QUESTION 1: HALO

The halo is the region populated by dark matter with a galaxy at its centre. One can show that the rotation curves of galaxies can be reproduced if the dark matter mass distribution scales scales like

$$\rho(r) = \frac{v_c^2}{4\pi G}\frac{1}{r^2} \qquad \Rightarrow \qquad M(r) = \frac{v_c^2 r}{G_N}$$

where v_c is a constant circular velocity and G_N is Newtons constant. Here the goal is to show that an ideal, isotropic gas in hydrostatic equilibrium with gravity with a Maxwell velocity distribution

$$f(\vec{v}) = N \exp\left(-\frac{1}{2}\frac{|\vec{v}|^2}{\sigma^2}\right) ,$$

predicts such a mass distribution.

a) Show that the mean square velocity squared $\langle v^2 \rangle$ is proportional to the temperature of an ideal gas.

b) Calculate the mean speed for the Maxwell Boltzmann distribution

$$\langle v^2 \rangle = \int_0^\infty v^2 f(v) dv$$

and solve for the velocity distribution σ using the result from a). Using the ideal gas law you should find the relation between the pressure $p(r)$ and the density $\rho(r)$,

$$p(r) = \sigma^2 \rho(r) .$$

Hint: $\int_{-\infty}^{\infty} e^{-x^2/b} dx = \sqrt{b\pi}$

c) Balance the force exerted by the hydrostatic pressure on a volume element with surface A perpendicular to r and infinitesimal thickness dr with the gravitational force. Solve the resulting differential equation for the matter-energy density using the ansatz $\rho(r) = a/r^2$ and fix the constant a. *Hint: You can use*

$$F_{\text{hydro}} = F_{\text{grav}}$$
$$(P(r+dr) - P(r))A = -G_N \frac{M(r)}{r^2} = -\rho A dr \frac{G_N M(r)}{r^2} .$$

and the relation for spherically symmetric mass distributions

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) .$$

QUESTION 2: CMB

The cosmic microwave background (CMB) radiation is the relic of the recombination era when photons for the first time were free to travel through the universe without constantly being scattered off free electrons and protons. Today the CMB is an almost perfect black body radiation with a temperature of $T_{\text{CMB}} = 2.725\,K$.

a) What is the wavelength and frequency of the CMB today? *hint: The CMB is black-body radiation. Use Wiens displacement law* $\lambda_{peak} = b/T$ *with* $b = 2.9 \times 10^{-3} mK$.

b) Why do we know that the temperature of the CMB at decoupling was 3000 K? What was its wavelength then? Would it have been visible to the human eye and can you guess its color?

c) How old was the universe when the night sky turned black? ***Hint:*** This should happen at the latest when the black-body spectrum has an average temperature of $T \approx 700$ K (the Draper point).

d) Go to https://lambda.gsfc.nasa.gov/bau/ and solve the CMB challenge. Understand what the parameter mean and try to get the fit right (don't click answer yet!) Why does the flatness of the universe depend on the first three parameters, but not on the Hubble constant, the spectral index and the re-ionization redshift?

QUESTION 3: COSMIC EXPANSION

The expansion of the Universe is described by a time-dependent solution to the Einstein equations

$$R_{\mu\nu}(t) - \frac{1}{2} g_{\mu\nu}(t) R(t) + \Lambda(t) = \frac{T_{\mu\nu}(t)}{M_{\text{Pl}}^2}\,, \tag{1}$$

where $R_{\mu\nu}(t)$ is the Ricci tensor, which measures how much curvature the spacetime has, $R(t) = R^{\mu}_{\mu}(t)$ is it's trace, $g_{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ is the stress-energy tensor and M_{Pl} is the Planck scale. The cosmological constant is a time-independent parameter in general relativity. The early Universe can be described as a uniform, relativistic fluid so that we can assume the stress energy tensor for a perfect fluid

$$T^{\mu\nu} = (\rho + P)\, u^{\mu} u^{\nu} - P g^{\mu\nu}\,,$$

where u_{μ} is the 4-velocity, ρ is the mass-energy density and P is the hydrostatic pressure.

a) Show that if the trace of the energy momentum tensor vanishes it follows that $P/\rho = 1/3$. The trace of the stress energy tensor is proportional to the mass scales in the theory. Explain why it is ok to set it to zero for a relativistic fluid. Argue that a cosmological constant with $T_{\mu\nu} = \rho_{\text{vac}} g_{\mu\nu}$ with $\rho_{\text{vac}} > 0$ corresponds to negative pressure. Finally, what is the relation for P/ρ for non-relativistic matter?

b) Using the Friedmann equation relating the mass-energy density and the evolution of the scale factor shown in the lecture

$$\rho(t) = \frac{3}{8\pi G_N} \left(\frac{\dot{a}(t)}{a(t)} \right)^2 \tag{2}$$

and the evolution of the scale factor dependence for a matter, radiation and a cosmological constant with density $\rho_{\text{vac}} = \Lambda M_{\text{Pl}}^2$ dominated Universe to show that

$$a(t) \propto \begin{cases} t^{-2/3} & \text{matter dominated}\,, \\ t^{-1/2} & \text{radiation dominated}\,, \\ e^{\sqrt{\Lambda}\, t} & \Lambda \text{ dominated}\,. \end{cases} \tag{3}$$

QUESTION 4: DIRECT DETECTION

Direct searches for Dark Matter are shielded detectors placed underground, which try to measure interactions of Dark Matter with targets made from different elements. A recent summary plot of direct detection bounds is shown in Fig. 1. Experimental bounds are shown as solid lines, projected bounds are shown as dashed lines. Possible signal candidates are shown as colored regions, while the thick, dashed orange line demarks the neutrino floor.

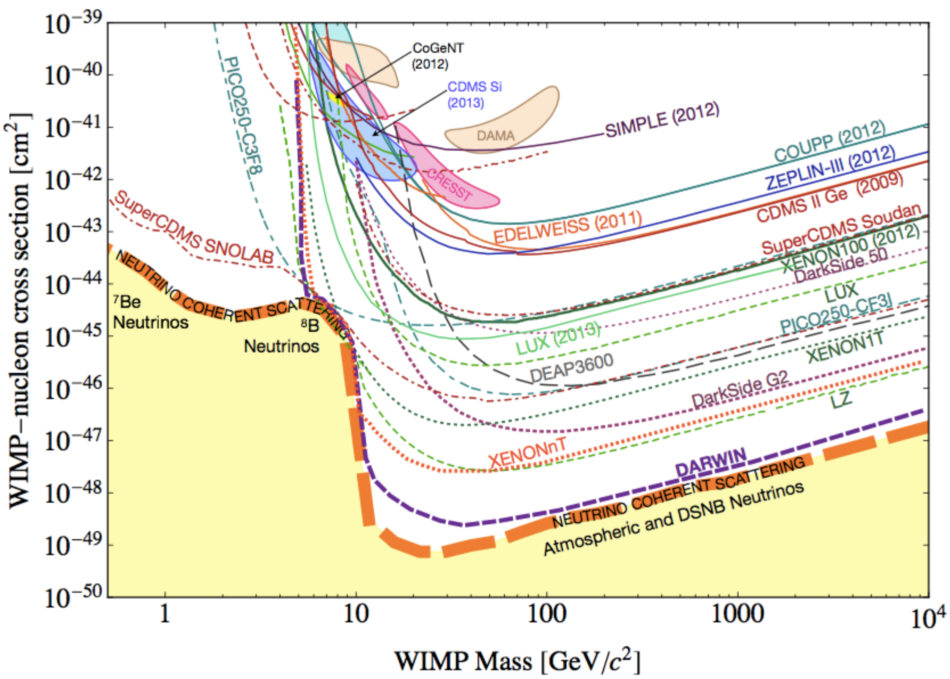

Fig 1: Direct Detection bounds from various experiments. (adapted from Snowmass, 1310.8327)

a) The local dark matter density is $\rho = 0.3$ GeV/cm^3. How many dark matter particles would you expect in a 2L bottle if the dark matter had a mass of 5 GeV or 500 GeV?

b) Explain the general shape of the exclusion limits (the sharp drop at a ~ 10 GeV and the linear slope for higher masses)! Can you explain the differences in the excluded regions from different experiments? What would you guess is the scaling for the sensitivity with mass and runtime of these experiments?

You are now an experimentalist working in a Direct Detection collaboration considering two detectors. The first is based on a Germanium target and the second has a Xenon target:

		Germanium	**Xenon**
Energy Threshold	E_t	1 keV	5 keV
Energy Interval	ΔE	(1-40) keV	(5-40) keV
Target Mass	M	1 kg	35 kg
Target Element Mass	m_A	65 GeV	122 GeV
Mass Number	A	73	131

The goal is to provide the best possible limit on a theory predicting a

1) light dark matter candidate $M_\chi = 5$ GeV ,

2) heavy dark matter candidate $M_\chi = 500$ GeV .

In both cases the commissioned runtime is $T = 100$ days.

c) Direct detection experiments are limited by the nuclear recoil energy threshold of the target material E_t. In terms of the velocity v of the Dark Matter particle and the scattering angle θ, the recoil energy is given by

$$E_R = v^2 \frac{\mu_N^2}{m_A}(1 - \cos\theta)\,, \tag{4}$$

in which the reduced mass is given by $\mu_N = \frac{m_A M_\chi}{m_A + M_\chi}$. Compute the minimal velocity $v_{\rm min}$ for the Germanium and Xenon detector and the two Dark Matter masses given above (note, that v is given as a fraction of $c = 3 \times 10^5$ km/s using the units in the table.)

d) The Dark Matter velocity distribution follows a Maxwell-Boltzmann distribution

$$f(\mathbf{v}) = N\, e^{-\mathbf{v}^2/v_0^2}\,, \tag{5}$$

with $v_0 = 220$ km/s the circular velocity of the Dark Matter halo and $N = 1/(\sqrt{\pi} v_0)^3$. Integrate over the solid angle to obtain the velocity distribution $f(|\mathbf{v}|)$. Plot this function and discuss what it implies for the velocities you have derived in c)? Does it even make sense to consider very fast Dark Matter particles or should the velocity distribution be cut off at a certain speed $v_{\rm max}$?

e) The expected rate for WIMP interactions can be expressed as

$$R \approx \frac{A^2}{2\mu_P^2\, M_\chi}\, \sigma_0 \rho_\chi \int_{v_{\rm min}}^{v_{\rm max}} \frac{f(v)}{v}\, dv \cdot \Delta E\,, \tag{6}$$

in which $\mu_P = \frac{m_N M_\chi}{m_N + M_\chi}$ is the reduced mass of the Dark Matter and a Nucleon (eitehr proton or neutron) $m_N \approx 1$ GeV. The local dark matter density $\rho_\chi = 0.3$ GeV/cm^3 and the velocity distribution given above are astrophysical inputs. The mass of the Dark Matter candidate and the cross section $\sigma_0 = 1 \cdot 10^{-38}$ cm^2 are quantities provided by your particle physics colleague. Compute the expected number of events $N = R \cdot T \cdot M$ for the two detectors and both the heavy and light Dark Matter candidate. (Make sure you are using consistent units!)

TABLE IV. 90% C.L. intervals for the Poisson signal mean μ, for total events observed n_0, for known mean background b ranging from 0 to 5.

$n_0 \backslash b$	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0
0	0.00, 2.44	0.00, 1.94	0.00, 1.61	0.00, 1.33	0.00, 1.26	0.00, 1.18	0.00, 1.08	0.00, 1.06	0.00, 1.01	*0.00, 0.98*
1	0.11, 4.36	0.00, 3.86	0.00, 3.36	0.00, 2.91	0.00, 2.53	0.00, 2.19	0.00, 1.88	0.00, 1.59	0.00, 1.39	0.00, 1.22
2	0.53, 5.91	0.03, 5.41	0.00, 4.91	0.00, 4.41	0.00, 3.91	0.00, 3.45	0.00, 3.04	0.00, 2.67	0.00, 2.33	0.00, 1.73
3	1.10, 7.42	0.60, 6.92	0.10, 6.42	0.00, 5.92	0.00, 5.42	0.00, 4.92	0.00, 4.42	0.00, 3.95	0.00, 3.53	0.00, 2.78
4	1.47, 8.60	1.17, 8.10	0.74, 7.60	0.24, 7.10	0.00, 6.60	0.00, 6.10	0.00, 5.60	0.00, 5.10	0.00, 4.60	0.00, 3.60
5	1.84, 9.99	1.53, 9.49	1.25, 8.99	0.93, 8.49	0.43, 7.99	0.00, 7.49	0.00, 6.99	0.00, 6.49	0.00, 5.99	0.00, 4.99
6	2.21,11.47	1.90,10.97	1.61,10.47	1.33, 9.97	1.08, 9.47	0.65, 8.97	0.15, 8.47	0.00, 7.97	0.00, 7.47	0.00, 6.47
7	3.56,12.53	3.06,12.03	2.56,11.53	2.09,11.03	1.59,10.53	1.18,10.03	0.89, 9.53	0.39, 9.03	0.00, 8.53	0.00, 7.53
8	3.96,13.99	3.46,13.49	2.96,12.99	2.51,12.49	2.14,11.99	1.81,11.49	1.51,10.99	1.06,10.49	0.66, 9.99	0.00, 8.99
9	4.36,15.30	3.86,14.80	3.36,14.30	2.91,13.80	2.53,13.30	2.19,12.80	1.88,12.30	1.59,11.80	1.33,11.30	0.43,10.30
10	5.50,16.50	5.00,16.00	4.50,15.50	4.00,15.00	3.50,14.50	3.04,14.00	2.63,13.50	2.27,13.00	1.94,12.50	1.19,11.50
11	5.91,17.81	5.41,17.31	4.91,16.81	4.41,16.31	3.91,15.81	3.45,15.31	3.04,14.81	2.67,14.31	2.33,13.81	1.73,12.81
12	7.01,19.00	6.51,18.50	6.01,18.00	5.51,17.50	5.01,17.00	4.51,16.50	4.01,16.00	3.54,15.50	3.12,15.00	2.38,14.00
13	7.42,20.05	6.92,19.55	6.42,19.05	5.92,18.55	5.42,18.05	4.92,17.55	4.42,17.05	3.95,16.55	3.53,16.05	2.78,15.05
14	8.50,21.50	8.00,21.00	7.50,20.50	7.00,20.00	6.50,19.50	6.00,19.00	5.50,18.50	5.00,18.00	4.50,17.50	3.59,16.50
15	9.48,22.52	8.98,22.02	8.48,21.52	7.98,21.02	7.48,20.52	6.98,20.02	6.48,19.52	5.98,19.02	5.48,18.52	4.48,17.52
16	9.99,23.99	9.49,23.49	8.99,22.99	8.49,22.49	7.99,21.99	7.49,21.49	6.99,20.99	6.49,20.49	5.99,19.99	4.99,18.99
17	11.04,25.02	10.54,24.52	10.04,24.02	9.54,23.52	9.04,23.02	8.54,22.52	8.04,22.02	7.54,21.52	7.04,21.02	6.04,20.02
18	11.47,26.16	10.97,25.66	10.47,25.16	9.97,24.66	9.47,24.16	8.97,23.66	8.47,23.16	7.97,22.66	7.47,22.16	6.47,21.16
19	12.51,27.51	12.01,27.01	11.51,26.51	11.01,26.01	10.51,25.51	10.01,25.01	9.51,24.51	9.01,24.01	8.51,23.51	7.51,22.51
20	13.55,28.52	13.05,28.02	12.55,27.52	12.05,27.02	11.55,26.52	11.05,26.02	10.55,25.52	10.05,25.02	9.55,24.52	8.55,23.52

Tab.1: Lower and upper bound on the expected events for a 90% confidence level and a given number of signal n_0 and background b events. (Feldman, Cousins, 9711021)

f) Now assume that you measured 3 signal events during the runtime, but also 2 background events. Table 1 gives the lower and upper bound on the expected events for a 90% confidence level and a given number of signal n_0 and background b events. Use these numbers and the formulas in d) to derive a bound on the cross section σ_0 for the two experiments and scenarios described above.

Nucleus	Z	Odd Nucleon	J	$\langle S_p \rangle$	$\langle S_n \rangle$	C_A^p/C_p	C_A^n/C_n
^{19}F	9	p	1/2	0.477	-0.004	9.10×10^{-1}	6.40×10^{-5}
^{23}Na	11	p	3/2	0.248	0.020	1.37×10^{-1}	8.89×10^{-4}
^{27}Al	13	p	5/2	-0.343	0.030	2.20×10^{-1}	1.68×10^{-3}
^{29}Si	14	n	1/2	-0.002	0.130	1.60×10^{-5}	6.76×10^{-2}
^{35}Cl	17	p	3/2	-0.083	0.004	1.53×10^{-2}	3.56×10^{-5}
^{39}K	19	p	3/2	-0.180	0.050	7.20×10^{-2}	5.56×10^{-3}
^{73}Ge	32	n	9/2	0.030	0.378	1.47×10^{-3}	2.33×10^{-1}
^{93}Nb	41	p	9/2	0.460	0.080	3.45×10^{-1}	1.04×10^{-2}
^{125}Te	52	n	1/2	0.001	0.287	4.00×10^{-6}	3.29×10^{-1}
^{127}I	53	p	5/2	0.309	0.075	1.78×10^{-1}	1.05×10^{-2}
^{129}Xe	54	n	1/2	0.028	0.359	3.14×10^{-3}	5.16×10^{-1}
^{131}Xe	54	n	3/2	-0.009	-0.227	1.80×10^{-4}	1.15×10^{-1}

Tab.2: Input values for spin-dependent cross sections. (Tovey et al., PLB 488 17(2000), 0005041)

g) You theorist friend comes back and is excited. A new model for dark matter could explain many of the observations. The theorist explains that the model includes a dark matter candidate χ with mass $M_\chi = 15$ GeV and a new particle, a scalar S that interacts with Standard Model particles and the dark matter and has a mass $M_S = 300$ GeV. The scalar S interacts exclusively with neutron spin with some coupling constant g. Use Table to find the best possible target material for a direct detection experiment to probe such a dark matter model. Now estimate the cross section. Draw a Feynman diagram and use dimensional analysis and your knowledge of Feynman rules to estimate $\sigma(\chi N \to \chi N)$. How long would it take you with a 10 kg target detector to be able to discover the signal of this model.

h) What is the neutrino floor (the dashed orange line in Fig.1)? Why does it have a similar shape as the exclusion limits? Should it be the same for different target elements? Look into (Billard, Strigari, Figueroa-Feliciano, 1307.5458).