k_T-factorization at NLO

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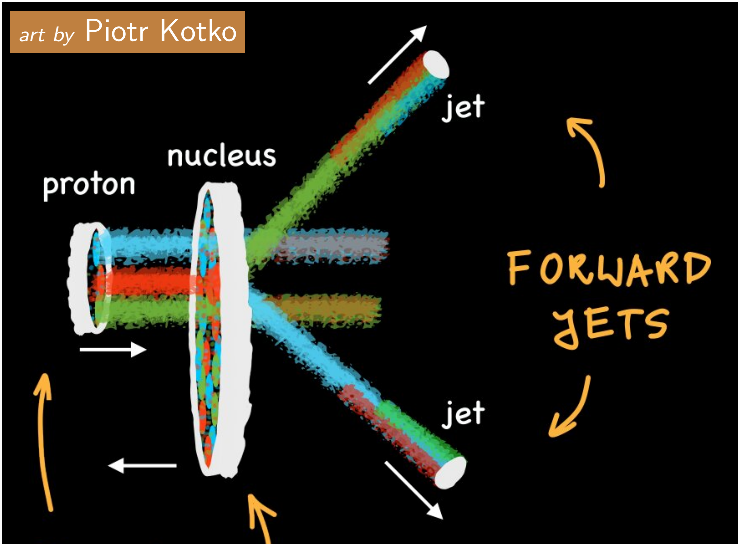
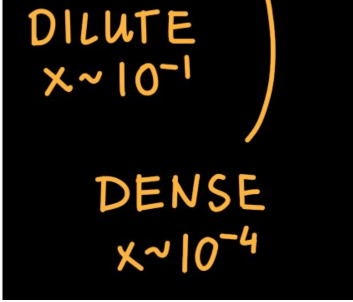
QCD evolution, dilute vs. dense, forward jets





A dilute system carries a few high-x partons contributing to the hard scattering.

A dense system carries many low-x partons.

At high density, gluons are imagined to undergo recombination, and to saturate.

This is modeled with non-linear evolution equations, involving explicit non-vanishing k_T.

Saturation implies the turnover of the gluon density, stopping it from growing indefinitely for small x.

Forward jets have large rapidities, and trigger events in which partons from the nucleus have small x.

Color Glass Condensate (CGC)

McLerran, Venugopalan 1994

introduction from Morreale, Salazar 2021

The CGC is an effective field theory for high energy QCD.

Partons carrying large hadron momentum fraction x are treated as static color sources ρ.

Their color charge distribution is non-perturbative and is dictated by a gauge invariant weight functional $W_{x_0}[\rho]$. The sources generate a current $J^{\mu,a}$.

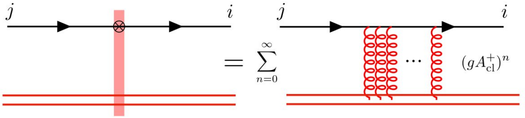
The partons carrying small x are treated as a dynamical classical field $A^{\mu,a}$.

Sources and fields are related by the Yang-Mills equations $[D_\mu, F_{\mu\nu}] = J_\nu$.

The expectation value $\langle \mathcal{O} \rangle_{x_0}$ of an observable $\mathcal{O}$ is calculated as the path integral $\mathcal{O}[\rho]$ in the presence of sources from $W_{x_0}[\rho]$, averaged over all possible configurations ρ.

The interaction of a highly energetic color charged particle with the classical field A in the eikonal approximation is encoded in the light-like Wilson lines

$$U(x_T) = \mathrm{P}\exp\left\{ ig \int_{-\infty}^{\infty} dx^+ A^{-,a}(x^+, x_T) t^a \right\}$$



Balitsky, Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner

Evolution in x of $W_x[\rho]$ implies an infinite hierarchy (known as the B-JIMWLK hierarchy) of non-linear coupled equations dictating the evolution of n-point Wilson line correlators.

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Cross section calculations involve particle wave functions and Wilson line correlators.

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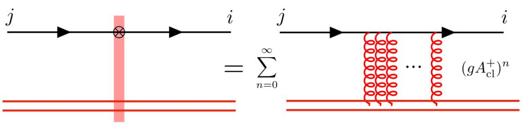

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ITMD Factorization

For forward dijet production in dilute-dense hadronic collisions

Generalized TMD factorization (Dominguez, Marquet, Xiao, Yuan 2011)

$$d\sigma_{AB\to X} = \int dk_T^2 \int dx_A \sum_i \int dx_B \sum_b \phi_{gb}^{(i)}(x_A, k_T, \mu)\, f_{b/B}(x_B, \mu)\, d\hat{\sigma}_{gb\to X}^{(i)}(x_A, x_B, \mu)$$

For $x_A \ll 1$ and $P_T \gg k_T \sim Q_s$ (jets almost back-to-back).
TMD gluon distributions $\phi_{gb}^{(i)}(x_A, k_T, \mu)$ satisfy non-linear evolution equations.
Partonic cross section $d\hat{\sigma}_{gb}^{(i)}$ is on-shell, but depends on color-structure i.

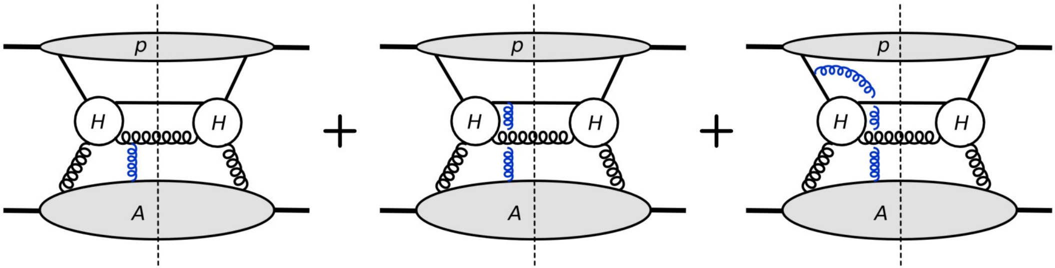
Improved TMD factorization (Kotko, Kutak, Marquet, Petreska, Sapeta, AvH 2015)

$$d\sigma_{AB\to X} = \int dk_T^2 \int dx_A \sum_i \int dx_B \sum_b \phi_{gb}^{(i)}(x_A, k_T, \mu)\, f_{b/B}(x_B, \mu)\, d\hat{\sigma}_{gb\to X}^{(i)}(x_A, x_B, k_T, \mu)$$

Originally a model interpolating between High Energy Factorization and Generalized TMD factorization: $P_T \gtrsim k_T \gtrsim Q_s$.
Partonic cross section $d\hat{\sigma}_{gb}^{(i)}$ is off-shell and depends on color-structure i.

ITMD formalism is obtained from the CGC formalism, by including so-called kinematic twist corrections (Antinoluk, Boussarie, Kotko 2019).

Definition of gluon TMDs



$+$ similar diagrams with $2, 3, \ldots$ gluon exchanges

Resummation of gluon exchanges leads to Wilson line $U_\gamma = \mathcal{P}\exp\left\{ -ig \int_\gamma dz \cdot A(z) \right\}$ acting as a gauge link for the gauge invariant definition of a TMD

$$\mathcal{F}_{g/A}(x, k_T) = 2 \int \frac{d^4\xi\, \delta(\xi^+)}{(2\pi)^3\, p_A^+} \exp\{ixp_A^+\xi^- - i\vec{k}_T \cdot \vec{\xi}_T\} \langle A | \mathrm{Tr}\{\hat{F}^{i+}(\xi) U_{\gamma(\xi,0)} \hat{F}^{i+}(0)\} | A \rangle$$

ITMD* factorization for more than 2 jets

Bury, Kotko, Kutak 2018

* only manifestly gauge invariant contribution included

Schematic hybrid (non-ITMD) factorization formula

$$d\sigma = \sum_{y=g,u,d,\dots} \int dx_1 d^2k_T \int dx_2 \, d\Phi_{g^*y\to n} \frac{1}{\text{flux}_{gy}} \mathcal{F}_g(x_1, k_T, \mu) \, f_y(x_2, \mu) \sum_{\text{color}} \left| \mathcal{M}^{(\text{color})}_{g^*y\to n} \right|^2$$

ITMD* formula: replace

$$\mathcal{F}_g \sum_{\text{color}} \left| \mathcal{M}^{(\text{color})} \right|^2 = \mathcal{F}_g \sum_{\sigma\in S_{n+2}} \sum_{\tau\in S_{n+2}} \mathcal{A}^*_\sigma \, \mathcal{C}_{\sigma\tau} \, \mathcal{A}_\tau \quad , \quad \mathcal{C}_{\sigma\tau} = N_c^{\lambda(\sigma,\tau)}$$

with "TMD-valued color matrix"

$$(N_c^2 - 1) \sum_{\sigma\in S_{n+2}} \sum_{\tau\in S_{n+2}} \mathcal{A}^*_\sigma \, \tilde{\mathcal{C}}_{\sigma\tau}(x, |k_T|) \, \mathcal{A}_\tau \quad , \quad \tilde{\mathcal{C}}_{\sigma\tau}(x, |k_T|) = N_c^{\bar{\lambda}(\sigma,\tau)} \tilde{\mathcal{F}}_{\sigma\tau}(x, |k_T|)$$

where each function $\tilde{\mathcal{F}}_{\sigma\tau}$ is one of 10 functions

$$\mathcal{F}^{(1)}_{qg} \quad , \quad \mathcal{F}^{(2)}_{qg} \quad , \quad \mathcal{F}^{(3)}_{qg}$$

$$\mathcal{F}^{(1)}_{gg} \quad , \quad \mathcal{F}^{(2)}_{gg} \quad , \quad \mathcal{F}^{(3)}_{gg} \quad , \quad \mathcal{F}^{(4)}_{gg} \quad , \quad \mathcal{F}^{(5)}_{gg} \quad , \quad \mathcal{F}^{(6)}_{gg} \quad , \quad \mathcal{F}^{(7)}_{gg}$$

ITMD* factorization for more than 2 jets

$$\mathcal{F}_{qg}^{(1)}(x,k_T) = \left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[-]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle \quad , \quad \left\langle \cdots \right\rangle = 2\int \frac{d^4\xi\,\delta(\xi_+)}{(2\pi)^3 P^+}\, e^{ik\cdot\xi}\left\langle P\middle|\cdots\middle|P\right\rangle$$

$$\mathcal{F}_{qg}^{(2)}(x,k_T) = \left\langle \frac{\mathrm{Tr}\left[\mathcal{U}^{[\square]}\right]}{N_c}\mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[+]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle$$

$$\mathcal{F}_{qg}^{(3)}(x,k_T) = \left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[+]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[\square]}\mathcal{U}^{[+]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(1)}(x,k_T) = \left\langle \frac{\mathrm{Tr}\left[\mathcal{U}^{[\square]\dagger}\right]}{N_c}\mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[-]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(2)}(x,k_T) = \frac{1}{N_c}\left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[\square]\dagger}\right]\mathrm{Tr}\left[\hat{F}^{i+}(0)\,\mathcal{U}^{[\square]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(3)}(x,k_T) = \left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[+]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(4)}(x,k_T) = \left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[-]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[-]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(5)}(x,k_T) = \left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[\square]\dagger}\mathcal{U}^{[+]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[\square]}\mathcal{U}^{[+]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(6)}(x,k_T) = \left\langle \frac{\mathrm{Tr}\left[\mathcal{U}^{[\square]}\right]}{N_c}\frac{\mathrm{Tr}\left[\mathcal{U}^{[\square]\dagger}\right]}{N_c}\mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[+]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(7)}(x,k_T) = \left\langle \frac{\mathrm{Tr}\left[\mathcal{U}^{[\square]}\right]}{N_c}\mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[\square]\dagger}\mathcal{U}^{[+]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle$$

ITMD gluons

Start with dipole distribution $\mathcal{F}_{qg}^{(1)}(x, k_T) = \left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[-]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle$ evolved via the BK equation formulated in momentum space supplemented with subleading corrections and fitted to F_2 data (Kutak, Sapeta 2012)

All other distribution appearing in dijet production, $\mathcal{F}_{qg}^{(2)}, \mathcal{F}_{gg}^{(1)}, \mathcal{F}_{gg}^{(2)}, \mathcal{F}_{gg}^{(6)}$, in the mean-field approximation (AvH, Marquet, Kotko, Kutak, Sapeta, Petreska 2016).

This is, at leading order in $1/N_c$. In this approximation, the same distributions suffice for trijets.

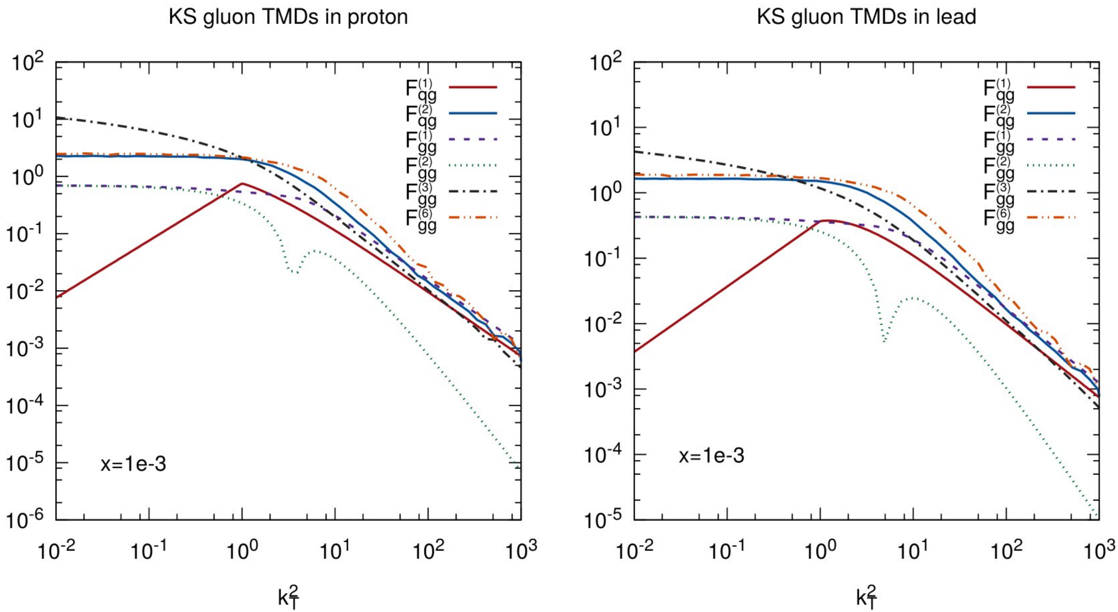
For DIS one only needs $\mathcal{F}_{gg}^{(3)}$

$$\mathcal{F}_{gg}^{(3)}(x, k_T) = \frac{\pi\alpha_s}{N_c k_T^2 S_\perp}\int_{k_T^2} dr_T^2 \ln\frac{r_T^2}{k_T^2}\int\frac{d^2q_T}{q_T^2}\,\mathcal{F}_{qg}^{(1)}(x, q_T)\,\mathcal{F}_{qg}^{(1)}(x, r_T - q_T)$$

where $S_\perp$ is the target's transverse area.

ITMD gluons

Bury, AvH, Kotko, Kutak 2020



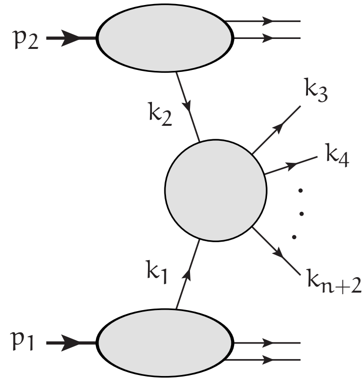
Dependence of $\mathcal{F}_{qg}^{(1)}$ on k_T below 1GeV approximated by power-like fall-off. For higher values of $|k_T|$ it is a solution to the BK equation.

TMDs decrease as $1/|k_T|$ for increasing $|k_T|$, except $\mathcal{F}_{gg}^{(2)}$, which decreases faster (even becomes negative, absolute value shown here).

Parton-level cross sections

Hadron-scattering process Y with partonic processes y contributing to multi-jet final state

$$d\sigma_Y(p_1, p_2; k_3, \dots, k_{2+n}) = \sum_{y \in Y} \int d^4k_1\, \mathcal{P}_{y_1}(k_1) \int d^4k_2\, \mathcal{P}_{y_2}(k_2)\, d\hat{\sigma}_y(k_1, k_2; k_3, \dots, k_{2+n})$$



Collinear factorization:

$$\mathcal{P}_{y_i}(k_i) = \int \frac{dx_i}{x_i}\, f_{y_i}(x_i, \mu)\, \delta^4(k_i - x_i p_i)$$

k_T-dependent factorization factorization:

$$\mathcal{P}_{y_i}(k_i) = \int \frac{d^2\mathbf{k}_{iT}}{\pi} \int \frac{dx_i}{x_i}\, \mathcal{F}_{y_i}(x_i, |\mathbf{k}_{iT}|, \mu)\, \delta^4(k_i - x_i p_i - k_{iT})$$

Differential partonic cross section:

$$d\hat{\sigma}_y(k_1, k_2; k_3, \dots, k_{2+n}) = d\Phi_Y(k_1, k_2; k_3, \dots, k_{2+n})\, \Theta_Y(k_3, \dots, k_{2+n}) \times \text{flux}(k_1, k_2) \times \mathcal{S}_y\, |\mathcal{M}_y(k_1, \dots, k_{2+n})|^2$$

Parton-level phase space:

$$d\Phi_Y(k_1, k_2; k_3, \dots, k_{2+n}) = \left(\prod_{i=3}^{n+2} d^4k_i \delta_+(k_i^2 - m_i^2) \right) \delta^4(k_1 + k_2 - k_3 - \dots - k_{n+2})$$

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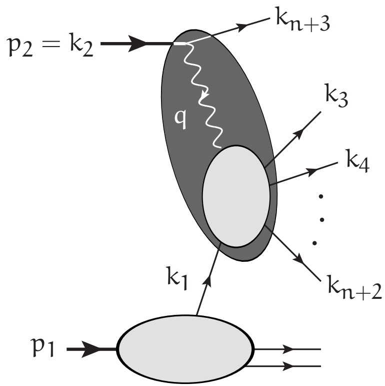

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KaTie https://bitbucket.org/hameren/katie

- parton level tree level event generator, like ALPGEN, HELAC, MADGRAPH, etc.
- arbitrary hadron-hadron or hadron-lepton processes within the standard model (including effective Higgs-gluon coupling) with several final-state particles.
- **0, 1, or 2 space-like initial states.**
- produces (partially un)weighted event files, for example in the LHEF format.
- requires LHAPDF. TMD PDFs can be provided as files containing rectangular grids, or with TMDlib (Hautmann, Jung, Krämer, Mulders, Nocera, Rogers, Signori 2014).
- a calculation is steered by a single input file.
- employs an optimization stage in which the pre-samplers for all channels are optimized.
- during the generation stage several event files can be created in parallel.
- event files can be processed further by parton-shower program like CASCADE.
- (evaluation of) matrix elements separately available.

Hybrid k_T-factorization at NLO

one side space-like
the other side on-shell

M. A. Nefedov, *Computing one-loop corrections to effective vertices with two scales in the EFT for Multi-Regge processes in QCD*, Nucl. Phys. B 946 (2019) 114715, [1902.11030].

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F. Bergabo and J. Jalilian-Marian, *Single inclusive hadron production in DIS at small x: next to leading order corrections*, JHEP 01 (2023) 095, [2210.03208].

P. Taels, Forward production of a Drell-Yan pair and a jet at small x at next-to-leading order, JHEP 01 (2024) 005. [2308.02449].

T. Altinoluk, N. Armesto, A. Kovner, and M. Lublinsky, *Single inclusive particle production at next-to-leading order in proton-nucleus collisions at forward rapidities: Hybrid approach meets TMD factorization*, Phys. Rev. D 108 (2023), no. 7 074003, [2307.14922].

Collinear factorization in QCD at NLO

general: $K^\mu = x_K P^\mu + \bar{x}_K \bar{P}^\mu + K_\perp^\mu$

one in-state: $k_\chi^\mu = x P^\mu$

other in-state: $k_{\bar{\chi}}^\mu = \bar{x}\bar{P}^\mu$

$$d\sigma^{\mathrm{LO}} = \int dx\, d\bar{x}\, f_\chi(x)\, f_{\bar{\chi}}(\bar{x})\, dB(x,\bar{x})$$

$$\begin{aligned} d\sigma^{\mathrm{NLO}} = \int dx\, d\bar{x} \Bigg\{ & f_\chi(x)\, f_{\bar{\chi}}(\bar{x}) \left[\frac{\alpha_s}{2\pi} dV(x,\bar{x}) + \frac{\alpha_s}{2\pi} dR(x,\bar{x})\right]_{\text{cancelling}} \\ & + \left[f_\chi(x)\, \frac{-\alpha_s}{2\pi\epsilon} \int_{\bar{x}}^1 d\bar{z}\, \mathcal{P}_{\bar{\chi}}(\bar{z})\, \frac{1}{\bar{z}} f_{\bar{\chi}}\!\left(\frac{\bar{x}}{\bar{z}}\right) \right. \\ & \left. + f_{\bar{\chi}}(\bar{x})\, \frac{-\alpha_s}{2\pi\epsilon} \int_x^1 dz\, \mathcal{P}_\chi(z)\, \frac{1}{z} f_\chi\!\left(\frac{x}{z}\right) \right] dB(x,\bar{x}) \\ & + \left[\frac{\alpha_s}{2\pi} f_\chi^{\mathrm{NLO}}(x)\, f_{\bar{\chi}}(\bar{x}) + f_\chi(x)\, \frac{\alpha_s}{2\pi} f_{\bar{\chi}}^{\mathrm{NLO}}(\bar{x}) \right] dB(x,\bar{x}) \Bigg\} \end{aligned}$$

$$f_\chi^{\mathrm{NLO}}(x) - \frac{1}{\epsilon} \int_x^1 dz\, \mathcal{P}_\chi(z)\, \frac{1}{z} f_\chi\!\left(\frac{x}{z}\right) = \text{finite}$$

$$f_{\bar{\chi}}^{\mathrm{NLO}}(\bar{x}) - \frac{1}{\epsilon} \int_{\bar{x}}^1 d\bar{z}\, \mathcal{P}_\chi(\bar{z})\, \frac{1}{\bar{z}} f_{\bar{\chi}}\!\left(\frac{\bar{x}}{\bar{z}}\right) = \text{finite}$$

a subtraction method at NLO for real radiation in k_T-factorization

Giachino, AvH, Ziarko 2023

Notation

for a subtraction method at NLO
for real radiation in k_T-factorization

Giachino, AvH, Ziarko 2023

The Born-level formula for the cross section in hybrid k_T-factorization:

$$\sigma_B = \frac{1}{\mathcal{S}_n}\int[dQ]\int d\Phi\big(Q;\{p\}_n\big)\,\mathcal{L}\big(Q;\{p\}_n\big)\,\big|\mathcal{M}\big|^2\big(Q;\{p\}_n\big)\,J_B\big(\{p\}_n\big)$$

Initial-state variables:

$$\int[dQ] = \int_0^1 dx\int_0^1 d\bar{x}\int d^2k_\perp\;,\quad Q^\mu = k_\chi^\mu + k_{\bar{\chi}}^\mu\;,\quad \begin{cases} k_\chi^\mu = xP^\mu + k_\perp^\mu & P^\mu = (E,0,0,E)\\ k_{\bar{\chi}}^\mu = \bar{x}\bar{P}^\mu & \bar{P}^\mu = (\bar{E},0,0,-\bar{E})\end{cases}$$

Differential phase space for the final-state momenta $\{p\}_n$

$$d\Phi\big(Q;\{p\}_n\big) = \left(\prod_{l=1}^{n}\frac{d^4p_l}{(2\pi)^3}\delta_+(p_l^2-m_l^2)\right)\frac{1}{(2\pi)^4}\,\delta\Big(Q-\sum_{l=1}^{n}p_l\Big)$$

The PDFs and flux factor:

$$\mathcal{L}\big(Q;\{p\}_n\big) = \frac{F_\chi\big(x,k_\perp,\mu_F(\{p\}_n)\big)\,f_{\bar{\chi}}\big(\bar{x},\mu_F(\{p\}_n)\big)}{8x\bar{x}E\bar{E}}$$

$\big|\mathcal{M}\big|^2\big(Q;\{p\}_n\big)$ tree-level matrix element without symmetry factors and averageing factors, they are captured by $\mathcal{S}_n$. Finally $J_B\big(\{p\}_n\big)$ denotes the jet function.

Singular limits at NLO: jets

The symbol J_B includes the decision if there are *enough* jets for Born-level. For the real radiation, the jet function J_R does not avoid all singularities of the tree-level squared matrix element anymore, but allows one pair of partons to become collinear,

$$\begin{aligned} \text{one pair of partons to become collinear:}\quad & p_r \| p_i \quad \Leftrightarrow \quad \vec{n}_r - \vec{n}_i \to \vec{0} \\ \text{one parton to become soft:}\quad & p_r \to \text{soft} \quad \Leftrightarrow \quad E_r \to 0 \end{aligned}$$

The jet function behaves in those limits such that

$$\begin{aligned} J_R\big(\{p\}_{n+1}\big) &\xrightarrow{p_r \to \text{soft}} J_B\big(\{p\}_n^{\not r}\big) \quad , \\ J_R\big(\{p\}_{n+1}\big) &\xrightarrow{p_r \| p_i} J_B\big(\{p\}_n^{\not r;i}\big) \quad , \\ J_R\big(\{p\}_{n+1}\big) &\xrightarrow{p_r \| P, \bar{P}} J_B\big(\{p\}_n^{\not r}\big) \quad , \end{aligned}$$

where

$\{p\}_n^{\not r}$ *is obtained from* $\{p\}_{n+1}$ *by removing momentum* p_r ,

$\{p\}_n^{\not r;i}$ *is obtained by additionally replacing* p_i *with* $(1+z_{ri})p_i \qquad z_{ri} = E_r/E_i$

(We assume p_r and also p_i to be light-like.)

Singular limits at NLO: matrix elements

Matrix elements are constructed from external momenta that must satisfy mometum conservation. When $(Q;\{p\}_{n+1})$ satisfies momentum conservation, then $(Q;\{p\}_n^{\not r})$ and $(Q;\{p\}_n^{\not r;i})$ do not. We must introduce deformed momenta to even write down the limits:

$$|\mathcal{M}|^2(Q;\{p\}_{n+1}) \xrightarrow{p_r\to\text{soft}} \hat{\mathcal{R}}^{\text{soft}}(p_r)\otimes\hat{\mathcal{A}}^{\text{soft}}(\tilde{Q};\{\tilde{p}\}_n^{\not r})$$

$$|\mathcal{M}|^2(Q;\{p\}_{n+1}) \xrightarrow{p_r\|p_i} \hat{\mathcal{R}}^{\text{F,col}}_{ir}(p_r)\otimes\hat{\mathcal{A}}^{\text{F,col}}_{ir}(\tilde{Q};\{\tilde{p}\}_n^{\not r;i})$$

$$|\mathcal{M}|^2(Q;\{p\}_{n+1}) \xrightarrow{p_r\|P} \hat{\mathcal{R}}^{\text{I,col}}_{\chi,r}(p_r)\otimes\hat{\mathcal{A}}^{\text{I,col}}_{\chi,r}(\tilde{Q}-x_rP;\{\tilde{p}\}_n^{\not r})$$

In k_T-factorization, we can choose to just deform the initial-state momenta:

$$|\mathcal{M}|^2(Q;\{p\}_{n+1}) \xrightarrow{p_r\to\text{soft}} \hat{\mathcal{R}}^{\text{soft}}(p_r)\otimes\hat{\mathcal{A}}^{\text{soft}}(Q-p_r;\{p\}_n^{\not r})$$

$$|\mathcal{M}|^2(Q;\{p\}_{n+1}) \xrightarrow{p_r\|p_i} \hat{\mathcal{R}}^{\text{F,col}}_{ir}(p_r)\otimes\hat{\mathcal{A}}^{\text{F,col}}_{ir}(Q-p_r+z_{ri}p_i;\{p\}_n^{\not r;i})$$

$$|\mathcal{M}|^2(Q;\{p\}_{n+1}) \xrightarrow{p_r\|P/\bar{P}} \hat{\mathcal{R}}^{\text{I,col}}_{\chi/\bar{\chi},r}(p_r)\otimes\hat{\mathcal{A}}^{\text{I,col}}_{\chi/\bar{\chi},r}(Q-p_r;\{p\}_n^{\not r})$$

This opens the possibility to construct subtraction terms with only deformed initial-state moenta.

Subtraction method

Frixione, Kunszt, Signer 1996
Catani, Seymour 1997

Real radiation contribution within dimensional regularization

$$\sigma_{\mathrm{R}}(\epsilon) = \frac{1}{\mathcal{S}_{n+1}} \int [dQ] \int d\Phi\big(\epsilon; Q; \{p\}_{n+1}\big)\mathcal{L}\big(Q; \{p\}_{n+1}\big)\big) \left|\mathcal{M}\right|^2\big(Q; \{p\}_{n+1}\big)\, J_{\mathrm{R}}\big(\{p\}_{n+1}\big)$$

We want to split the real-radiation integral into a finite part and a divergent part that can be explicitly expressed as a Laurent expansion in ϵ within dimensional regularization

$$\sigma_{\mathrm{R}}(\epsilon) = \sigma_{\mathrm{R}}^{\mathrm{div}}(\epsilon) + \sigma_{\mathrm{R}}^{\mathrm{fin}} + \mathcal{O}(\epsilon)$$

We define the finite "subtracted-real" integral as

$$\sigma_{\mathrm{R}}^{\mathrm{fin}} = \frac{1}{\mathcal{S}_{n+1}} \int [dQ] \int d\Phi\big(Q; \{p\}_{n+1}\big) \Big\{ \mathcal{L}\big(Q; \{p\}_{n+1}\big)\big) \left|\mathcal{M}\right|^2\big(Q; \{p\}_{n+1}\big)\, J_{\mathrm{R}}\big(\{p\}_{n+1}\big) - \sum_r \mathrm{Subt}_r\big(Q; \{p\}_{n+1}\big) \Big\} \,,$$

that can be integrated numerically, and

$$\sigma_{\mathrm{R}}^{\mathrm{div}}(\epsilon) = \frac{1}{\mathcal{S}_{n+1}} \sum_r \int [dQ] \int d\Phi\big(\epsilon; Q; \{p\}_{n+1}\big) \mathrm{Subt}_r\big(Q; \{p\}_{n+1}\big) \,,$$

that should be integrable analytically.

Subtraction terms

largely following Somogyi, Trócsányi 2006

but with parameters E_0, ζ_0, ξ_0 to restrict the phase space where the terms are active.

Final-state terms, with arguments $\left(Q - p_r + z_{ri} p_i; \{p\}_n^{\not r;i}\right)$ for amplitudes $\mathcal{M}$:

$$\mathcal{R}_{ir}^{F,col} \otimes \mathcal{A}_{ir}^{F,col} = \frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\ \theta(n_r\cdot n_i < 2\zeta_0)\ \frac{\theta(E_r < E_i)}{p_i\cdot p_r}\,\mathcal{Q}_{ir}(z_{ri}) \otimes \left|\mathcal{M}_{ir}\right|^2$$

$$\mathcal{R}_{i}^{F,soft} \otimes \mathcal{A}_{i}^{F,soft} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\ \theta(E_r < E_0)\ \frac{2}{n_i\cdot p_r}\sum_b \frac{n_i\cdot n_b}{n_i\cdot p_r + n_b\cdot p_r}\left(\mathcal{M}\right)^2_{color(i,b)}$$

$$\mathcal{R}_{i}^{F,soco} \otimes \mathcal{A}_{i}^{F,soco} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\ \theta(E_r < E_0)\theta(n_r\cdot n_i < 2\zeta_0)\ \frac{2C_i}{p_i\cdot p_r}\frac{1}{z_{ri}}\left|\mathcal{M}\right|^2$$

Initial-state terms, with arguments $\left(Q - p_r; \{p\}_n^{\not r}\right)$ for amplitudes $\mathcal{M}$:

$$\mathcal{R}_{\chi r}^{I,col} \otimes \mathcal{A}_{\chi r}^{I,col} = \frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\ \theta\left(\bar{x}_r < \xi_0 x_r\right)\ \frac{-2}{S\bar{x}_r x}\,\mathcal{Q}_{\chi r}(-x_r/x) \otimes \left|\mathcal{M}_{\chi r}\right|^2$$

$$\mathcal{R}_{\chi}^{I,soft} \otimes \mathcal{A}_{\chi}^{I,soft} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\ \theta(E_r < E_0)\ \frac{2}{n_\chi\cdot p_r}\sum_b \frac{n_\chi\cdot n_b}{n_\chi\cdot p_r + n_b\cdot p_r}\left(\mathcal{M}\right)^2_{color(\chi,b)}$$

$$\mathcal{R}_{\chi}^{I,soco} \otimes \mathcal{A}_{\chi}^{I,soco} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\ \theta(E_r < E_0)\theta\left(\bar{x}_r < \xi_0 x_r\right)\ \frac{4C_\chi}{S x_r \bar{x}_r}\left|\mathcal{M}\right|^2$$

Subtraction terms

largely following Somogyi, Trócsányi 2006

but with parameters E_0, ζ_0, ξ_0 to restrict the phase space where the terms are active.

While $k_\chi^\mu = xP^\mu + k_\perp^\mu$, there is an initial-state singularity related to the space-like gluon if the radiative momentum becomes collinear to P, with splitting function

$$\mathcal{Q}_{\chi r}(\zeta) = \frac{2C_g}{\zeta(1+\zeta)^2} \quad \Leftrightarrow \quad \mathcal{P}_{\chi r}(z) \equiv -z\mathcal{Q}_\chi(z-1) = \frac{2C_g}{z(1-z)}$$

Initial-state terms, with arguments $(Q - p_r; \{p\}'_n)$ for amplitudes $\mathcal{M}$:

$$\mathcal{R}_{\chi r}^{\mathrm{I,col}} \otimes \mathcal{A}_{\chi r}^{\mathrm{I,col}} = \frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\,\theta\big(\bar{x}_r < \xi_0 x_r\big) \quad \frac{-2}{S\bar{x}_r x}\,\mathcal{Q}_{\chi r}(-x_r/x) \otimes \big|\mathcal{M}_{\chi r}\big|^2$$

$$\mathcal{R}_{\chi}^{\mathrm{I,soft}} \otimes \mathcal{A}_{\chi}^{\mathrm{I,soft}} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\,\theta(E_r < E_0)\,\frac{2}{n_\chi \cdot p_r}\sum_b \frac{n_\chi \cdot n_b}{n_\chi \cdot p_r + n_b \cdot p_r}\,(\mathcal{M})^2_{\mathrm{color}(\chi,b)}$$

$$\mathcal{R}_{\chi}^{\mathrm{I,soco}} \otimes \mathcal{A}_{\chi}^{\mathrm{I,soco}} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\,\theta(E_r < E_0)\,\theta\big(\bar{x}_r < \xi_0 x_r\big) \quad \frac{4C_\chi}{S x_r \bar{x}_r}\,|\mathcal{M}|^2$$

Subtraction method

$$\sigma_R(\epsilon) = \sigma_R^{div}(\epsilon) + \sigma_R^{fin} + \mathcal{O}(\epsilon)$$

We define the finite "subtracted-real" integral as

$$\sigma_R^{fin} = \frac{1}{\mathcal{S}_{n+1}} \int [dQ] \int d\Phi(Q;\{p\}_{n+1}) \Big\{ \mathcal{L}(Q;\{p\}_{n+1}) \, |\mathcal{M}|^2(Q;\{p\}_{n+1}) \, J_R(\{p\}_{n+1}) - \sum_r \mathrm{Subt}_r(Q;\{p\}_{n+1}) \Big\},$$

where the r-sum is over all final-state partons, and where $\mathrm{Subt}_r(Q;\{p\}_{n+1})$ is given by

$$\begin{aligned}
&\sum_i \mathcal{L}(Q - p_r + z_{ri}p_i;\{p\}_n^{\not r;i}) \; \mathcal{R}_{ir}^F(p_r) \otimes \mathcal{A}_{ir}^F(Q - p_r + z_{ri}p_i;\{p\}_n^{\not r;i}) \, J_B(\{p\}_n^{\not r;i}) \\
&+ \sum_{a\in\{\chi,\bar\chi\}} \mathcal{L}(Q - p_r;\{p\}_n^{\not r}) \; \mathcal{R}_a^{I,soft}(p_r) \otimes \mathcal{A}_a^{I,soft}(Q - p_r;\{p\}_n^{\not r}) \, J_B(\{p\}_n^{\not r}) \\
&+ \sum_{a\in\{\chi,\bar\chi\}} \mathcal{L}(Q - p_r;\{p\}_n^{\not r}) \; \mathcal{R}_a^{I,soco}(p_r) \otimes \mathcal{A}_a^{I,soco}(Q - p_r;\{p\}_n^{\not r}) \, J_B(\{p\}_n^{\not r}) \\
&+ \mathcal{L}(Q - \bar x_r \bar P - p_{\perp r};\{p\}_n^{\not r}) \; \mathcal{R}_{\chi,r}^{I,col}(p_r) \otimes \mathcal{A}_{\chi,r}^{I,col}(Q - p_r;\{p\}_n^{\not r}) \, J_B(\{p\}_n^{\not r}) \\
&+ \mathcal{L}(Q - x_r P - p_{\perp r};\{p\}_n^{\not r}) \; \mathcal{R}_{\bar\chi,r}^{I,col}(p_r) \otimes \mathcal{A}_{\bar\chi,r}^{I,col}(Q - p_r;\{p\}_n^{\not r}) \, J_B(\{p\}_n^{\not r})
\end{aligned}$$

where also the i-sum is over all final-state partons with $\mathcal{R}_{rr}^F(p_r) \equiv 0$.

Subtraction method

$$\sigma_R(\epsilon) = \sigma_R^{\text{div}}(\epsilon) + \sigma_R^{\text{fin}} + \mathcal{O}(\epsilon)$$

$$\sigma_R^{\text{div}}(\epsilon) = \frac{1}{\mathcal{S}_{n+1}} \sum_r \int [dQ] \int d\Phi\left(Q;\{p\}_n^{\not r}\right) \mathcal{L}\left(Q;\{p\}_n^{\not r}\right) J_B\left(\{p\}_n^{\not r}\right)$$
$$\times \left\{ \sum_i \mathcal{J}_{ir}^{F}\left(\epsilon, Q, \{p\}_n^{\not r}\right) \otimes \mathcal{A}_{ir}^{F}\left(Q;\{p\}_n^{\not r}\right) + \sum_{a\in\{\chi,\bar\chi\}} \mathcal{J}_{ar}^{I}\left(\epsilon, Q, \{p\}_n^{\not r}\right) \otimes \mathcal{A}_{ar}^{I}\left(Q;\{p\}_n^{\not r}\right) \right\},$$

with

$$\mathcal{J}_{ir}^{F}\left(\epsilon, Q, \{p\}_n^{\not r}\right) = \int \frac{d^{4-2\epsilon}p_r}{(2\pi)^{3-2\epsilon}}\, \delta_+(p_r^2)\,(1-z_{ri})\,\mathcal{R}_{ir}^{F}(p_r)\,\Theta(p_r - z_{ri}p_i)$$

$$\mathcal{J}_{a}^{I,\text{soft/soco}}\left(\epsilon, Q, \{p\}_n^{\not r}\right) = \int \frac{d^{4-2\epsilon}p_r}{(2\pi)^{3-2\epsilon}}\, \delta_+(p_r^2)\,\mathcal{R}_{a}^{I,\text{soft/soco}}(p_r)\,\Theta(p_r)$$

$$\mathcal{J}_{\chi r}^{I,\text{col}}\left(\epsilon, Q, \{p\}_n^{\not r}\right) = \int \frac{d^{4-2\epsilon}p_r}{(2\pi)^{3-2\epsilon}}\, \delta_+(p_r^2)\,\mathcal{R}_{\chi r}^{I,\text{col}}(p_r)\,\Theta(p_r)\,\frac{\mathcal{L}\left(Q + x_r P;\{p\}_n^{\not r}\right)}{\mathcal{L}\left(Q;\{p\}_n^{\not r}\right)}$$

and

$$\Theta(q) = \theta(-x < x_q < 1 - x)\,\theta(-\bar{x} < \bar{x}_q < 1 - \bar{x})$$

Only $\mathcal{J}_{\chi/\bar\chi,r}^{I,\text{col}}$ involve $\mathcal{L}$-function $\Longrightarrow$ “P”-operator, must be integrated numerically.

But the Θ restrictions obstruct confortable analytic integration also for the other terms.

Example integrated subtraction term

F,soft

$$\bar{\epsilon} = -2\epsilon \ , \quad \pi_\epsilon = \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}$$

We need to calculate

$$L_{ib}^{F,soft}(\epsilon) = \frac{-2}{\pi_\epsilon \mu^{\bar{\epsilon}}} \int d^{4+\bar{\epsilon}} p_r \, \delta_+(p_r^2) \, \frac{1}{n_i \cdot p_r} \, \frac{n_i \cdot n_b}{n_i \cdot p_r + n_b \cdot p_r} \, \theta(E_r < E_0) \, (1 - z_{ri}) \, \Theta(p_r - z_{ri} p_i)$$

but find it too complicated because of $\Theta(p_r - z_{ri} p_i)$.

Because $p_{r\perp} - z_{ri} p_{i\perp}$ vanishes both in the soft and the collinear limit, the integral

$$L_{ib,compl}^{F,soft,fin} = \frac{-2}{\pi} \int d^4 p_r \, \delta_+(p_r^2) \, \frac{1}{n_i \cdot p_r} \, \frac{n_i \cdot n_b}{n_i \cdot p_r + n_b \cdot p_r} \, \theta(E_r < E_0) \, (1 - z_{ri}) \, \Big[\Theta(p_r - z_{ri} p_i) - 1 \Big]$$

is finite and can be calculated numerically, while

$$L_{ib,compl}^{F,soft,div}(\epsilon) = \frac{-2}{\pi_\epsilon \mu^{\bar{\epsilon}}} \int d^{4+\bar{\epsilon}} p_r \, \delta_+(p_r^2) \, \frac{1}{n_i \cdot p_r} \, \frac{n_i \cdot n_b}{n_i \cdot p_r + n_b \cdot p_r} \, \theta(E_r < E_0) \, (1 - z_{ri})$$

can, in principle, be calculated analytically.

Still, the explicit appearance of $n_i \cdot p_r, n_b \cdot p_r$ and E_r makes it complicated.

Example integrated subtraction term

F,soft

$$\bar{\epsilon} = -2\epsilon\ , \quad \pi_\epsilon = \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}$$

Thus, we introduce

$$E_r^{(ib)} = \frac{n_b\cdot p_r}{n_i\cdot n_b} + \frac{n_i\cdot p_r}{n_i\cdot n_b} = E_r\,\frac{n_r\cdot n_b + n_i\cdot n_r}{n_i\cdot n_b}$$

which vanishes in the soft limit, and becomes equal to E_r in the collinear limit, so we can define

$$L_{ib}^{F,\text{soft,fin}} = \frac{-2}{\pi}\int d^4p_r\,\delta_+(p_r^2)\,\frac{1}{n_i\cdot p_r}\,\frac{n_i\cdot n_b}{n_i\cdot p_r + n_b\cdot p_r}$$
$$\times\left[\Theta(p_r - z_{ri}p_i)\,\theta\big(E_r < E_0\big)\left(1-\frac{E_r}{E_i}\right) - \theta\big(E_r^{(ib)} < E_0\big)\left(1-\frac{E_r^{(ib)}}{E_i}\right)\right]$$

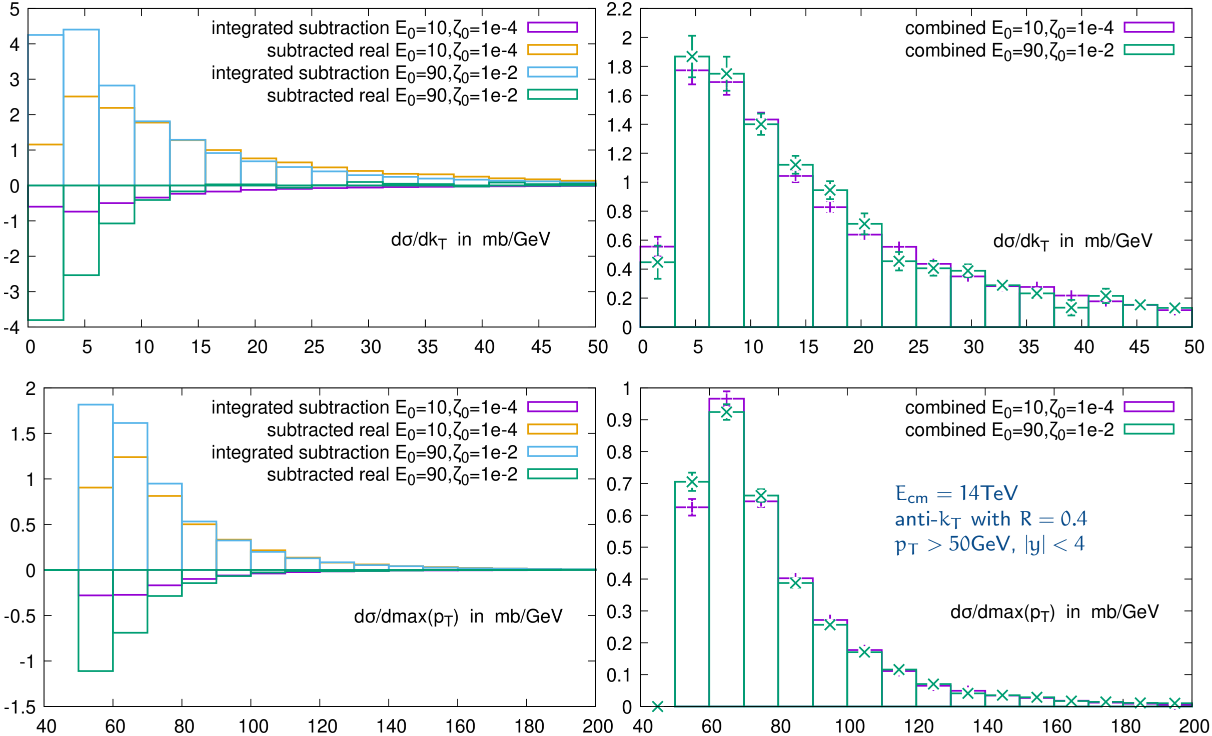
which can be calculated numerically, and

$$L_{ib}^{F,\text{soft,div}}(\epsilon) = \frac{-2}{\pi_\epsilon\mu^{\bar{\epsilon}}}\int d^{4+\bar{\epsilon}}p_r\,\delta_+(p_r^2)\,\frac{1}{n_i\cdot p_r}\,\frac{n_i\cdot n_b}{n_i\cdot p_r + n_b\cdot p_r}\,\theta\big(E_r^{(ib)} < E_0\big)\left(1-\frac{E_r^{(ib)}}{E_i}\right)$$

which is easier to calculate analytically.

Numerical results

for dijets, including: $gg^\star \to ggg$, $gg^\star \to u\bar{u}g$, $ug^\star \to ugg$, $ug^\star \to u\bar{u}d$, $ug^\star \to u\bar{u}u$, $(u \leftrightarrow d)$



k_T-dependent PDF: `PB-NLO-HERAI+II-2018-set2` Bermudez Martinez *et al.* 2019

Divergences

All poles in ϵ of the integrated subtraction terms are *the same as in the on-shell case*, except the initial-state collinear divergence

$$\sigma_{\chi r}^{\mathrm{I,col,div}} = \frac{1}{\mathcal{S}_n}\int[dQ]\int d\Phi\big(Q;\{p\}_n\big)\,\mathcal{L}\big(Q;\{p\}_n\big)\,\big|\mathcal{M}\big|^2\big(Q;\{p\}_n\big)\,J_B\big(\{p\}_n\big)$$
$$\times\frac{\alpha_s}{2\pi}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left\{\frac{C_{\chi r}}{\epsilon^2}-\frac{1}{\epsilon}\int_0^1 dz\,\mathcal{P}_{\chi r}^{\mathrm{reg}}(z)\,\frac{\ell_\chi(x/z)}{z^2}\,\theta(z>x)\right\}$$

with

$$\ell_\chi(y) = \frac{\mathcal{L}\big(yP+\bar{x}\bar{P}+k_\perp;\{p\}_n\big)}{\mathcal{L}\big(xP+\bar{x}\bar{P}+k_\perp;\{p\}_n\big)} = \frac{F_\chi\big(y,k_\perp,\mu_F(\{p\}_n)\big)/y}{F_\chi\big(x,k_\perp,\mu_F(\{p\}_n)\big)/x}\,.$$

$$\mathcal{P}_{\chi g}^{\mathrm{reg}}(z) = 2C_A\left[\frac{1}{[1-z]_+}+\frac{1}{z}\right]$$

compare with the collinear case

$$\ell_{\bar{\chi}}(y) = \frac{\mathcal{L}\big(xP+y\bar{P}+k_\perp;\{p\}_n\big)}{\mathcal{L}\big(xP+\bar{x}\bar{P}+k_\perp;\{p\}_n\big)} = \frac{f_{\bar{\chi}}\big(y,\mu_F(\{p\}_n)\big)/y}{f_{\bar{\chi}}\big(x,\mu_F(\{p\}_n)\big)/x}$$

$$\mathcal{P}_{\bar{\chi}g}^{\mathrm{reg}}(z) = 2C_A\left[\frac{1}{[1-z]_+}+\frac{1}{z}+z(1-z)-2\right]$$

Collinear factorization in QCD at NLO

$$d\sigma^{LO} = \int dx d\bar{x}\, f_\chi(x)\, f_{\bar{\chi}}(\bar{x})\, dB(x,\bar{x})$$

general: $K^\mu = x_K P^\mu + \bar{x}_K \bar{P}^\mu + K_\perp^\mu$

one in-state: $k_\chi^\mu = x P^\mu$

other in-state: $k_{\bar{\chi}}^\mu = \bar{x}\bar{P}^\mu$

$$\begin{aligned} d\sigma^{NLO} = \int dx d\bar{x} \Bigg\{ & f_\chi(x)\, f_{\bar{\chi}}(\bar{x}) \left[\frac{\alpha_s}{2\pi}\, dV(x,\bar{x}) + \frac{\alpha_s}{2\pi}\, dR(x,\bar{x})\right]_{\text{cancelling}} \\ & + \Bigg[f_\chi(x)\, \frac{-\alpha_s}{2\pi\epsilon} \int_{\bar{x}}^1 d\bar{z} \left[\mathcal{P}_{\bar{\chi}}^{\text{reg}}(\bar{z}) + \gamma_{\bar{\chi}}\delta(1-\bar{z})\right] \frac{1}{\bar{z}} f_{\bar{\chi}}\left(\frac{\bar{x}}{\bar{z}}\right) \\ & \quad + f_{\bar{\chi}}(\bar{x})\, \frac{-\alpha_s}{2\pi\epsilon} \int_x^1 dz \left[\mathcal{P}_\chi^{\text{reg}}(z) + \gamma_\chi\delta(1-z)\right] \frac{1}{z} f_\chi\left(\frac{x}{z}\right) \Bigg] dB(x,\bar{x}) \\ & + \left[\frac{\alpha_s}{2\pi} f_\chi^{NLO}(x)\, f_{\bar{\chi}}(\bar{x}) + f_\chi(x)\, \frac{\alpha_s}{2\pi} f_{\bar{\chi}}^{NLO}(\bar{x})\right] dB(x,\bar{x}) \Bigg\} \end{aligned}$$

$$f_\chi^{NLO}(x) - \frac{1}{\epsilon}\int_x^1 dz \left[\mathcal{P}_\chi^{\text{reg}}(z) + \gamma_\chi\delta(1-z)\right] \frac{1}{z} f_\chi\left(\frac{x}{z}\right) = \text{finite}$$

$$f_{\bar{\chi}}^{NLO}(\bar{x}) - \frac{1}{\epsilon}\int_{\bar{x}}^1 d\bar{z} \left[\mathcal{P}_{\bar{\chi}}^{\text{reg}}(z) + \gamma_{\bar{\chi}}\delta(1-\bar{z})\right] \frac{1}{\bar{z}} f_{\bar{\chi}}\left(\frac{\bar{x}}{\bar{z}}\right) = \text{finite}$$

Auxiliary parton method

AvH, Kotko, Kutak 2013

$$k_\chi^\mu = xP^\mu + k_\perp^\mu \qquad k_{\bar\chi}^\mu = \bar{x}\bar{P}^\mu$$

We desire to obtain the matrix element with one space-like gluon for the process

$$g^\star(k_\chi)\,\omega_{\bar\chi}(k_{\bar\chi}) \;\to\; \omega_1(p_1)\,\omega_2(p_2)\,\cdots\,\omega_n(p_n) \qquad \text{e.g.} \quad g^\star(k_\chi)\,g(k_{\bar\chi}) \;\to\; g(p_1)\,g(p_2)\,g(p_3)$$

and do so by replacing the space-like gluon with an *on-shell auxiliary* quark pair

$$q\big(k_1(\Lambda)\big)\,\omega_{\bar\chi}(k_{\bar\chi}) \;\to\; q\big(k_2(\Lambda)\big)\,\omega_1(p_1)\,\omega_2(p_2)\,\cdots\,\omega_n(p_n)$$

with special momenta

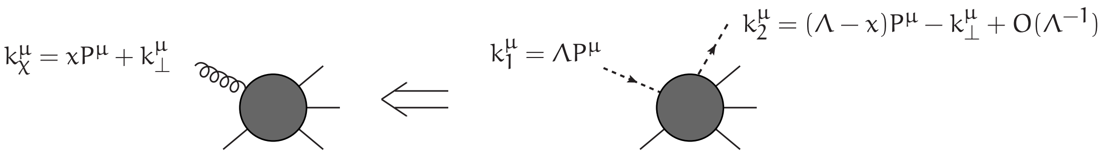
$$k_1^\mu = \Lambda P^\mu \quad , \quad k_2^\mu = p_\Lambda{}^\mu = (\Lambda - x)P^\mu - k_\perp^\mu + \frac{|k_\perp|^2}{2(\Lambda - x)P\cdot\bar{P}}\,\bar{P}^\mu$$

such that, while individually on-shell, their difference is

$$k_1^\mu - k_2^\mu = xP^\mu + k_\perp^\mu + \mathcal{O}\big(\Lambda^{-1}\big) = k_\chi^\mu + \mathcal{O}\big(\Lambda^{-1}\big)$$

The matrix element with the space-like gluon is obtained by taking $\Lambda \to \infty$

$$\big|\overline{\mathcal{M}}^\star\big|^2(k_\chi, k_{\bar\chi}; \{p\}_n) \;=\; \lim_{\Lambda\to\infty} \frac{1}{g_s^2 C_{aux}}\,\frac{x^2|k_\perp|^2}{\Lambda^2}\big|\overline{\mathcal{M}}^{aux}\big|^2(\Lambda P, k_{\bar\chi}; p_\Lambda, \{p\}_n)$$



Auxiliary parton method

$k_\chi^\mu = xP^\mu + k_\perp^\mu \qquad k_{\bar\chi}^\mu = \bar{x}\bar{P}^\mu$

AvH, Kotko, Kutak 2013

We desire to obtain the matrix element with one space-like gluon for the process

$g^\star(k_\chi)\,\omega_{\bar\chi}(k_{\bar\chi}) \;\to\; \omega_1(p_1)\,\omega_2(p_2)\,\cdots\,\omega_n(p_n)$ e.g. $g^\star(k_\chi)\,g(k_{\bar\chi}) \;\to\; g(p_1)\,g(p_2)\,g(p_3)$

and do so by replacing the space-like gluon with an *on-shell auxiliary* quark pair

$q\big(k_1(\Lambda)\big)\,\omega_{\bar\chi}(k_{\bar\chi}) \;\to\; q\big(k_2(\Lambda)\big)\,\omega_1(p_1)\,\omega_2(p_2)\,\cdots\,\omega_n(p_n)$

with special momenta

$$k_1^\mu = \Lambda P^\mu \quad , \quad k_2^\mu = p_\Lambda{}^\mu = (\Lambda - x)P^\mu - k_\perp^\mu + \frac{|k_\perp|^2}{2(\Lambda - x)P\cdot\bar{P}}\,\bar{P}^\mu$$

such that, while individually on-shell, their difference is

$$k_1^\mu - k_2^\mu = xP^\mu + k_\perp + \mathcal{O}\big(\Lambda^{-1}\big) = k_\chi^\mu + \mathcal{O}\big(\Lambda^{-1}\big)$$

The matrix element with the space-like gluon is obtained by taking $\Lambda \to \infty$

$$\big|\overline{\mathcal{M}}^\star\big|^2(k_\chi, k_{\bar\chi}; \{p\}_n) \;=\; \lim_{\Lambda\to\infty} \frac{1}{g_s^2 C_{\mathrm{aux}}}\,\frac{x^2|k_\perp|^2}{\Lambda^2}\,\big|\overline{\mathcal{M}}^{\mathrm{aux}}\big|^2(\Lambda P, k_{\bar\chi}; p_\Lambda, \{p\}_n)$$

The factor $x^2|k_\perp|^2$ ensures the correct on-shell limit, $1/\Lambda^2$ selects the leading power, $1/g_s^2$ corrects the power of the coupling.

One can use auxiliary quarks, as well as gluons, by including the color-correction factor

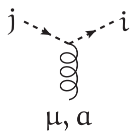
$$C_{\text{aux-q}} = \frac{N_c^2 - 1}{N_c} \quad , \quad C_{\text{aux-g}} = 2N_c$$

Auxiliary parton method

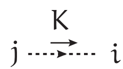
AvH, Kotko, Kutak 2013

$$k_\chi^\mu = xP^\mu + k_\perp^\mu \qquad k_{\bar\chi}^\mu = \bar{x}\bar{P}^\mu$$

the auxiliary parton method can be applied to Feynman graphs, from which one can derive eikonal Feynman rules for the auxiliary partons

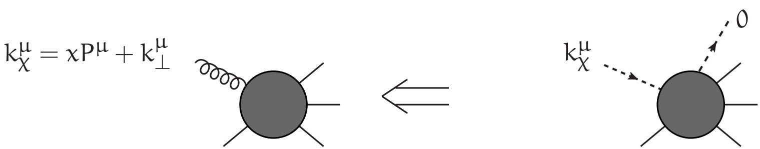


$= -\mathrm{i}\, T^a_{i,j}\, P^\mu$



$= \delta_{i,j}\,\dfrac{\mathrm{i}}{P\cdot K}$

$$\left|\overline{\mathcal{M}}^{\star}\right|^2\!\left(k_\chi, k_{\bar\chi}; \{p\}_n\right) \;=\; \frac{1}{g_s^2 C_{\mathrm{aux}}}\, x^2 |k_\perp|^2 \left|\overline{\mathcal{M}}^{\mathrm{aux}}\right|^2\!\left(k_\chi, k_{\bar\chi}; 0\;, \{p\}_n\right)$$



Auxiliary partons at one loop

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Λ effectively works as a regulator for linear denominators

$$\frac{1}{P\cdot K} \xleftarrow{\Lambda\to\infty} \frac{2\Lambda}{(\Lambda P + K)^2} \quad \Longrightarrow \quad \ln\Lambda \text{ in loop integrals}$$

One-loop amplitudes turn out to depend non-trivially on the type of auxiliary parton.

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For example, apply Λ limit on $A^{\mathrm{loop}}(1_{\bar{Q}}, 6_Q, 2_{\bar{q}}, 3_q, 4_{e^+}, 5_{e^-})$ (Bern, Dixon, Kosower 1998) to get $A^{\mathrm{loop}}(1^\star, 2_{\bar{q}}, 3_q, 4_{e^+}, 5_{e^-})$. The pole-part is proportional to the tree-level amplitude with factor

$$\left\{-\frac{1}{\epsilon^2}\left[\left(\frac{\mu^2}{-s_{p3}}\right)^\epsilon + \left(\frac{\mu^2}{-s_{p2}}\right)^\epsilon\right] - \frac{3}{2\epsilon}\right\} A^{\mathrm{tree}}(1^\star, 2_{\bar{q}}, 3_q, 4_{e^+}, 5_{e^-}) \ ,$$

with s_{p2} and s_{p3} involving only the longitudinal part of $k_1 = p + k_\perp$.

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$$\mathcal{V}_{\mathsf{aux}} = \left(\frac{\mu^2}{|k_\perp|^2}\right)^\epsilon \left[\frac{2}{\epsilon}\ln\frac{\Lambda}{x} - i\pi + \bar{\mathcal{V}}_{\mathsf{aux}}\right] + \mathcal{O}(\epsilon) + \mathcal{O}\big(\Lambda^{-1}\big)$$

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$$\bar{\mathcal{V}}_{\text{aux-q}} = \frac{1}{\epsilon}\frac{13}{6} + \frac{\pi^2}{3} + \frac{80}{18} + \frac{1}{N_c^2}\left[\frac{1}{\epsilon^2} + \frac{3}{2}\frac{1}{\epsilon} + 4\right] - \frac{n_f}{N_c}\left[\frac{2}{3}\frac{1}{\epsilon} + \frac{10}{9}\right]$$

$$\bar{\mathcal{V}}_{\text{aux-g}} = -\frac{1}{\epsilon^2} + \frac{\pi^2}{3}$$

Auxiliary partons at one loop

More-or-less proven using known universal collinear limits of one-loop amplitudes (Bern, Chalmers 1995, Bern, Del Duca, Kilgore, Schmidt 1999).

Before the large-Λ, the small-$|k_\perp|$ corresponds to a collinear limit of auxiliary partons. While the large-Λ and small-$|k_\perp|$ limit commute at tree-level, they do not at one loop.

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Real radiation with auxiliary partons

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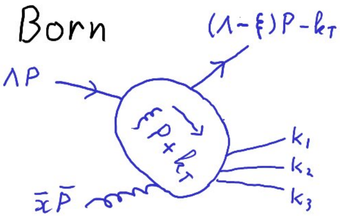

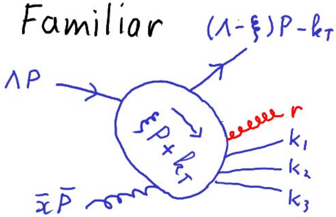

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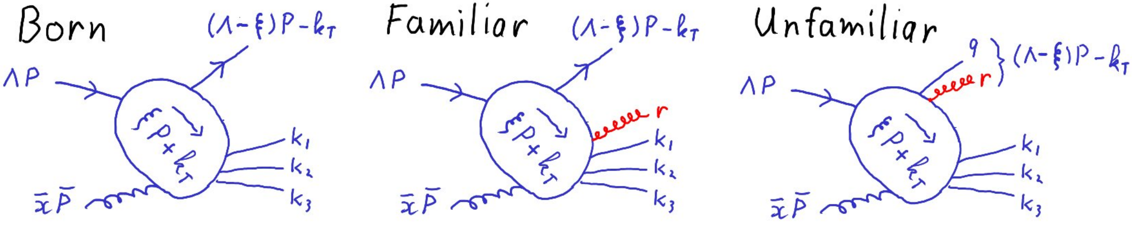

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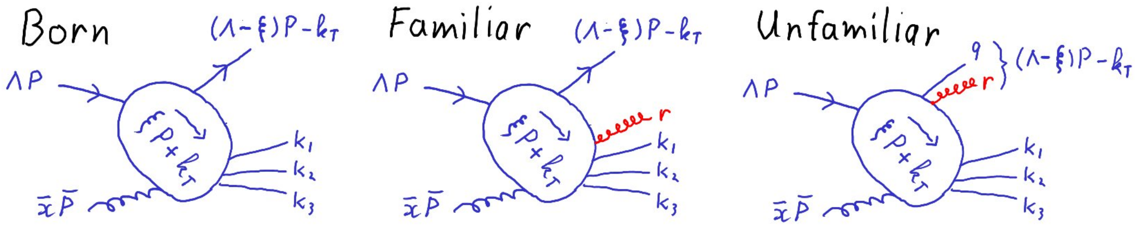
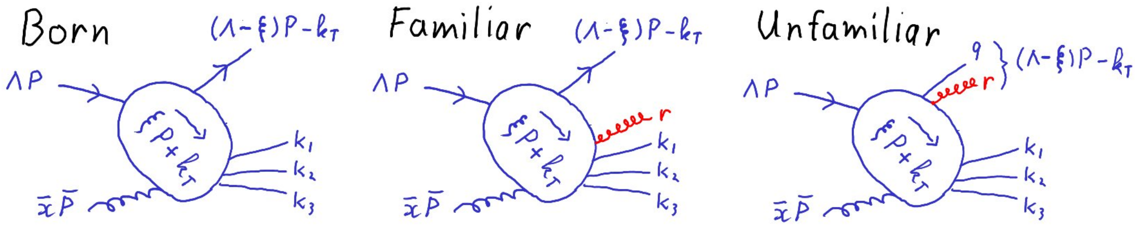


The differential phase space and the matrix element factorize for the *unfamiliar* case, where the radiative gluon participates in the consumption of Λ.

Real radiation with auxiliary partons



The differential phase space and the matrix element factorize for the *unfamiliar* case, where the radiative gluon participates in the consumption of Λ.

$$\frac{1}{\mathcal{C}_{\mathrm{aux}}}\left|\overline{\mathcal{M}}^{\mathrm{aux}}\right|^2\big((\Lambda+x)P,k_{\bar{x}}\,;x_r\Lambda P+r_\perp+\bar{x}_r\bar{P}\,,\,x_q\Lambda P+q_\perp+\bar{x}_q\bar{P}\,,\,\{p_i\}_{i=1}^n\big)$$

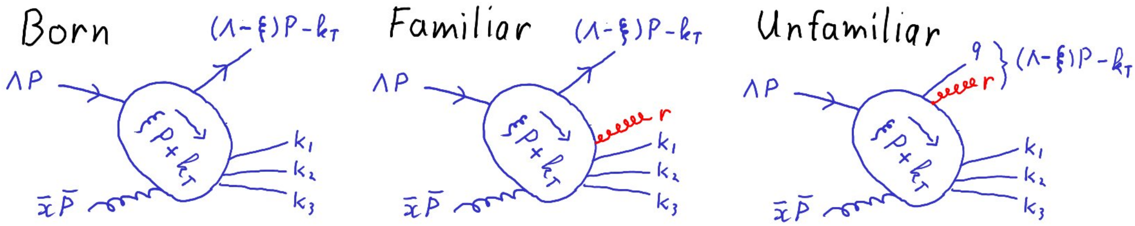
$$\overset{\Lambda\to\infty}{\longrightarrow}\;\mathcal{Q}_{\mathrm{aux}}(x_q,q_\perp,x_r,r_\perp)\,\frac{\Lambda^2\left|\overline{\mathcal{M}}^\star\right|^2(xP-q_\perp-r_\perp,k_{\bar{x}}\,;\{p_i\}_{i=1}^n)}{x^2|q_\perp+r_\perp|^2}$$

$$\mathcal{Q}_{\mathrm{aux}}(x_q,q_\perp,x_r,r_\perp)=x_qx_r\,\mathcal{P}_{\mathrm{aux}}(x_q,x_r)\,|q_\perp+r_\perp|^2$$

$$\times\left[\frac{c_{\bar{q}}}{|q_\perp|^2|r_\perp|^2}+\frac{1}{x_r|q_\perp|^2+x_q|r_\perp|^2-x_qx_r|q_\perp+r_\perp|^2}\left(\frac{c_r\,x_r^2}{|r_\perp|^2}+\frac{c_q\,x_q^2}{|q_\perp|^2}\right)\right]$$

Can be integrated analytically and is proportional to the Born result.
Like the unfamiliar virtual, it is proportional to $\left(\mu^2/|k_\perp|^2\right)^\epsilon$, produces $\ln\Lambda$, and depends on the auxiliary parton types.

Real radiation with auxiliary partons



The differential phase space and the matrix element factorize for the *unfamiliar* case, where the radiative gluon participates in the consumption of Λ.

Precise separation of *familiar* and *unfamiliar* phase space via the demand that in the latter case, the radiation must not become collinear to P in the terms with $1/x_r$

$$\frac{|r_\perp|}{\nu\sqrt{\Lambda}} < x_r < \frac{|r_\perp|}{|r_\perp + k_\perp|} \quad \text{for terms with } 1/x_r$$

Ciafaloni, Colferai 1999

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Complete unfamiliar contribution

Combining the unfamiliar contributions and organizing them suggestively, we can write

$$\mathrm{d}R^{\star\,\mathrm{unf}} + \mathrm{d}V^{\star\,\mathrm{unf}} = \Delta_{\mathrm{unf}}\,\mathrm{d}B^{\star}\ ,$$

where

$$\Delta_{\mathrm{unf}} = \frac{a_{\epsilon} N_c}{\epsilon}\left(\frac{\mu^2}{|k_{\perp}|^2}\right)^{\epsilon}\left[\mathcal{J}_{\mathrm{aux}} + \mathcal{J}_{\mathrm{univ}} + \mathcal{J}_{\mathrm{univ}} - 2\ln\frac{2P\cdot\bar{P}x}{|k_{\perp}|^2}\right]\ ,$$

with

$$\mathcal{J}_{\mathrm{univ}} = \frac{11}{6} - \frac{n_f}{3N_c} - \frac{\mathcal{K}}{N_c}(-\epsilon) \quad \text{writing} \quad \mathcal{K} = N_c\left(\frac{67}{18} - \frac{\pi^2}{6}\right) - \frac{5n_f}{9}\ ,$$

and

$$\mathcal{J}_{\mathrm{aux\text{-}q}} = \frac{3}{2} - \frac{1}{2}(-\epsilon) \quad , \quad \mathcal{J}_{\mathrm{aux\text{-}g}} = \frac{11}{6} + \frac{n_f}{3N_c^3} + \frac{n_f}{6N_c^3}(-\epsilon)\ .$$

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- Target impact factor corrections as in Ciafaloni, Colferai 1999.

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- No $\ln\Lambda$ present. $\mathcal{O}(\alpha_s)$ contribution to the space-like gluon Regge trajectory.
- Target impact factor corrections as in Ciafaloni, Colferai 1999.
- Collinear divergence, cancels against familiar virtual divergence.

Summary

AvH, Motyka, Ziarko 2022

$$d\sigma^{NLO} = \int dx\, d^2k_\perp\, d\bar{x} \Big\{ F(x,k_\perp)\, f(\bar{x}) \Big[dV^\star(x,k_\perp,\bar{x}) + dR^\star(x,k_\perp,\bar{x}) \Big]_{\text{cancelling}}$$
$$+ \Big[F^{NLO}(x,k_\perp) + F(x,k_\perp)\Delta_{unf}(x,k_\perp) + \Delta^\star_{coll}(x,k_\perp) \Big] f(\bar{x})\, dB^\star(x,k_\perp,\bar{x})$$
$$+ \Big[f^{NLO}(\bar{x}) + \Delta_{\overline{coll}}(\bar{x}) \Big] F(x,k_\perp)\, dB^\star(x,k_\perp,\bar{x}) \Big\}$$

$$\Delta_{\overline{coll}}(\bar{x}) = -\frac{a_\epsilon}{\epsilon} \int_{\bar{x}}^1 dz\, \left[\mathcal{P}^{reg}_{\bar{\chi}}(z) + \gamma_{\bar{\chi}}\delta(1-z) \right] \frac{1}{z} f\!\left(\frac{\bar{x}}{z}\right)$$

$$\Delta^\star_{coll}(x,k_\perp) = -\frac{a_\epsilon}{\epsilon} \int_x^1 dz \left[\frac{2N_c}{[1-z]_+} + \frac{2N_c}{z} + \gamma_g\delta(1-z) \right] \frac{1}{z} F\!\left(\frac{x}{z}, k_\perp\right)$$

$$\Delta_{unf}(x,k_\perp) = \frac{a_\epsilon N_c}{\epsilon} \left(\frac{\mu^2}{|k_\perp|^2} \right)^\epsilon \left[\text{impactFactCorr} + \mathcal{J}_{univ} - 2\ln\frac{2P\cdot\bar{P}x}{|k_\perp|^2} \right]$$

$$f^{NLO}(\bar{x}) + \Delta_{\overline{coll}}(\bar{x}) = \text{finite}$$

$$F^{NLO}(x,k_\perp) + F(x,k_\perp)\Delta_{unf}(x,k_\perp) + \Delta^\star_{coll}(x,k_\perp) \stackrel{?}{=} \text{finite}$$

Backup

On-shell limit

Space-like (LO) matrix elements have desired on-shell limit only after azimuthal integration:

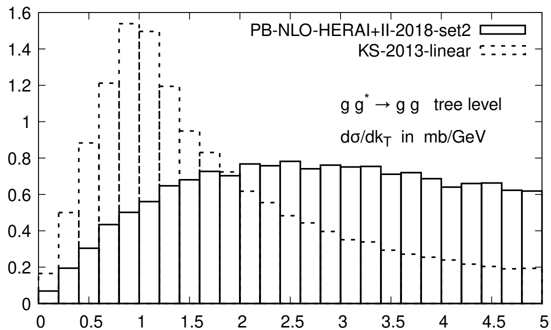
$$\left|\mathcal{M}(k_\perp)\right|^2 \xrightarrow{|k_\perp|\to 0} \mathcal{M}^*_\mu(0)\,\frac{k_\perp^\mu k_\perp^\nu}{|k_\perp|^2}\,\mathcal{M}_\nu(0) \xrightarrow{\int d\varphi_\perp} \left|\mathcal{M}(0)\right|^2$$

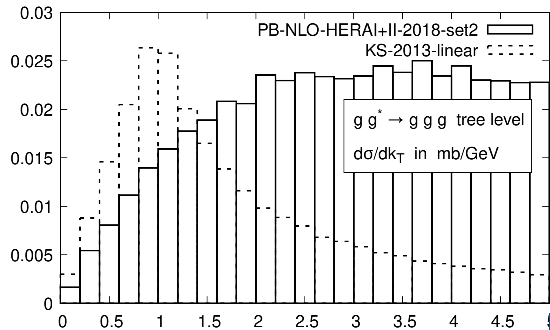
As a consequence, point-wise cancellation of singularities fails at $|k_\perp| = 0$:

$$\left|\mathcal{M}(k_\perp, r_\perp)\right|^2 \xrightarrow{|k_\perp|\to 0} \mathcal{M}^*_\mu(0, r_\perp)\,\frac{k_\perp^\mu k_\perp^\nu}{|k_\perp|^2}\,\mathcal{M}_\nu(0, r_\perp) \xrightarrow{|r_\perp|\to 0} \text{Singular} \times \mathcal{M}^*_\mu(0)\,\frac{k_\perp^\mu k_\perp^\nu}{|k_\perp|^2}\,\mathcal{M}_\nu(0)$$

$$\text{Singular} \times \left|\mathcal{M}(k_\perp - r_\perp)\right|^2 \xrightarrow{|k_\perp|\to 0} \text{Singular} \times \left|\mathcal{M}(-r_\perp)\right|^2 \xrightarrow{|r_\perp|\to 0} \text{Singular} \times \mathcal{M}^*_\mu(0)\,\frac{r_\perp^\mu r_\perp^\nu}{|r_\perp|^2}\,\mathcal{M}_\nu(0)$$

Fortunately, the measure of the problematic phase space vanishes





ITMD* factorization for more than 2 jets

We want to establish a similar factorization for more than 2 jets.

However, the ITMD formalism does not account for linearly polarized gluons in unpolarized target.

Such a contribution is absent for massless 2-particle production in CGC theory, but does appear in heavy quark production (Marquet, Roiesnes, Taels 2018), in the correlation limit for 3-parton final-states (Altinoluk, Boussarie, Marquet, Taels 2020), and can be concluded to be present from 3-jet formulae in CGC (Iancu, Mulian 2019).

This contribution cannot staightforwardly be formulated in terms of gauge-invariant off-shell hard scattering amplitudes

$$\sum_{i,j} \mathcal{M}_i^* \left(\frac{k_T^{(i)} k_T^{(j)}}{2|\vec{k}_T|^2} (\mathcal{F} + \mathcal{H}) + \frac{q_T^{(i)} q_T^{(j)}}{2|\vec{q}_T|^2} (\mathcal{F} - \mathcal{H}) \right) \mathcal{M}_j \quad , \quad \vec{q}_T \cdot \vec{k}_T = 0$$

$\sum_i \mathcal{M}_i k_T^{(i)}$ is gauge invariant while $\sum_i \mathcal{M}_i q_T^{(i)}$ is not. For dijets, it happens that $\mathcal{F} = \mathcal{H}$.

In the following only the manifestly gauge-invariant contribution is included, hence the designation ITMD*.

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Howe ... ıpolarized target

Such ... but does appea ... tion limit for 3- ... :oncluded to be

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$\sum_i \mathcal{M}$ ... $\mathcal{F} = \mathcal{H}$.

In the ... ıence the designation ITMD*.

Using the axial gauge with gluon propagator

$$\frac{-i}{K^2}\left(g^{\mu\nu} - \frac{P^\mu K^\nu + K^\mu P^\nu}{P\cdot K}\right) \qquad P^\mu \text{ hadron momentum}$$

the amplitude $\mathcal{M}$ for a process involving an off-shell gluon with momentum $xP^\mu + k_T^\mu$ can be written as

$$\mathcal{M} = k_T^\mu \mathcal{M}_\mu = -\sum_{i=1}^{2} k_T^{(i)} \mathcal{M}_i$$

where $\mathcal{M}_\mu$ is obtained from the usual Feynman graphs indeed with one gluon simply left "off-shell". The role of "polarization vector" is played by k_T^μ.