the "Higgs" field, with hypercharge $Y_\phi = 1$ to mediate SSB.

Gauge invariant Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - \lambda\left(\phi^\dagger\phi - \frac{v^2}{2}\right)^2$$

where the final term is the scalar potential, and the covariant derivative is

$$D_\mu \phi = (\partial_\mu - ig\frac{\sigma_a}{2}W^a_\mu - i\frac{g'}{2}Y_\phi B_\mu)\phi$$

where $Y_\phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and g and g' are the coupling constants corresponding to the gauge interactions.

The minimum of the potential is at

$$|\langle\phi_1\rangle|^2 + |\langle\phi_2\rangle|^2 = \frac{1}{2}v^2$$

Under $SU(2)_L$ $U\langle\phi\rangle \neq \langle\phi\rangle$ in general ($U = e^{i\theta^a T^a} = I + i\theta^a T^a + ...$).

If we choose min $\langle\phi\rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$ then we get SSB EW symmetry breaking.

Note that $Q = \frac{\sigma^3}{2} + \frac{Y_\phi}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ does not change the vacuum state $U\langle\phi\rangle = (1+Q)\langle\phi\rangle = \langle\phi\rangle$, $\left(Q\langle\phi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$ so SSB leaves a remnant U(1) symmetry unbroken.

$$SU(2)_L \times U(1)_Y \to U(1)_{EM}$$

Now expand about the vacuum $\langle W^a_\mu\rangle = 0$, $\langle B_\mu\rangle = 0$, $\langle\phi\rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$.

$$\phi = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} + \frac{H(x)}{\sqrt{2}} \end{pmatrix}$$

(note: we have gone to the "unitary gauge" where the Goldstone bosons disappear from the spectrum).

$$D_\mu\phi = \partial_\mu\phi + \left[-\frac{ig}{2}W^1_\mu\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{ig}{2}W^2_\mu\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \frac{ig}{2}W^3_\mu\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{ig'}{2}B_\mu\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} + \frac{H(x)}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{ig}{2\sqrt{2}}(W^1_\mu - iW^2_\mu)(v + H(x)) \\ -\frac{ig}{2\sqrt{2}}(g'B_\mu - gW^3_\mu)(v + H(x)) + \frac{1}{\sqrt{2}}\partial_\mu H(x) \end{pmatrix}$$

Introduce $W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \mp iW^3_\mu)$ so that $(W^-_\mu)^* = W^*_\mu$, $Z_\mu = \frac{1}{\sqrt{g^2+g'^2}}(gW^3_\mu - g'B_\mu)$ and $A_\mu = \frac{1}{\sqrt{g^2+g'^2}}(gB_\mu + g'W^3_\mu$. Then we can write

$$D_\mu\phi = \begin{pmatrix} -\frac{igv}{2}W^+_\mu \\ \frac{1}{\sqrt{2}}\partial_\mu H(x) + \frac{i\sqrt{g^2+g'^2}}{2\sqrt{2}}vZ_\mu \end{pmatrix} + \begin{pmatrix} -\frac{ig}{2}W^+_\mu H \\ \frac{i\sqrt{g^2+g'^2}}{2\sqrt{2}}Z_\mu H(x) \end{pmatrix}$$

where the first term is linear in excitations whilst the second term is the second order terms. To second order in the excitations:

$$\mathcal{L} \subset [(D_\mu\phi)^\dagger(D_\mu\phi)]^{(2)} = \frac{1}{2}(\partial_\mu H)^2 + \frac{g^2v^2}{4}W^+_\mu W^{\mu-} + \frac{1}{2}\frac{(g^2+g'^2)}{4}v^2 Z_\mu Z^\mu$$

where $(D_\mu\phi)^\dagger$ is (1×2) and $D_\mu\phi$ is (2×1). So we have massive vector bosons $W^\pm_\mu$ and Z_μ with masses

$$m_W = \frac{gv}{2}, \quad m_Z = \frac{\sqrt{g^2+g'^2}}{2}v$$

One can derive by expanding the scalar potential $V(\phi)$ to second order in $H(x)$ that

$$m_H = \sqrt{2\lambda}v$$

...and don't forget the massless photon A_μ!

Introduce the Weinberg mixing angle

$$\cos\theta_w = \frac{g}{\sqrt{g^2+g'^2}}, \quad \sin\theta_w = \frac{g'}{\sqrt{g^2+g'^2}}$$

then we get $m_Z = \frac{m_W}{\cos\theta_w}$.

Experimentally $\sin^2\theta_w \approx 0.23$- measured independently (via γ, W, Z, with quarks and leptons) $\rightarrow$ precision test of SM.

Quarks and leptons under EW symmetry:

Leptons e_L^- and ν_{eL} form a doublet under $SU(2)_L$

$$\psi_L = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \text{ and also } \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}, \ \psi_L \rightarrow \psi_L' = U\psi_L$$

and transform under hypercharge with charge $Y_L = 1$. ($\psi_L \rightarrow e^{iY_L\theta(x)}\psi_L$)

e_R is $SU(2)_L$ singlet and has hypercharge $Y_R = 2$. As for neutrinos that are massless- no need for ν_R.

$$Q_{EM} = \frac{\sigma^3}{2} + \frac{Y}{2}I = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow Q_{EM}e_L = -1, \ \ Q_{EM}\nu_L = 0$$

Note $\bar{e}_L e_R$ would not be gauge invariant but

$$\mathcal{L}_{yukawa,e^-} = -h_e\bar{\psi}_L^{i=1,2}\phi^i e_R - h_e\bar{e}_R\psi^{\dagger i}\psi_L^i$$

where h_e is the coupling constant is allowed under gauge invariance and gives an effective mass term after SSB:

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \rightarrow m_e = \frac{h_e v}{\sqrt{2}}$$

Quarks LH quarks form doublets under $SU(2)_L$ and have $Y_{q_L} = \frac{1}{3}$

$$\begin{pmatrix} u_L \\ d_L' \end{pmatrix}, \ \begin{pmatrix} c_L \\ s_L' \end{pmatrix}, \ \begin{pmatrix} t_L \\ b_L' \end{pmatrix}$$

(Note we need CKM matrix to go to the mass eigenstate basis).

RH quarks are $SU(2)_L$ singlets with $Y_{u_R} = \frac{4}{3}$ and $Y_{d_R} = -\frac{2}{3}$