Entanglement and Nonlocality in systems with additive observables

Merton College, Oxford 2024

Collab. with
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(Work in progress, about to appear)

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$$|\psi\rangle \in \mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$$

☆ $|\psi\rangle$ entangled $\Leftrightarrow$ $\rho_A = \mathrm{Tr}_B\, \rho$ mixed $\Leftrightarrow$ $\begin{cases} \mathrm{Tr}\, \rho_A^2 \neq 1 \\ S_{\mathrm{vN}} = -\,\mathrm{Tr}\, \rho_A \ln \rho_A \neq 0 \\ \cdots \end{cases}$

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☆ $|\psi\rangle$ entangled $\Leftrightarrow$ $|\psi\rangle$ nonlocal (always violates a CHSH inequality)

Gisin 91
Popescu, Rohrlich 92

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Entanglement

Peres-Horodecki criterion (ρ^{T_2} contains negative eigenvalues)

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Nonlocality

No general rule beyond low dimension, $\mathscr{H}_2 \otimes \mathscr{H}_2$ Horodeckis 95

(see however J Moreno's talk on $\mathscr{H}_2 \otimes \mathscr{H}_d$)

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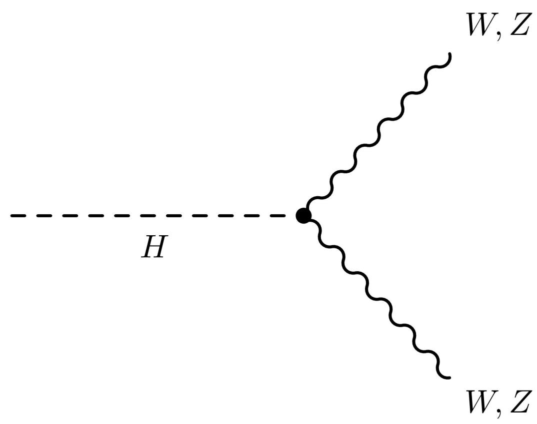

E.g. $H \longrightarrow WW,\, ZZ$

Conservation of $J_Z = 0$ implies

$$\rho = \begin{pmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&\boxed{a}&0&\boxed{b}&0&\boxed{c}&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&\boxed{b^*}&0&\boxed{d}&0&\boxed{f}&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&\boxed{c^*}&0&\boxed{f^*}&0&\boxed{g}&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{pmatrix} \quad \Rightarrow \quad \rho^{T_2} = \begin{pmatrix} 0&0&0&0&0&0&0&0&\boxed{c} \\ 0&0&0&0&0&\boxed{b}&0&0&0 \\ 0&0&\boxed{a}&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&\boxed{f}&0 \\ 0&0&0&0&\boxed{d}&0&0&0&0 \\ 0&\boxed{b^*}&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&\boxed{g}&0&0 \\ 0&0&0&\boxed{f^*}&0&0&0&0&0 \\ \boxed{c^*}&0&0&0&0&0&0&0&0 \end{pmatrix}$$

Still, in some relevant physical scenarios, one can find more general rules.

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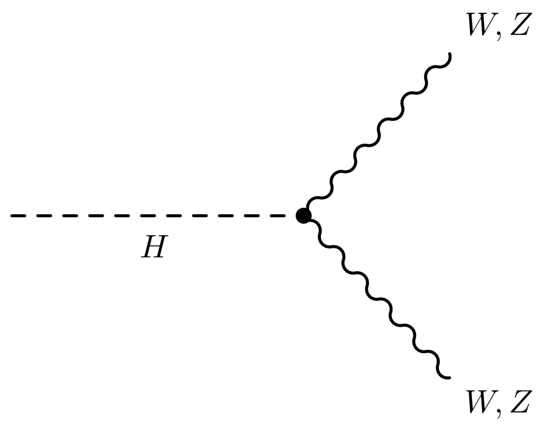

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Eigenvalues of ρ^{T_2} : $a, d, g, \pm|b|, \pm|c|, \pm|f|$ $\Longrightarrow$ Peres-Horodecki criterion **sufficient and necessary**

Aguilar-Saavedra, Bernal, JAC and Moreno 2022

- ☆ Can we generalize this result to more general systems?

- ☆ What about Bell inequalities?

Consider a generic bipartite system, $\mathscr{H}_A \otimes \mathscr{H}_B$ for which there exists an additive observable

$$\hat{J} = \hat{J}_A + \hat{J}_B$$

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- $H \longrightarrow WW,\, ZZ$ $\qquad \hat{J} \equiv \hat{J}_Z$
- Meson decays
- Spin chains ($\hat{J} \equiv$ Magnetization)
- Atom-cavity systems ($\hat{J} \equiv$ Energy)

This setup remains along unitary evolution if such global quantities are conserved

Entanglement

$$\hat{J} = \hat{J}_A + \hat{J}_B \ , \qquad J \text{ well defined}$$

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Use bases of eigenstates of $\hat{J}_A, \hat{J}_B$

$$\mathscr{H}_A \longrightarrow \{\,|m\rangle\}, \qquad \text{eigenvalues } M \text{ (possibly degenerate)}$$

$$\mathscr{H}_B \longrightarrow \{\,|p\rangle\}, \qquad \text{Eigenvalues } P \text{ (possibly degenerate)}$$

$$\mathscr{H}_B \otimes \mathscr{H}_B \longrightarrow \{\,|m\ p\rangle\}, \qquad \text{Eigenvalues } J = M + P \text{ (possibly deg.)}$$

Density matrix elements

$$\rho_{(mp)(nq)} \qquad \text{may be} \quad \neq 0 \text{ only if } \ M + P = N + Q = J$$

Entanglement

☆ The partial-transposed matrix

$$\rho^{T_2}_{(mq)(np)} = \rho_{(mp)(nq)}$$

is block-diagonal: one block for each pair (M, P)

Eigenvalues of $\hat{J}_A, \hat{J}_B$

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Eigenvalues of $\hat{J}_A, \hat{J}_B$

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☆ Two different kinds of blocks, depending on the corresponding diagonal elements,

$$\rho^{T_2}_{(mq)(mq)} \begin{cases} M+Q=J \longrightarrow & \rho^{T_2}_{(mq)(mq)} \text{ may be } \neq 0 \\ M+Q \neq J \longrightarrow & \rho^{T_2}_{(mq)(mq)} = 0 \end{cases}$$

Entanglement

$$M + Q = J \longrightarrow \quad \rho^{T_2}_{(mq)(mq)} \neq 0 \text{ (possibly)}$$

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Row:

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$$\implies N = M,\ P = Q$$

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$$\rho^{T_2}_{(mq)(np)} = \rho_{(mp)(nq)} \quad \text{may be } \neq 0 \text{ only if } M + P = N + Q = J$$

$$\Longrightarrow N = M,\ P = Q$$

Denoting $(m\ q)_\alpha$ the pairs with eigenvalues M, Q :

$$\begin{pmatrix} \rho^{T_2}_{(mq)_1(mq)_1} & \rho^{T_2}_{(mq)_1(mq)_2} & \rho^{T_2}_{(mq)_1(mq)_3} & \cdots \\ \rho^{T_2}_{(mq)_2(mq)_1} & \rho^{T_2}_{(mq)_2(mq)_2} & \rho^{T_2}_{(mq)_2(mq)_3} & \cdots \\ \rho^{T_2}_{(mq)_3(mq)_1} & \rho^{T_2}_{(mq)_3(mq)_2} & \rho^{T_2}_{(mq)_3(mq)_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Principal submatrix of dimension

$(\deg M \cdot \deg Q) \times (\deg M \cdot \deg Q)$

Entanglement

$M + Q = J \longrightarrow \quad \rho^{T_2}_{(mq)(mq)} \neq 0$ (possibly)

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$$\rho^{T_2}_{(mq)(np)} = \rho_{(mp)(nq)}$$ may be $\neq 0$ only if $M + P = N + Q = J$

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Principal submatrix of dimension

$(\deg M \cdot \deg Q) \times (\deg M \cdot \deg Q)$

If M, Q non-degenerate $\implies$ just one (diagonal) entry

Entanglement

$$M + Q \neq J \longrightarrow \quad \rho^{T_2}_{(mq)(mq)} = 0$$

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$\rho^{T_2}_{(mq)(np)} = \rho_{(mp)(nq)}$ may be $\neq 0$ only if $M + P = N + Q = J$

$$\Longrightarrow \; N = J - Q, \;\; P = J - M$$

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Denoting $(m\ q)_\beta$ the pairs with M, Q and $(n\ p)_\gamma$ the pairs with N, P

$$\begin{array}{c} \text{deg } M \cdot \text{deg } Q \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ \text{deg } N \cdot \text{deg } P \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \end{array} \left(\begin{array}{ccc|ccc} \rho^{T_2}_{(mq)_1(mq)_1} & \rho^{T_2}_{(mq)_1(mq)_2} & \cdots & \rho^{T_2}_{(mq)_1(np)_1} & \rho^{T_2}_{(mq)_1(np)_2} & \cdots \\ \rho^{T_2}_{(mq)_2(mq)_1} & \rho^{T_2}_{(mq)_2(mq)_2} & \cdots & \rho^{T_2}_{(mq)_2(np)_1} & \rho^{T_2}_{(mq)_2(np)_1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ \hline \rho^{T_2}_{(np)_1(mq)_1} & \rho^{T_2}_{(np)_1(mq)_2} & \cdots & \rho^{T_2}_{(np)_1(np)_1} & \rho^{T_2}_{(np)_1(n\vec{p})_2} & \cdots \\ \rho^{T_2}_{(np)_2(mq)_1} & \rho^{T_2}_{(np)_2(mq)_2} & \cdots & \rho^{T_2}_{(np)_2(np)_1} & \rho^{T_2}_{(np)_2(np)_2} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{array} \right)$$

Entanglement

$$
\left(\begin{array}{ccc|ccc}
\rho^{T_2}_{(mq)_1(mq)_1} & \rho^{T_2}_{(mq)_1(mq)_2} & \cdots & \rho^{T_2}_{(mq)_1(np)_1} & \rho^{T_2}_{(mq)_1(np)_2} & \cdots \\
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\hline
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$$

$$
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\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\hline
\rho^{T_2}_{(np)_1(mq)_1} & \rho^{T_2}_{(np)_1(mq)_2} & \cdots & 0 & 0 & \cdots \\
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\end{array}\right)
$$

any of these entries $\neq 0$ $\Longrightarrow$ entangled state

Entanglement

Any element $\rho_{(mp)(nq)} = \rho^{T_2}_{(mq)(np)} \neq 0$ (with $M + Q \neq J$)

$\Longrightarrow$ entangled state (by Peres-Horodecki)

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"crossed off-diagonal entry"

Entanglement

Higgs-boson decays

$H \longrightarrow WW,\, ZZ$

$\hat{J} \equiv \hat{J}_Z$

$$\mathcal{H}_A = \mathrm{span}\left\{ |1\rangle_A \,, |0\rangle_A \,, |-1\rangle_A \right\}, \quad \mathcal{H}_B = \mathrm{span}\left\{ |1\rangle_B \,, |0\rangle_B \,, |-1\rangle_B \right\}$$

non-degenerate eigenvalues $\Longrightarrow$ All non-diagonal entries of are *crossed off-diagonal*

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$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{(1,-1)(1,-1)} & 0 & \rho_{(1,-1)(0,0)} & 0 & \rho_{(1,-1)(-1,1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{(0,0)(1,-1)} & 0 & \rho_{(0,0)(0,0)} & 0 & \rho_{(0,0)(-1,1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{(-1,1)(1,-1)} & 0 & \rho_{(-1,1)(0,0)} & 0 & \rho_{(-1,1)(-1,1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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If any of them are $\neq 0$ $\Longleftrightarrow$ *the state is entangled*

Entanglement

General case

When all the crossed off-diagonal entries are zero, the state can only be entangled if there are degenerate eigenvalues M, Q with $M + Q = J$

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Recall

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Principal submatrix of dimension

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Principal submatrix of dimension $(\deg M \cdot \deg Q) \times (\deg M \cdot \deg Q)$

- If only one of the eigenvalues M, Q is degenerate, the state is separable
- If $(\deg M, \deg Q) = (2,2), (2,3), (3,2)$ the Peres-Horodecki criterion is sufficient and necessary

Entanglement

SUMMARY

For bipartite system with an additive observable $\hat{J} = \hat{J}_A + \hat{J}_B$

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☆ The Peres-Hor. criterion for entanglement is sufficent and neccesary if for any pair J_A, J_B satisfying $J_A + J_B = J$

1. $(\deg J_A,\ \deg J_B) = (1,\ d)$ or $(d,\ 1)$

2. $(\deg J_A,\ \deg J_B) = (2,2), (2,3), (3,2)$

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Always the case in qubit-qudit systems

1. $(\deg\ J_A,\ \deg\ J_B) = (1,\ d)$ or $(d,\ 1)$

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Bell nonlocality

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Definition:

For bipartite system with an additive observable $\hat{J} = \hat{J}_A + \hat{J}_B$

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we say that $\rho_{(m_0 p_0)(n_0 q_0)} \neq 0$ is an anchor entry if

- $M_0 + P_0 = N_0 + Q_0 = J$

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- $M_0 \neq N_0 \ , \ \ P_0 \neq Q_0$

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- $M_0 \neq N_0\ ,\ \ P_0 \neq Q_0$

} Crossed off-diagonal term

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- $M_0 \neq N_0 \, , \;\; P_0 \neq Q_0$

} Crossed off-diagonal term

- (M_0 , P_0) or (N_0 , Q_0) non-degenerate

Bell nonlocality

Definition:

For bipartite system with an additive observable $\hat{J} = \hat{J}_A + \hat{J}_B$

which has a definite value, J

we say that $\rho_{(m_0 p_0)(n_0 q_0)} \neq 0$ is an anchor entry if

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- $(M_0 \,, P_0)$ or $(N_0 \,, Q_0)$ non-degenerate

Result:

If the density matrix contains an anchor entry, the state violates a CHSH inequality

Bell nonlocality

Recall:

For (binary) observables, A_1 , A_2 (Alice's) and B_1 , B_2 (Bob's)

CHSH: $$F_{\text{CHSH}} = E(A_1(B_1 + B_2)) + E(A_2(B_1 - B_2)) \leq 2$$

Bell nonlocality

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For (binary) observables, A_1 , A_2 (Alice's) and B_1 , B_2 (Bob's)

CHSH: $$F_{\mathrm{CHSH}} = E(A_1(B_1 + B_2)) + E(A_2(B_1 - B_2)) \leq 2$$

$$F_{\mathrm{CHSH}}(\rho) = \mathrm{Tr}\left\{ \rho \left[A_1 \otimes (B_1 + B_2) \right] \right\} + \mathrm{Tr}\left\{ \rho \left[A_2 \otimes (B_1 - B_2) \right] \right\}$$

Bell nonlocality

Sketch of the demonstration:

Suppose that $\rho_{(m_0 p_0)(n_0 q_0)} \neq 0$ is the anchor entry, with N_0 , Q_0 non-degenerate

Bell nonlocality

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- Re-order Alice's and Bob's bases:

$$\mathscr{H}_A : \quad \left\{ |m_0\rangle, |n_0\rangle, |s_1\rangle, \ldots, |s_{d_A-2}\rangle \right\}_{s_i \neq m_0, n_0},$$

$$\mathscr{H}_B : \quad \left\{ |p_0\rangle, |q_0\rangle, |t_1\rangle, \ldots, |t_{d_B-2}\rangle \right\}_{t_j \neq p_0, q_0},$$

Bell nonlocality

Sketch of the demonstration:

Suppose that $\rho_{(m_0 p_0)(n_0 q_0)} \neq 0$ is the anchor entry, with N_0 , Q_0 non-degenerate

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$$\mathscr{H}_B: \quad \left\{ |p_0\rangle, |q_0\rangle, |t_1\rangle, \ldots, |t_{d_B-2}\rangle \right\}_{t_j \neq p_0, q_0},$$

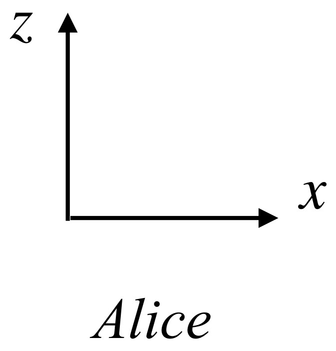
- Define A_1, A_2, B_1, B_2 observables as

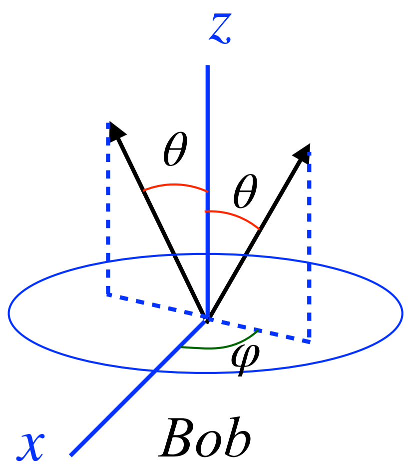
$$A_i = \begin{pmatrix} \hat{a}_i \vec{\sigma} & 0 \\ 0 & \mathbb{1}_{d_A-2} \end{pmatrix} \qquad B_j = \begin{pmatrix} \hat{b}_j \vec{\sigma} & 0 \\ 0 & \mathbb{1}_{d_B-2} \end{pmatrix}$$

(Inspired in Popescu & Rohrlich construction, 92)

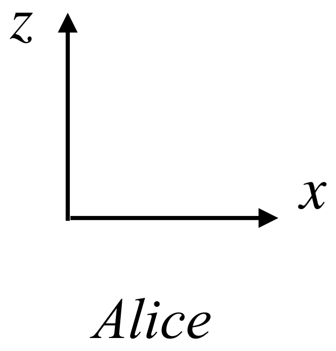
$$\begin{cases} \hat{a}_1 = (0,0,1), \\ \hat{a}_2 = (1,0,0) \end{cases} \qquad \begin{cases} \hat{b}_1 = (\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta) \\ \hat{b}_2 = (-\sin\theta\sin\varphi, -\sin\theta\cos\varphi, \cos\theta) \end{cases}$$

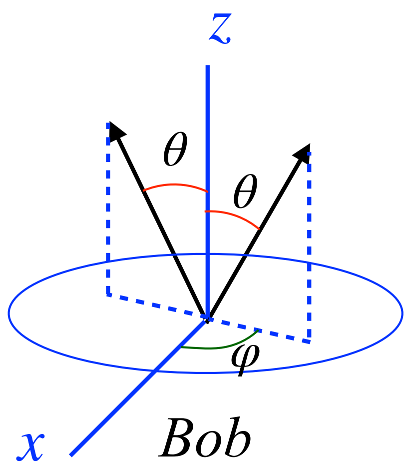
Bell nonlocality





Bell nonlocality





After some algebra…

$$\max_{\theta,\varphi}\left\{F_{\rm CHSH}(\rho)\right\} = 2\left[1+\sqrt{4\,|\rho_{(m_0p_0)(n_0q_0)}|^2+\langle\mathcal{O}_z\rangle_\rho^2}-\langle\mathcal{O}_z\rangle_\rho\right] > 2 \iff \rho_{(m_0p_0)(n_0q_0)} \neq 0.$$

Bell nonlocality

$$\mathcal{O}_z = \left(\begin{array}{cc|ccc}
\begin{pmatrix} \sigma_z & \mathbf{0} \\ \mathbf{0} & \mathbb{O}_{d_B-2} \end{pmatrix} & \mathbb{O}_{d_B} & \mathbb{O}_{d_B} & \cdots & \mathbb{O}_{d_B} \\
\mathbb{O}_{d_B} & -\begin{pmatrix} \sigma_z & \mathbf{0} \\ \mathbf{0} & \mathbb{O}_{d_B-2} \end{pmatrix} & \mathbb{O}_{d_B} & \cdots & \mathbb{O}_{d_B} \\
\hline
\mathbb{O}_{d_B} & \mathbb{O}_{d_B} & \begin{pmatrix} \sigma_z & \mathbf{0} \\ \mathbf{0} & \mathbb{O}_{d_B-2} \end{pmatrix} & \cdots & \mathbb{O}_{d_B} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbb{O}_{d_B} & \mathbb{O}_{d_B} & \mathbb{O}_{d_B} & \cdots & \begin{pmatrix} \sigma_z & \mathbf{0} \\ \mathbf{0} & \mathbb{O}_{d_B-2} \end{pmatrix}
\end{array}\right)$$

$$\mathbb{O}_{d_B} = d_B \times d_B \text{ null matrix}$$

Bell nonlocality

Application to Higgs decays:

$H \longrightarrow \tau\tau, WW, ZZ$ Additive observable $J_z = 0$

All J_z^A, J_z^B sectors are non-degenerate

Bell nonlocality

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All non-diagonal elements in ρ are crossed off-diagonal and anchor entries

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All $J_z^A,\ J_z^B$ sectors are non-degenerate

All non-diagonal elements in ρ are crossed off-diagonal and anchor entries

The presence of one off-diagonal entry in ρ is sufficient to certify entanglement and the existence of Bell non-locality

Bell nonlocality

$H \longrightarrow ZZ$

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & 0 & a_{12} & 0 & a_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{12}^* & 0 & a_{22} & 0 & a_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{13}^* & 0 & a_{23}^* & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with

$$a_{12} = a_{23} = \frac{1}{3} C_{2,1,2,-1}, \quad a_{11} = a_{33} = a_{13} = \frac{1}{3} C_{2,2,2,-2}.$$

$$\rho = \frac{1}{9} \left[\mathbb{1}_3 \otimes \mathbb{1}_3 + A^1_{LM}\; T^L_M \otimes \mathbb{1}_3 + A^2_{LM}\; \mathbb{1}_3 \otimes T^L_M + C_{L_1 M_1 L_2 M_2}\; T^{L_1}_{M_1} \otimes T^{L_2}_{M_2} \right]$$

(basis of irreducible tensor operators)

Bell nonlocality

$H \longrightarrow ZZ$

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & 0 & a_{12} & 0 & a_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{12}^* & 0 & a_{22} & 0 & a_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{13}^* & 0 & a_{23}^* & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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(basis of irreducible tensor operators)

Bell nonlocality

$H \longrightarrow ZZ$

$$F(\rho)_{12} = F(\rho)_{23} = 2 + 2(1 - a_{13}) \left(\sqrt{1 + \left(\frac{2a_{12}}{1 - a_{13}} \right)^2} - 1 \right),$$

$$F(\rho)_{13} = 2 + 4a_{13} \left(\sqrt{2} - 1 \right).$$

Bell nonlocality

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min m_{Z_2}

	0	10 GeV	20 GeV	30 GeV
N	450	418	312	129
a_{12}	-0.33 ± 0.10	-0.32 ± 0.11	-0.35 ± 0.13	-0.35 ± 0.20
a_{13}	0.20 ± 0.12	0.21 ± 0.13	0.25 ± 0.14	0.27 ± 0.21
$F(\rho)_{12}$	2.47	2.46	2.55	2.57
	$> 2\ (3.2\sigma)$	$> 2\ (2.9\sigma)$	$> 2\ (2.7\sigma)$	$> 2\ (1.7\sigma)$
$F(\rho)_{13}$	2.33	2.35	2.41	2.45
	$> 2\ (1.7\sigma)$	$> 2\ (1.6\sigma)$	$> 2\ (1.8\sigma)$	$> 2\ (1.3\sigma)$

Table 5.1: Values of the crossed off-diagonal terms a_{12} and a_{13} signaling quantum entanglement, as obtained from 1000 pseudoexperiments with $L = 300\ \text{fb}^{-1}$ in ref. [15].

(From Aguilar-Saavedra, Bernal, JAC and Moreno 2022)

Bell nonlocality

$H \longrightarrow ZZ$

$$F(\rho)_{12} = F(\rho)_{23} = 2 + 2(1 - a_{13})\left(\sqrt{1 + \left(\frac{2a_{12}}{1 - a_{13}}\right)^2} - 1\right),$$

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Bell nonlocality

$H \longrightarrow ZZ$

$\min\ m_{Z_2}$

	0	10 GeV	20 GeV	30 GeV
N	4500	4180	3120	1290
a_{12}	-0.32 ± 0.03	-0.33 ± 0.03	-0.35 ± 0.04	-0.35 ± 0.06
a_{13}	0.20 ± 0.04	0.21 ± 0.04	0.25 ± 0.05	0.28 ± 0.07
$F(\rho)_{12}$	2.44	2.49	2.54	2.56
	$> 2\ (9.5\sigma)$	$> 2\ (10.0\sigma)$	$> 2\ (8.7\sigma)$	$> 2\ (5.5\sigma)$
$F(\rho)_{13}$	2.33	2.35	2.41	2.46
	$> 2\ (5.0\sigma)$	$> 2\ (5.3\sigma)$	$> 2\ (5.3\sigma)$	$> 2\ (4.2\sigma)$

Table 5.2: The same as table 5.1, but for a luminosity $L = 3\ \text{ab}^{-1}$.

(From Aguilar-Saavedra, Bernal, JAC and Moreno 2022)

Bell nonlocality

$$H \longrightarrow ZZ$$

Note:

- The CHSH violation found is not the maximal Bell-violation
- Other (CGLMP) inequalities, which are also violated, have been recently found in A. Bernal, P. Caban and J. Rembielinski, 2024

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For bipartite system with an additive observable $\hat{J} = \hat{J}_A + \hat{J}_B$

which has a definite value, J

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☆ The Peres-Hor. criterion for entanglement is sufficent and neccesary if for any pair J_A, J_B satisfying $J_A + J_B = J$

1. $(\deg J_A,\ \deg J_B) = (1,\ d)$ or $(d,\ 1)$

2. $(\deg J_A,\ \deg J_B) = (2,2), (2,3), (3,2)$

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☆ This holds for $H \longrightarrow \tau\tau,\ WW,\ ZZ$, where the presence of any non-vanishing off-diagonal entry of ρ is a sufficient and necessary condition for entanglement and nonlocality

Backup slides

Observables B_1 , B_2:

$$F(\rho) = 2\left[\langle \mathcal{O}_0\rangle_\rho + \sin\theta\sin\varphi\langle \mathcal{O}_x\rangle_\rho + \sin\theta\cos\varphi\langle \mathcal{O}_y\rangle_\rho + \cos\theta\langle \mathcal{O}_z\rangle_\rho\right]. \quad (4.16)$$

Note that the three last terms are simply the dot product $\vec{\hat{b}}_1 \cdot \langle \vec{\mathcal{O}}\rangle_\rho$ with $\langle \vec{\mathcal{O}}\rangle_\rho = (\langle \mathcal{O}_x\rangle_\rho, \langle \mathcal{O}_y\rangle_\rho, \langle \mathcal{O}_z\rangle_\rho)$. The maximum of this expression is obtained when the θ, φ angles are in such a way that the unitary vector $\hat{b}_1$ is aligned with $\langle \vec{\mathcal{O}}\rangle_\rho$, and it is equal to the modulus of this vector. Hence

$$\max_{\theta,\varphi}\{F(\rho)\} = 2\left[\langle \mathcal{O}_0\rangle_\rho + \sqrt{\langle \mathcal{O}_x\rangle_\rho^2 + \langle \mathcal{O}_y\rangle_\rho^2 + \langle \mathcal{O}_z\rangle_\rho^2}\right], \quad (4.17)$$

$$\mathcal{O}_0 = (\sigma_z \oplus \mathbb{1}_{d_A-2}) \otimes (\mathbb{O}_2 \oplus \mathbb{1}_{d_B-2}), \quad \mathcal{O}_z = (\sigma_z \oplus \mathbb{1}_{d_A-2}) \otimes (\sigma_z \oplus \mathbb{O}_{d_B-2}),$$
$$\mathcal{O}_x = (\sigma_x \oplus \mathbb{1}_{d_A-2}) \otimes (\sigma_x \oplus \mathbb{O}_{d_B-2}), \quad \mathcal{O}_y = (\sigma_x \oplus \mathbb{1}_{d_A-2}) \otimes (\sigma_y \oplus \mathbb{O}_{d_B-2}).$$

Bell nonlocality

$H \longrightarrow ZZ$

At $L = 300\ \text{fb}^{-1}$ both, entanglement and nonlocality, certified at $\sim 3\sigma$

At $L = 3\ \text{ab}^{-1}$ both, entanglement and nonlocality, certified at $\sim 10\sigma$