Violating Bell inequality in qubit-qudit systems: necessary and sufficient conditions

Oxford
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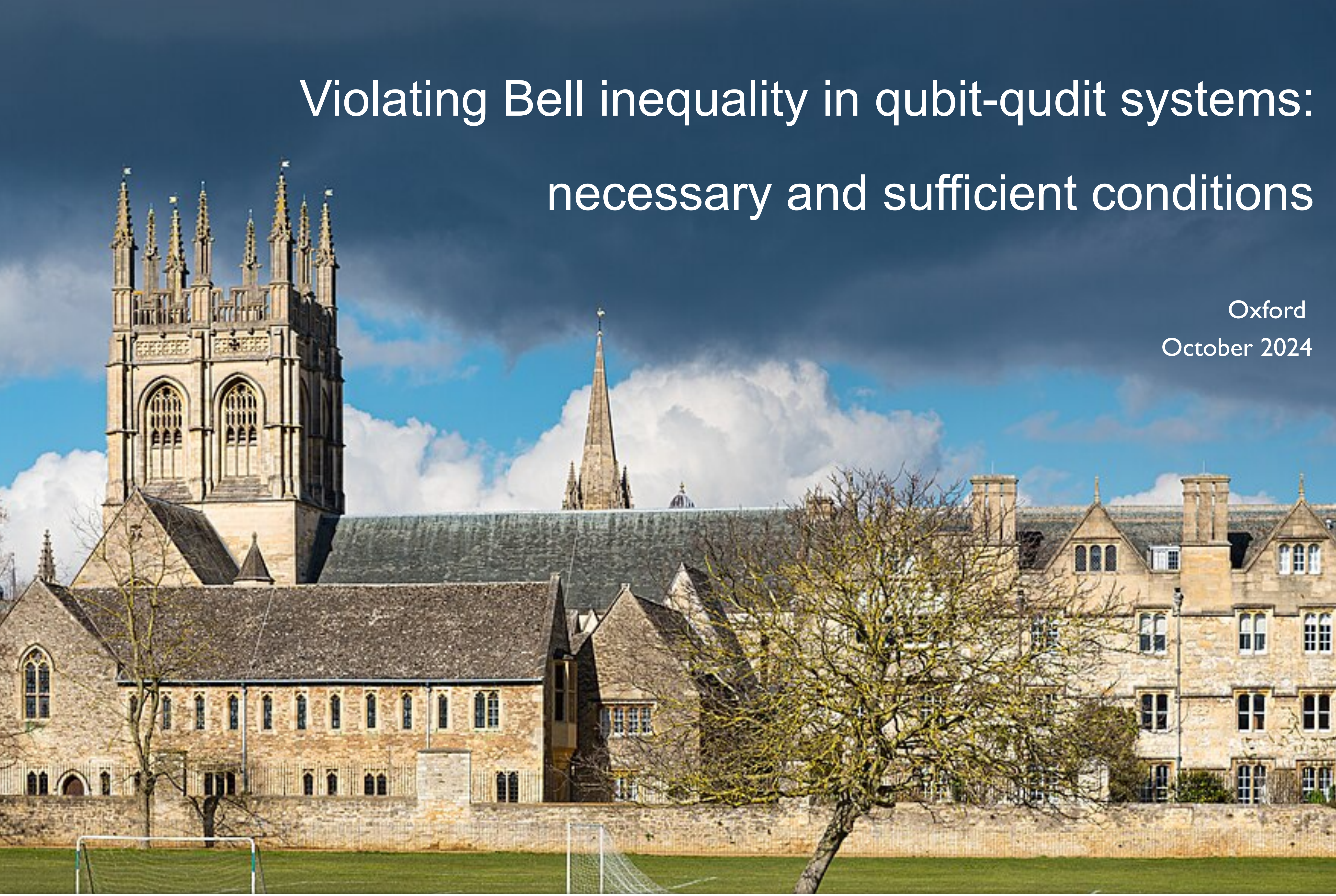

Picture credits: J. Herzog

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OUTLINE

- Introduction & motivation
- Review of Horodecki's result for qubit-qubit systems
- Main result:
  Necessary and sufficient condition for ~~Bell~~ in qubit-qubit systems
- Applications
  - Quantum non locality in the atom - cavity system
  - Quantum non locality in random matrices sets
  - Quantum non locality in the t W system at the LHC
- Summary

Based on work done in collaboration with A. Bernal, J.A. Casas, 2024

Introduction

Qubits play a main role in entanglement tests at low energies (Quantum optics, atomic physics) & high energies (ATLAS and CMS entanglement in t-tbar states)

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Bell non locality is fully characterized in qubit-qubit systems
What about Bell non locality in **qubits-<u>qudits</u>** states?

Question from JA Aguilar-Saavedra

qubit-qubit system: Horodecki's result

A generic state in $\mathcal{H}_2 \otimes \mathcal{H}_2$ can be decomposed as

$$\rho = \frac{1}{4}\left(\mathbb{1}_2 \otimes \mathbb{1}_2 + \boldsymbol{a}.\boldsymbol{\sigma} \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \boldsymbol{b}.\boldsymbol{\sigma} + \sum_{i,j=1}^{3} C_{ij}\sigma_i \otimes \sigma_j \right)$$

Bell CHSH-like inequalities (two measurements, two outcomes {+1,-1}) for this system

$$\mathcal{O}_{\text{Bell}} = AB + AB' + A'B - A'B'$$

are tight: they capture the facets of the Local Hidden Polytope. Notice that Tr$(\rho\mathcal{O}_{Bell})$ only depends on the spin correlations, C_{ij}

Horodecki's result:

$$\langle \mathcal{O}_{\text{Bell}} \rangle_{\max} = 2\sqrt{\kappa_1 + \kappa_2}$$

R. Horodecki, P. Horodecki, M. Horodecki, 1995

κ_i are the CC^T eigenvalues

From qubit-qubit to qubit-qudit

Naive generalization to $\mathscr{H}_2 \otimes \mathscr{H}_d$ using SU(d) Gell-Mann matrices

$$\rho = \frac{1}{2d}\left(\mathbb{1}_2 \otimes \mathbb{1}_d + \mathbf{a}\boldsymbol{\sigma} \otimes \mathbb{1}_d + \mathbb{1}_2 \otimes \mathbf{b}\boldsymbol{\lambda} + \sum_{i=1}^{3}\sum_{\alpha=1}^{d^2-1} C_{i\alpha}\sigma_i \otimes \lambda_\alpha\right)$$

doesn't work. Some drawbacks:

- Tr$\left(\rho O_{Bell}\right)$ depends on the spin polarization for d ≥ 3
- The B set of $d \times d$ operators is more subtle
    - Several signatures $\{+1, -1\}$ for d ≥ 4
    - $B \neq \vec{u}\vec{\lambda}$ for d ≥ 3

We find useful this **alternative** parameterization of a generic $\mathscr{H}_2 \otimes \mathscr{H}_d$ state

$$\rho = \frac{1}{2}\begin{pmatrix} \beta_0 + \beta_3 & \beta_1 - i\beta_2 \\ \beta_1 + i\beta_2 & \beta_0 - \beta_3 \end{pmatrix}$$

← "Schmidt decomposition" using Pauli matrices (≈ extracting matrix blocks)

$$= \frac{1}{2}\left(\mathbb{1}_2 \otimes \beta_0 + \sigma_i \otimes \beta_i\right)$$

← $\beta_0 =$ Tr$_A\ \rho$, hermitian

Choosing a different representation $\sigma \to U^\dagger \sigma U,\ \ \sigma_i \to \mathscr{R}_{ij}\sigma_j$ will entail a different - $SO(3)$ rotated - $\{\beta_i\}$ set, $\beta_i \to \mathscr{R}_{ij}\beta_j$

qubit-qudit: MAIN RESULT

$\|\mathscr{M}\|_1 = \sum eigen(\mathscr{M})$

$\langle \mathcal{O}_{\rm Bell} \rangle_{\max}$ is a function of the Trace Norms, $\| \ \|_1$ of $\{\beta_i\}$

$$\langle O_{\rm Bell} \rangle_{\max} = 2 \max_{\mathcal{R}} \sqrt{\|\beta_1\|_1^2 + \|\beta_2\|_1^2}$$

A. Bernal, J.A. Casas and JMM arXiv:2404.02092

- Analytical & compact. Generalizes the well-known 2×2 result !
- Easy to implement numerically
- MANY APPLICATIONS for NONLOCAL TESTS!

Sketch of the proof

It is convenient to define the following vectors:

$$
\begin{aligned}
\vec{r}_A &= (\mathrm{Tr}(\sigma_1 A), \mathrm{Tr}(\sigma_2 A), \mathrm{Tr}(\sigma_3 A)) \,, \\
\vec{r}_B &= (\mathrm{Tr}(\beta_1 B), \mathrm{Tr}(\beta_2 B), \mathrm{Tr}(\beta_3 B))
\end{aligned}
$$

Then

$$\langle \mathcal{O}_{\mathrm{Bell}} \rangle = \frac{1}{2}\vec{r}_A(\vec{r}_B + \vec{r}_{B'}) + \frac{1}{2}\vec{r}_{A'}(\vec{r}_B - \vec{r}_{B'})$$

Now, since $\mathrm{Tr}\, A^2 = \mathrm{Tr}\, A'^2 = \mathrm{Tr}\, \mathbb{1}_2 = 2$, we get $\|\vec{r}_A\|^2 = \|\vec{r}_{A'}\|^2 = 4$.

$$\langle \mathcal{O}_{\mathrm{Bell}} \rangle_{\max} = \max_{B,B'} \Big\{ \|\vec{r}_B + \vec{r}_{B'}\| + \|\vec{r}_B - \vec{r}_{B'}\| \Big\}.$$

Sketch of the proof

- Lemma I

  Let $\vec{v}, \vec{w}$ be two arbitrary vectors in a plane. Consider a simultaneous rotation of angle φ of both vectors and call the new vectors $\vec{v}(\varphi), \vec{w}(\varphi)$, so that $\vec{v} = \vec{v}(0)$, $\vec{w} = \vec{w}(0)$. Then the following identity takes place:

$$\left(\|\vec{v}+\vec{w}\|+\|\vec{v}-\vec{w}\|\right)^2 = 4\max_{\varphi}\left\{v_1(\varphi)^2 + w_2(\varphi)^2\right\}, \qquad \varphi = \frac{1}{2}\arctan\frac{2w_1w_2}{v_1^2+w_2^2-w_1^2}$$

Then

$$\left(\|\vec{v}+\vec{w}\|+\|\vec{v}-\vec{w}\|\right)^2 = 4\max_{\mathcal{R}}\left\{(\mathcal{R}\vec{v})_1^2 + (\mathcal{R}\vec{w})_2^2\right\}$$

Applying it to our case:

$$\langle \mathcal{O}_{\rm Bell}\rangle_{\max} = 2\max_{B,B',\mathcal{R}}\sqrt{|(\mathcal{R}\vec{r}_B)_1|^2 + |(\mathcal{R}\vec{r}_{B'})_2|^2}\ .$$

$$\langle \mathcal{O}_{\rm Bell}\rangle_{\max} = 2\max_{B,B',\mathcal{R}}\sqrt{\left|{\rm Tr}\left[(\mathcal{R}\vec{\beta})_1\cdot B\right]\right|^2 + \left|{\rm Tr}\left[(\mathcal{R}\vec{\beta})_2\cdot B'\right]\right|^2}\ .$$

$$\begin{aligned}\langle \mathcal{O}_{\rm Bell}\rangle_{\max} &= 2\max_{\mathcal{R}}\sqrt{\|(\mathcal{R}\vec{\beta})_1\|_1^2+\|(\mathcal{R}\vec{\beta})_2\|_1^2} \\ &= 2\max_{\mathcal{R}}\left[\left(\sum_{i=1}^{d}|\lambda_i^{(1)}(\mathcal{R})|\right)^2 + \left(\sum_{i=1}^{d}|\lambda_i^{(2)}(\mathcal{R})|\right)^2\right]^{1/2}\end{aligned}$$

A note on the numerical implementation

$\langle \mathcal{O}_{\rm Bell} \rangle_{\rm max}$ from a FULLY NUMERICAL (A,A', B,B') scan

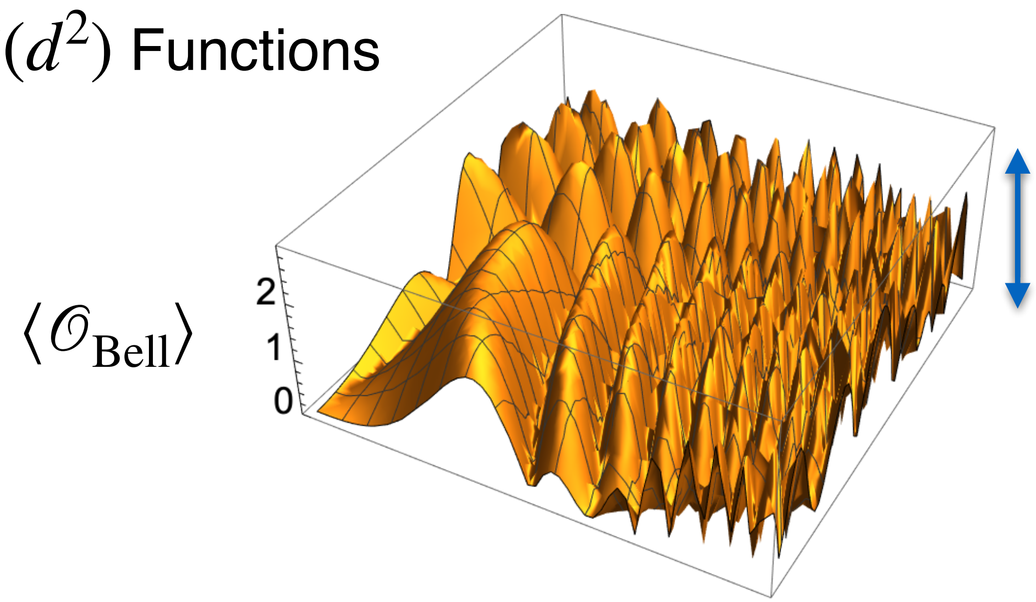
- Scan in (B, B'):

  $2(d^2 - 1)$ parameters

  $\mathcal{O}(d^2)$ signatures to be explored

- Wide range of $\mathcal{O}_{\rm Bell}$ values, many local extrema when increasing d

$\#\mathcal{O}(d^2)$ Functions



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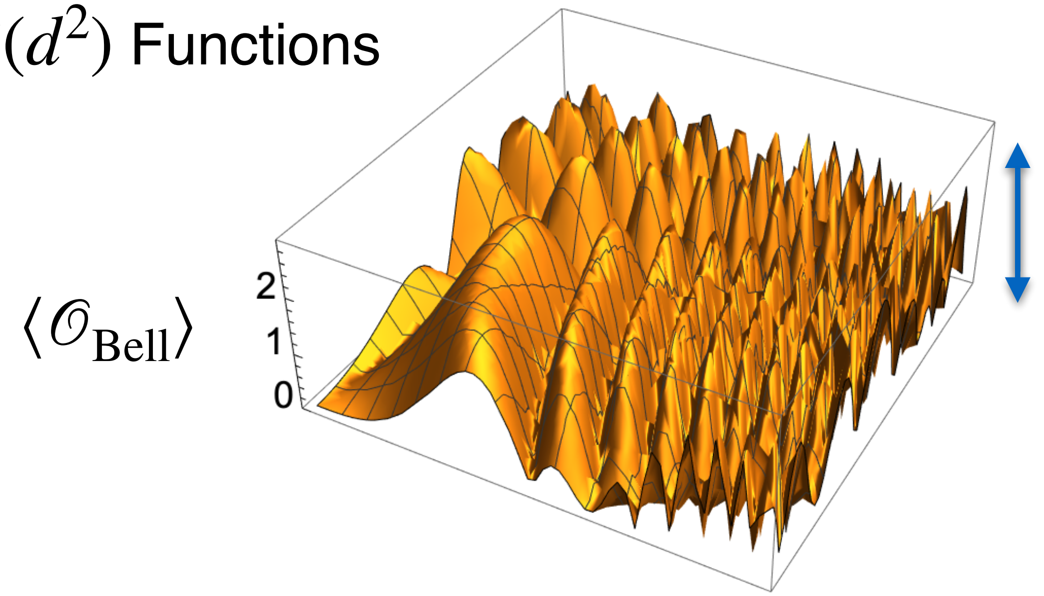


$\#\mathcal{O}(d^2)$ variables

$\langle \mathcal{O}_{\rm Bell} \rangle_{\rm max}$ from our ANALYTICAL expression (+ numerical scan of SO(3))

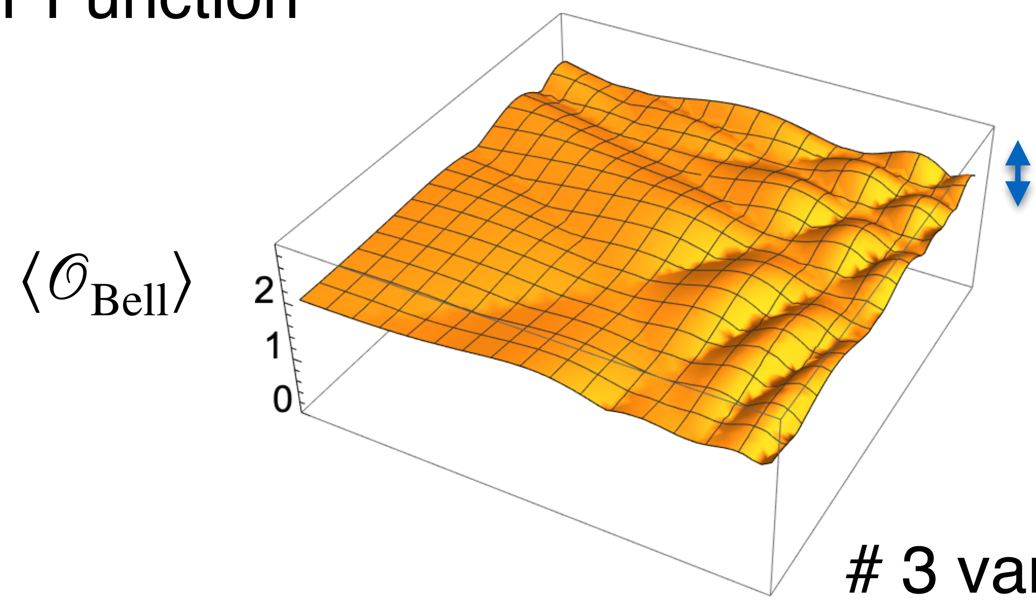
- Scan in $\mathscr{R}$, rotations

  3 parameters

  the signature is nothing but Sign(Eigen)

- Narrow range of $\mathcal{O}_{\rm Bell}$ values when increasing d

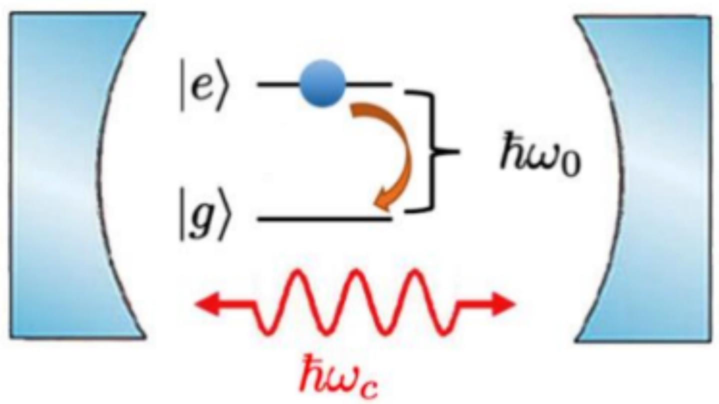
1 Function



3 variables

Quantum non locality in atom - cavity systems

Consider a **two level atom** coupled to an **electromagnetic cavity** ($\approx$ harmonic oscillator)



The Hilbert space is $\mathscr{H} = \mathscr{H}_2 \otimes \mathscr{H}_\infty$

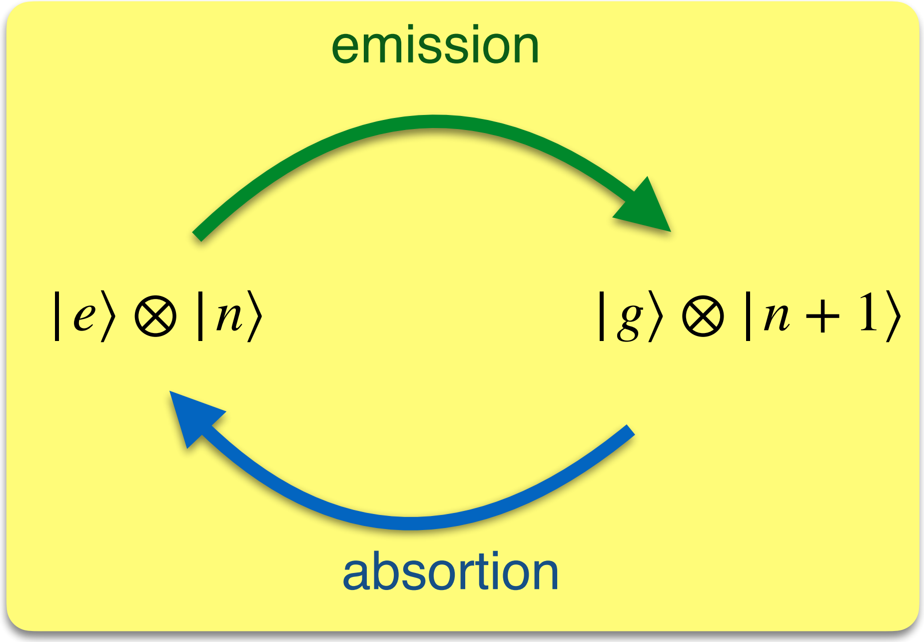
In practice, $\mathscr{H} = \mathscr{H}_2 \otimes \mathscr{H}_d$

It is interesting to work in the resonant regime, $\omega_0 = \omega_c$. Then the atom-cavity interaction is described by the Jaynes-Cummings Hamiltonian

$$H_I = \hbar\lambda\,(\sigma_+ \otimes a + \sigma_- \otimes a^\dagger)$$

that couples pairs of states, producing oscillations between them, characterised by the Rabi frequency

$$\Omega_n = 2\lambda\sqrt{n+1}$$



Quantum non locality in atom - cavity

A pure state is given by its two projections on the atom subsystem

$$|\psi\rangle = |e\rangle \otimes |\varphi_e\rangle + |g\rangle \otimes |\varphi_g\rangle$$

QUESTION:

- Suppose we start from a pure, separable state, with the atom in its excited level:

$$|\psi\rangle = |e\rangle \otimes |\varphi_e\rangle \quad \text{at} \quad t=0$$

- It will evolve in time into a pure, in general entangled, state

$$|\psi(t)\rangle = |e\rangle \otimes |\varphi_e(t)\rangle + |g\rangle \otimes |\varphi_g(t)\rangle$$

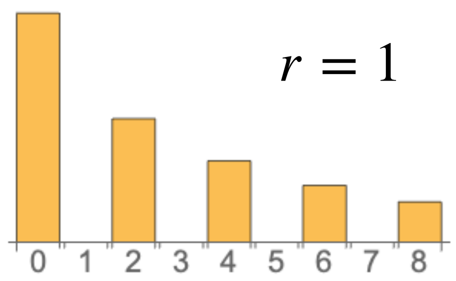
WHAT about its NON-LOCALITY? , ie

WHAT is the $\langle \mathcal{O}_{\text{Bell}} \rangle_{\max}$ value for a given time?

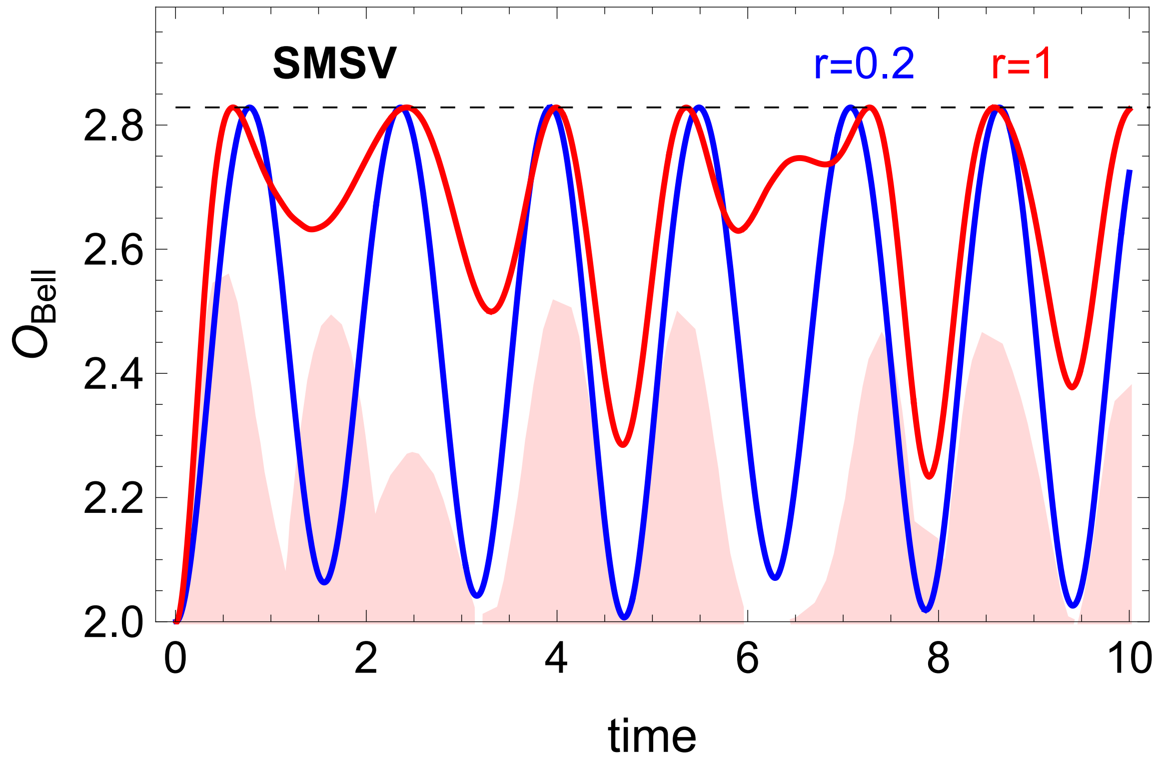
Atom-cavity: Initial Squeeze state

At t=0

$$|\psi\rangle = |e\rangle \otimes |\varphi_{\text{Squeezed}}\rangle \ , \qquad \text{with } |\varphi_{\text{Squeezed}}\rangle = (-1)^{\frac{n}{2}} \frac{\sqrt{n!}}{2^{\frac{n}{2}} \frac{n}{2}!} \frac{\left(e^{i\theta}\tanh(r)\right)^{\frac{n}{2}}}{\sqrt{\cosh(r)}} \ \text{(n even)}$$



Our results improve the ones obtained by *Halder et al* using pseudo spin operators



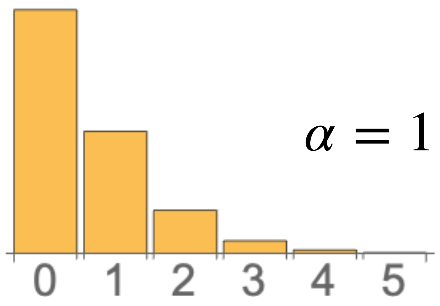
A. Bernal, J.A. Casas and JMM
To appear

P. Halder, R. Banerjee, S.Roy, A. Sen(De)
2023

Atom-cavity: Initial Coherent state

At t=0

$$|\psi\rangle = |e\rangle \otimes |\varphi_{\text{Coherent}}\rangle \ , \qquad \text{with } |\varphi_{\text{Coherent}}\rangle = \sum \frac{1}{\sqrt{n!}} e^{-\frac{|\alpha|^2}{2}} \alpha^n |n\rangle$$



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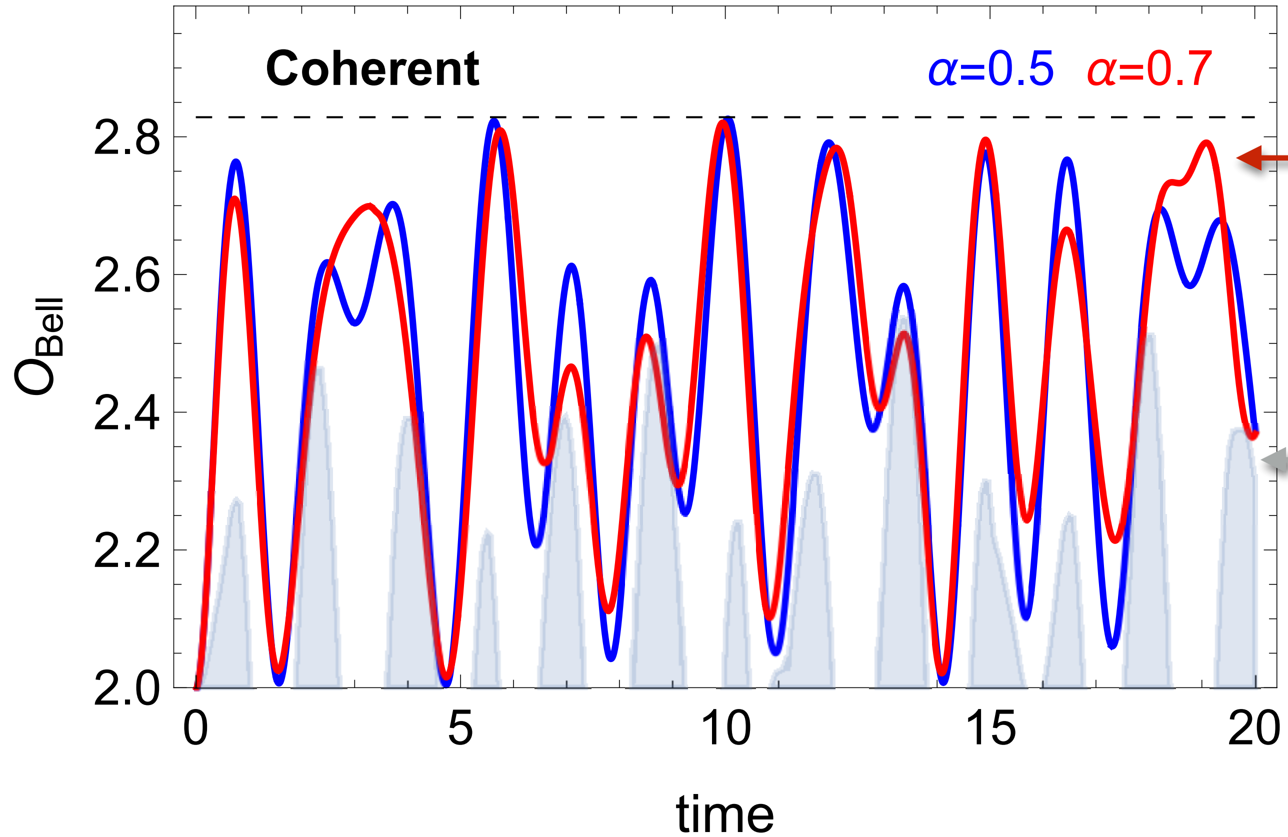

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Atom-cavity: Initial Coherent state with noise

At t=0

$$|\psi\rangle = |e\rangle \otimes |\varphi_{\text{Coherent}}\rangle \ ,\qquad \text{with } |\varphi_{\text{Coherent}}\rangle = \sum \frac{1}{\sqrt{n!}} e^{-\frac{|\alpha|^2}{2}} \alpha^n |n\rangle$$

Assume that there is some gaussian noise in the atom-cavity coupling, λ

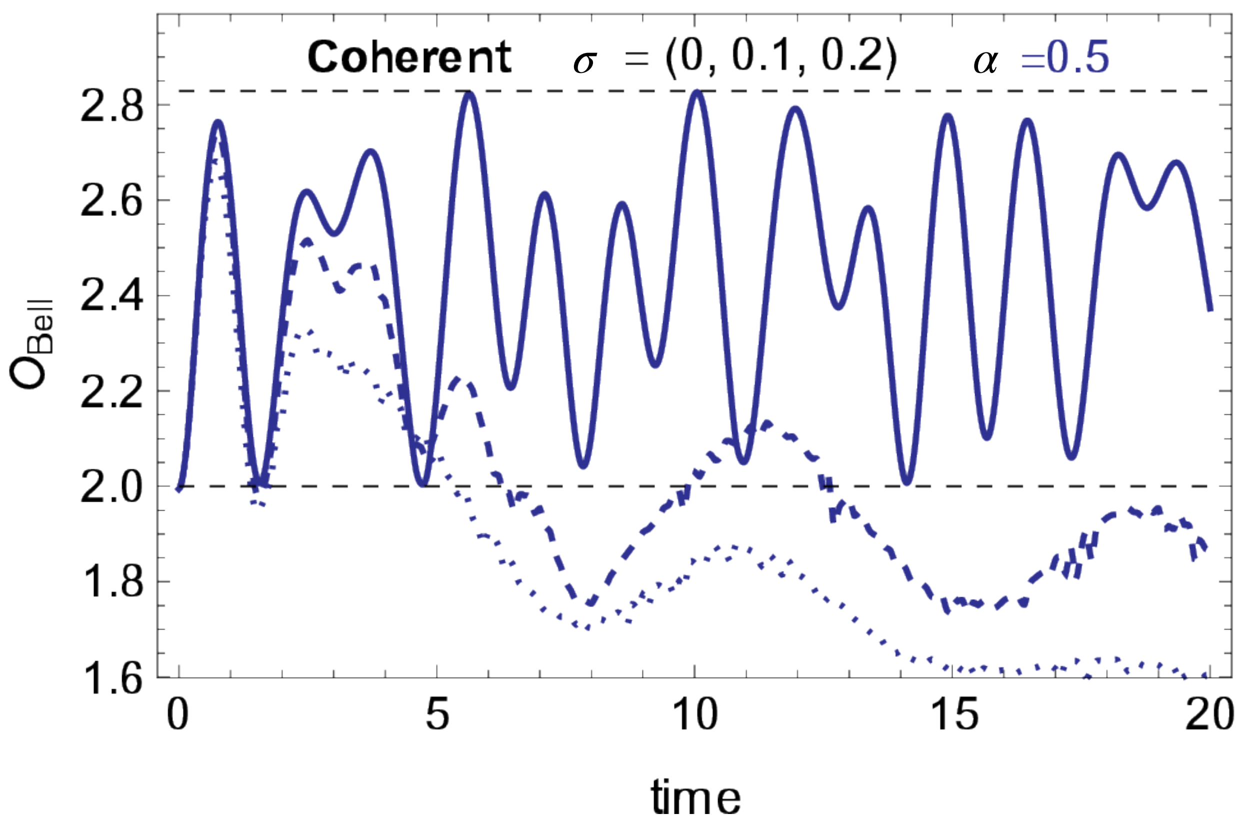
$$H_I = \hbar\lambda\,(\sigma_+ \otimes a + \sigma_- \otimes a^\dagger)$$

$$P(\lambda) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(\lambda-\bar{\lambda})^2}{2\sigma^2}}$$

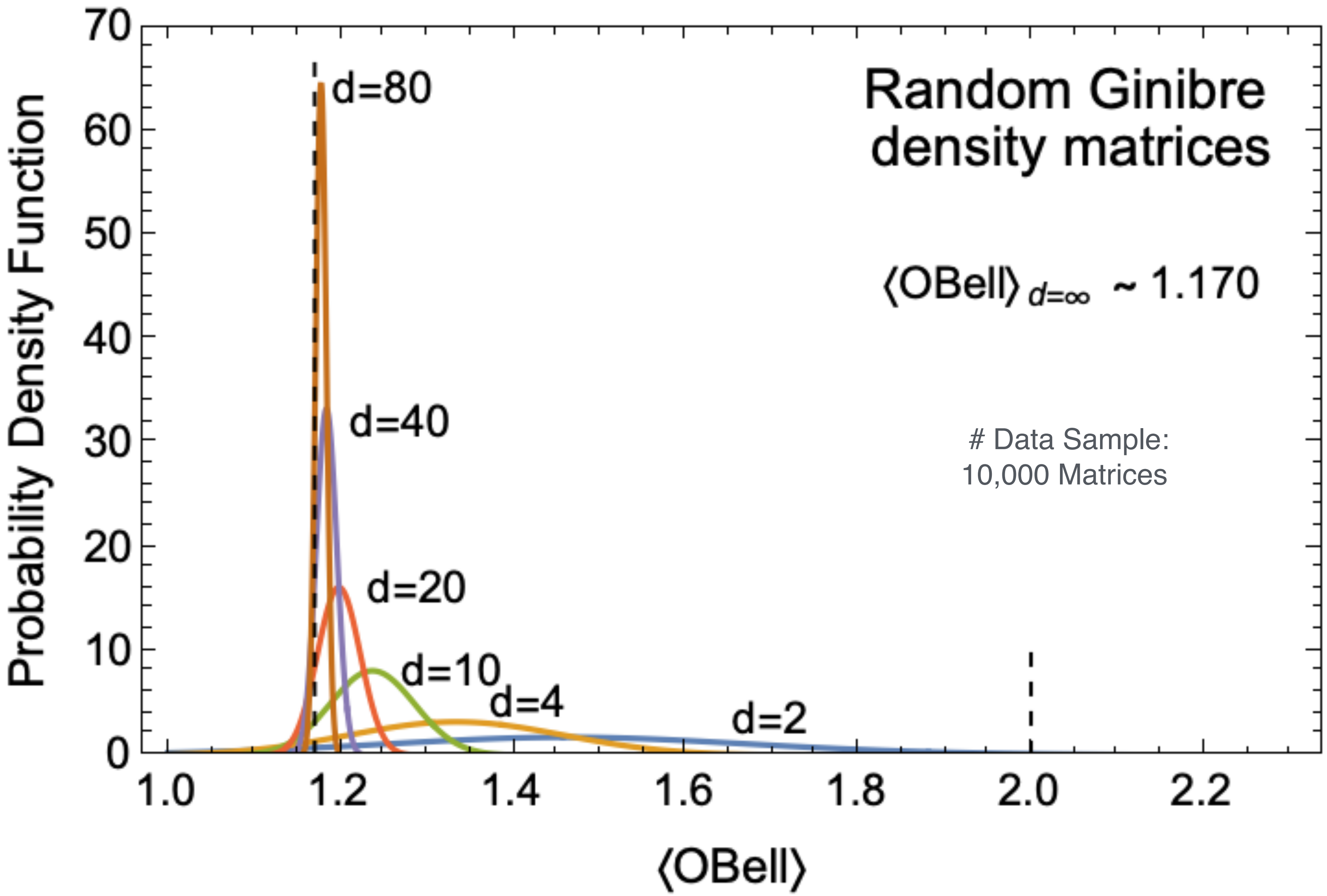
This can be incorporating by dealing with the mixed state:

$$\rho(t) = \int d\lambda\, P(\lambda)\, |\psi(\lambda,t)\rangle\langle\psi(\lambda,t)|,$$

The quantum non locality is, in general, lost after some time



Non locality Random Density Matrices

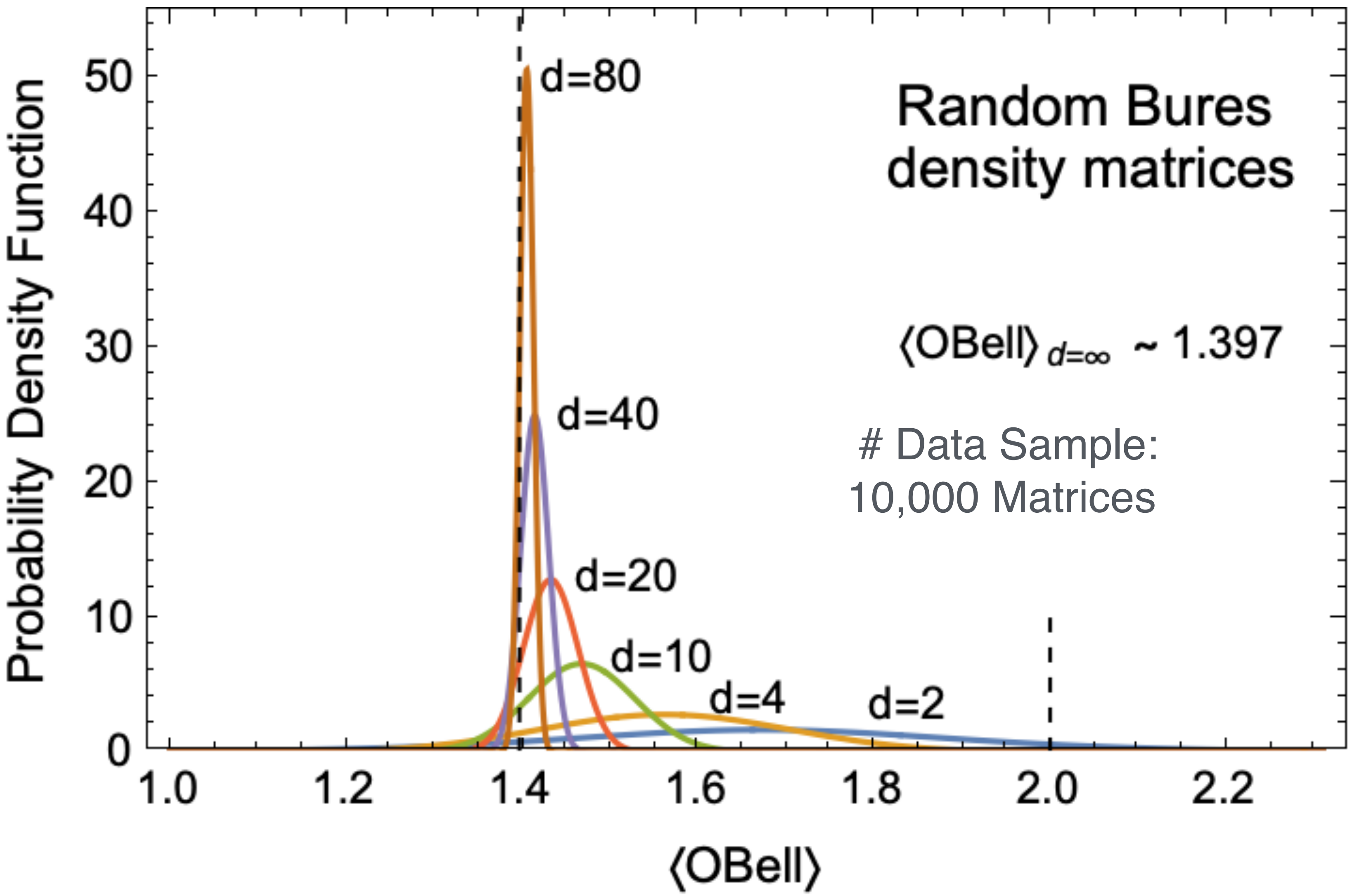


GINIBRE set

G Matrix with random gaussian independent complex entries

$$\rho_{\text{Ginibre}} = \frac{GG^\dagger}{\mathrm{Tr}\left[GG^\dagger\right]}$$

Non locality in Random Density Matrices



BURES set

U Unitary matrix randomly chosen according to the Haar measure

$$\rho_{\text{Bures}} = \frac{(\mathbb{1} + U)GG^\dagger(\mathbb{1} + U^\dagger)}{\text{Tr}\left[(\mathbb{1} + U)GG^\dagger(\mathbb{1} + U^\dagger)\right]}$$

V.A. Osipov et al, 2010

Non locality in tW at the LHC

We can study the tW system, described by a mixed $\mathscr{H}_2 \otimes \mathscr{H}_3$ state

Thanks to JA Aguilar-Saavedra for the data !

Kinem. **Reg I** $\rho = \begin{pmatrix} 0.2372 & 0 & 0 & 0 & 0.1089 & 0 \\ 0 & 0.1667 & 0 & 0.0121 & 0 & 0.1089 \\ 0 & 0 & 0.0961 & 0 & 0.0121 & 0 \\ 0 & 0.0121 & 0 & 0.0961 & 0 & 0 \\ 0.1089 & 0 & 0.0121 & 0 & 0.1667 & 0 \\ 0 & 0.1089 & 0 & 0 & 0 & 0.2372 \end{pmatrix}$ $\langle \mathcal{O}_{\text{Bell}} \rangle_{\text{max}} = 0.89$

PRELIMINARY

Kinem. **Reg II** $\rho = \begin{pmatrix} 0.2566 & 0 & 0 & 0 & 0.3538 & 0 \\ 0 & 0.1074 & 0 & 0.0303 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0303 & 0 & 0.0467 & 0 & 0 \\ 0.3538 & 0 & 0 & 0 & 0.5893 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $\langle \mathcal{O}_{\text{Bell}} \rangle_{\text{max}} = 2.07$

Balance between cuts / statistics

Summary

- The study of **non-locality in qubit-qudit systems** has been addressed
- We have obtained a **closed, compact expression** for $\langle \mathcal{O}_{\rm Bell} \rangle_{\rm max}$
- **Easy to implement** numerically. Allows to address the Large d limit
- **Many applications** (Hybrid - discrete / continuous - models, Random matrices sets, Non locality in tW)

THANKS