Quantum Field Theory - Lecture 2

Let us now talk about classical fields. A field is a function that acts on spacetime: it takes in a spacetime point x^μ and it outputs a value. That value may be

- a number (scalar field)
- a vector (vector field)
- a spinor (spinor field)
- ...

Examples: Temperature in this room → scalar
Velocity of a fly flying around → vector

Fields obey field equations, e.g.

$$\vec{\nabla}\cdot\vec{B} = 0, \qquad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}, \qquad \frac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$$

Key point: Field equations can be found from a Lagrangian or Hamiltonian.

Lagrangian

We will focus on a single scalar field $\phi(x)$,

$$\phi(x) = \phi(x^0, x^i).$$

We will only consider Lagrangians without explicit time dependence. Then with ϕ thought of as a generalised coordinate q, L will depend on

$$\phi(x), \quad \partial_\mu\phi(x) = (\partial_0\phi(x^0,x^j), \partial_i\phi(x^0,x^j))$$

When we discussed the Lagrangian in classical mechanics our generalised coordinates were only a function of time. In classical field theory they are a function of spacetime, and we can similarly define a quantity that, when integrated over spacetime, gives us the action:

$$S(\phi) = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi).$$

This curly $\mathcal{L}$ is called the Lagrangian density. The principle of least action now gives

$$\delta S = 0 \implies \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

This is the Euler-Lagrange equation or equation of motion for a general scalar field theory.

<u>Example</u>: Free massive scalar field

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \\
&= \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \\
&= \frac{1}{2} \partial_0 \phi \, \partial_0 \phi - \frac{1}{2} \sum_{i=1}^{3} \partial_i \phi \, \partial_i \phi - \frac{1}{2} m^2 \phi^2 \\
&= \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \vec{\nabla} \phi \cdot \vec{\nabla} \phi - \frac{1}{2} m^2 \phi^2
\end{aligned}$$

Euclidean dot product in space

We have

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} &= \frac{\partial}{\partial(\partial_\mu \phi)} \left(\frac{1}{2} \eta^{\nu\rho} \partial_\nu \phi \, \partial_\rho \phi \right) \\
&= \frac{1}{2} \eta^{\nu\rho} \left(\delta_\nu{}^\mu \partial_\rho \phi + \partial_\nu \phi \, \delta_\rho{}^\mu \right) \\
&= \frac{1}{2} \left(\eta^{\mu\rho} \partial_\rho \phi + \eta^{\nu\mu} \partial_\nu \phi \right)
\end{aligned}$$

$$= \partial^\mu \phi$$

and so

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \Rightarrow \partial_\mu(\partial^\mu \phi) + m^2 \phi = 0 .$$

This is the Klein-Gordon equation.

Hamiltonian

Introduce

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

and define the Hamiltonian density

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L}$$

Example: For $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$ one can show (see problems) that $\pi = \dot{\phi}$ and

$$\mathcal{H} = \frac{1}{2} \left(\pi^2 + \vec{\nabla}\phi \cdot \vec{\nabla}\phi + m^2 \phi^2 \right)$$

Energy-momentum tensor

Suppose we have a Lagrangian that possesses a symmetry, meaning it changes at most by a total derivative under a group of transformations. Then, Noether's theorem tells us that there exists a corresponding conserved current.

For a theory with a scalar field, consider translations

$$x^\mu \longrightarrow x^\mu + \varepsilon^\mu$$

fixed 4-vector considered infinitesimal

Then, $\phi(x) \rightarrow \phi(x+\varepsilon) = \phi(x) + \varepsilon^\mu \partial_\mu \phi(x) + \dots .$

It is then straightforward to compute that

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}$$

is conserved, i.e. $\partial_\mu T^{\mu\nu} \overset{\text{upon using the equation of motion}}{=} 0, \quad \nu = 0, 1, 2, 3.$ Indeed, if $\delta x^\mu = \varepsilon^\mu$, then

$$\delta \mathcal{L} = \underbrace{\varepsilon^\mu \partial_\mu \mathcal{L}}_{\text{total derivative, so } \delta S = 0}$$

for arbitrary ϕ, and also

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) = \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta \phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \phi \right).$$

If the Euler-Lagrange equations are satisfied, then

$$\varepsilon^\mu \partial_\mu \mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \underbrace{\varepsilon^\nu \partial_\nu \phi}_{\delta \phi} \right) \Rightarrow \partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} \right)}_{T^{\mu\nu}} = 0.$$

Components of $T^{\mu\nu}$ correspond to energy and momentum of the field:

$$E = \int d^3x \, T^{00},$$

$$p^i = \int d^3x \, T^{0i}.$$

<u>Example</u>: For a free scalar field one may compute

$$T^{00} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^0 \phi - \eta^{00} \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right)$$

$$= (\partial^0 \phi)^2 - \frac{1}{2} (\partial_0 \phi)^2 + \frac{1}{2} \vec{\nabla} \phi \cdot \vec{\nabla} \phi + \frac{1}{2} m^2 \phi^2$$

$$= \frac{1}{2} \left(\dot{\phi}^2 + \vec{\nabla} \phi \cdot \vec{\nabla} \phi + m^2 \phi^2 \right).$$

With $\dot{\phi} = \pi$ this is exactly the Hamiltonian density. For T^{0i} and p^i see problems.

Summary

Lagrangians and Hamiltonians are objects that can be generalised from classical mechanics to classical field theory.

$$H, \ L \longrightarrow \mathcal{H}, \ \mathcal{L}$$

densities $\int d^3x \, \mathcal{L} = L$

We prefer the Lagrangian as it is Lorentz invariant.

We analysed the real massive scalar field and saw how energy is derived from the energy-momentum tensor.