The Bell-Clauser-Horne-Shimony-Holt inequality in (for now still free) Quantum Field Theory: numerical and formal study via bumpified Haar wavelets

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Based on

- D. Dudal, P. De Fabritiis, M. S. Guimaraes, I. Roditi and S. P. Sorella, Phys. Rev. D **108**, L081701 (2023).

- D. Dudal and K. Vandermeersch, work in progress.

- Useful lecture notes:

  M. S. Guimaraes, I. Roditi and S. P. Sorella, *Introduction to Bell's inequality in Quantum Mechanics*, [arXiv:2409.07597 [quant-ph]].

Overview

Motivation for this work

Maximal violation of the Bell-CHSH correlation function via bumpified Haar wavelets: numerically

Maximal violation of the Bell-CHSH correlation function via bumpified Haar wavelets: formally

Outlook

Bell: Classically

The famous Bell inequality is well known in quantum mechanics. Let us first consider a classical counterpart.

Indeed, consider Alice and Bob, measuring in their “far apart” lab’s (read: space-like separation). Both measure two binary (dichotomic) quantities, let’s say (A, A') and (B, B'). They carry out the experiment n times. Each measurement, viz. a_j, a'_j, b_j, b'_j, gives as value ± 1.

Then, for the classical average, one has

$$\left\langle (A+A')B + (A-A')B' \right\rangle = \frac{1}{n}\sum_i \left((a_i + a'_i)b_i + (a_i - a'_i)b'_i\right) \leq 2$$

Bell: Quantum Mechanically

Quantum mechanically, this gets replaced by the Bell-Clauser-Horne-Shimony-Holt[1] inequality,

$$\langle\psi|\,\mathcal{C}\,|\psi\rangle = \langle\psi|\,(A+A')B+(A-A')B'\,|\psi\rangle \ ,$$

where $|\psi\rangle$ is the quantum state of the system and we now consider four bounded (dichotomic) Hermitian operators with

$$A^2 = A'^2 = 1\,, \quad B^2 = B'^2 = 1\,, \quad [A\vee A', B\vee B'] = 0\,, \quad [A,A'] \neq 0 \neq [B,B']$$

One speaks of a violation of the Bell-CHSH inequality whenever

$$|\,\langle\psi|\,\mathcal{C}\,|\psi\rangle\,| > 2$$

whilst there is a *maximal violation* (Tsirelson's bound) [2]

$$|\,\langle\psi|\,\mathcal{C}\,|\psi\rangle\,| \leq 2\sqrt{2}\,.$$

[1] Bell, Physics Physique Fizika **1**, 195 (1964); Clauser, Horne, Shimony, Holt, Phys. Rev. Lett. **23**, 880 (1969).

[2] Tsirelson, Lett. Math. Phys. 93 (1980).

Bell: Quantum Mechanically

Using *entangled* states, numerous examples of the violation have been found.

Needless to mention are the experimental studies of Aspect, Clauser and Zeilinger of the Bell inequalities in quantum systems.

During this workshop, another type of experimental evidence will be discussed, now coming from the world of high energy physics, see other talks.

Speaking about high energy physics, the appropriate language is quantum field theory.

Bell: Quantum Field Theory

Pioneering work of Summers & Werner[3] for free (non-interacting) QFTs: maximal violation is reached, even in the vacuum state!

Their result is rooted in Algebraic Quantum Field Theory, heavily relying on the language of C^*-operator (von Neumann) algebras and analysis (Tomita-Takesaki modular theory). I will not go into this, but I can give a hint where it comes in.

[3] Summers, Werner, J. Math. Phys. **28**, 2440 (1987); 2448 (1987).

Bell: Quantum Field Theory

One starts from a free spinor field in $(1+1)$-dimensions, with action

$$S = \int d^2x \left[\bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi\right].$$

The plane wave basis gives

$$\psi(t,x) = \int \frac{dk}{2\pi}\frac{m}{\omega_k}\left[u(k)c_k\, e^{-ik_\mu x^\mu} + v(k)d_k^\dagger\, e^{+ik_\mu x^\mu}\right],$$

where $k_\mu x^\mu = \omega_k t - kx$ and $\omega_k = \sqrt{k^2+m^2}$.

For the creation and annihilation operators' algebra, we get

$$\{c_k, c_q^\dagger\} = \{d_k, d_q^\dagger\} = 2\pi\frac{\omega_k}{m}\delta(k-q).$$

Bell: Quantum Field Theory

One must smear the quantum fields to get well-defined operator-valued distributions[4], with two-component spinor test functions $h_\alpha(x) = (h_1(x), h_2(x))^t$, where h_1, h_2 are commuting test functions belonging to the space $C_0^\infty(\mathbb{R}^2)$ of infinitely differentiable functions with compact support (aka. bump functions). Then,

$$\psi(h) = \int d^2x\, \bar{h}^\alpha(x)\psi_\alpha(x); \qquad \psi^\dagger(h) = \int d^2x\, \bar{\psi}^\alpha(x) h_\alpha(x)$$

so that

$$\psi(h) = c_h + d_h^\dagger, \quad \psi^\dagger(h) = c_h^\dagger + d_h,$$

and

$$c_h = \int \frac{dk}{2\pi}\frac{m}{\omega_k}\bar{h}(k)u(k)c_k; \; d_h = \int \frac{dk}{2\pi}\frac{m}{\omega_k}\bar{v}(k)h(-k)d_k,$$

next to

$$\{c_h, c_{h'}^\dagger\} = \int \frac{dk}{2\pi}\frac{1}{2\omega_k}\bar{h}(k)\,(\not{k} + m)\,h'(k),$$
$$\{d_h, d_{h'}^\dagger\} = \int \frac{dk}{2\pi}\frac{1}{2\omega_k}\bar{h}'(-k)\,(\not{k} - m)\,h(-k),$$

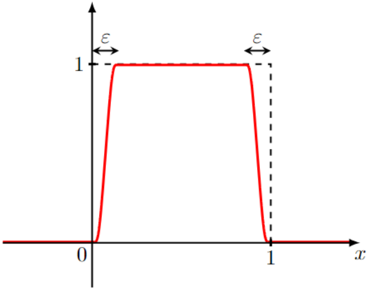
[4] Haag, *Local quantum physics: Fields, particles, algebras*, Springer-Verlag, 1992.

Example of a 1d bump function

Consider the Planck taper window[5],

$$\sigma_0(x,\varepsilon) = \begin{cases} \left[1+\exp\left(\frac{\varepsilon(2x-\varepsilon)}{x(x-\varepsilon)}\right)\right]^{-1}, \text{ if } x \in (0,\varepsilon), \\ +1, \text{ if } x \in [\varepsilon, 1-\varepsilon], \\ \left[1+\exp\left(\frac{\varepsilon(-2x-\varepsilon+2)}{(x-1)(x+\varepsilon-1)}\right)\right]^{-1}, \text{ if } x \in (1-\varepsilon, 1), \\ 0, \text{ otherwise.} \end{cases}$$

which defines a smoothened rectangle.



[5] McKechan et al, Class. Quant. Grav. **27** (2010), 084020.

Bell: Quantum Field Theory

The dichotomic Bell operators are eventually given by

$$\mathcal{A}_h = \psi(h) + \psi^\dagger(h)$$

with

$$\langle 0|\mathcal{A}_h \mathcal{A}_h'|0\rangle = \langle h|h'\rangle$$

. To be more precise, the vev $\mathcal{A}_h \mathcal{A}_{h'}$ corresponds to a inner product,

$$\langle h|h'\rangle = \int \frac{dk}{2\pi} \frac{1}{2\omega_k} \left[\bar{h}(k)\,(\not{k}+m)\,h'(k) + \bar{h}'(-k)\,(\not{k}-m)\,h(-k)\right],$$

For the Bell-CHSH correlator in the vacuum, one finally gets

$$\begin{aligned} \langle \mathcal{C} \rangle &= \langle 0|\,(\mathcal{A}_f + \mathcal{A}_{f'})\,\mathcal{A}_g + (\mathcal{A}_f - \mathcal{A}_{f'})\,\mathcal{A}_{g'}\,|0\rangle, \\ &= \langle f|g\rangle + \langle f|g'\rangle + \langle f'|g\rangle - \langle f'|g'\rangle. \end{aligned}$$

The problem of finding a violation is thus reduced to finding proper test functions. In general, by a magical mapping of the free QFT case to a QM case (and much more than that of course), Summers & Werner exactly showed this, confirming the asymptotic reaching of the Tsirelson bound. The explicit form of these test functions is however unknown, to the best of our knowledge.

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Outlook

Our work

We shall numerically construct proper test functions. The procedure can then (hopefully) be generalized in the future to the interacting QFT case, for which the Summers & Werner magic is unknown.

In practice, we explicitly implement relativistic causality by considering the hypersurface $t = 0$ (read: a fine-smeared version of $\delta(t)$.) and adopt the supports of Alice-(f, f') to $x < 0$, and Bob-(g, g') to $x > 0$.

For the litmus test, we will take $m \to 0$, for which in k-space

$$\langle f|g\rangle = \int \frac{dk}{2\pi} \left[(1 + \text{sgn}(k))\, f_1^*(k) g_1(k) + (1 - \text{sgn}(k))\, f_2^*(k) g_2(k) \right],$$

or, in x-space, $\langle f|g\rangle = I_1 + I_2$, where

$$I_1 = \int dx \, [f_1^*(x) g_1(x) + f_2^*(x) g_2(x)] \,,$$

$$I_2 = -\frac{i}{\pi} \int dxdy \left(\frac{1}{x-y} \right) [f_1^*(x) g_1(y) - f_2^*(x) g_2(y)] \,.$$

Step 1: a Haar wavelet solution

Consider the Daubechies db2 wavelets[6], aka. Haar wavelets[7],

$$\psi_{n,k}(x) = 2^{n/2}\psi(2^n x - k)$$

descending from its mother wavelet ψ

$$\psi(x) = \begin{cases} +1, \text{ if } x \in \left[0, \frac{1}{2}\right), \\ -1, \text{ if } x \in \left[\frac{1}{2}, 1\right), \\ 0, \text{ otherwise.} \end{cases}$$

These wavelets functions provide an orthonormal basis, $\int dx\, \psi_{n,k}(x)\psi_{m,\ell}(x) = \delta_{nm}\delta_{k\ell}$, for the square-integrable functions on the real line and, moreover, have a compact support whose maximum size can be controlled. More precisely, $\psi_{n,k}$ has support $I_{n,k} = \left[k2^{-n}, (k+1)2^{-n}\right)$ and is piecewise constant, giving $+2^{\frac{n}{2}}$ on the first half of $I_{n,k}$ and $-2^{\frac{n}{2}}$ on the second half. We use these to expand the would-be test function entering the Bell-CHSH inequality

$$\widetilde{f}_j(x) = \sum_{n=n_i}^{n_f} \sum_{k=k_i}^{k_f} f_j(n,k)\,\psi_{n,k}(x),$$

The $\{n_i, n_f, k_i, k_f\}$ set the range and resolution of the Haar wavelet expansion.

[6] Daubechies, Commun. Pure Appl. Math. **41**, 909 (1988); *Ten Lectures on Wavelets*, (SIAM, Philadelphia, 1992).

[7] Lepik, Hein, *Haar Wavelets: With Applications* (Springer International publishing, Switzerland, 2014).

Step 1: a Haar wavelet solution

What do we gain using the Haar wavelets?

- The integrals entering the inner products/norms can be executed in (lengthy but) closed form, of great assistance for the sought for numerical precision ($\rightarrow$ it is hard to get close to $2\sqrt{2}$!)
- Inspired by Summers & Werner, we shall impose

$$\left\langle \tilde{f} | \tilde{f} \right\rangle = \left\langle \tilde{f}' | \tilde{f}' \right\rangle = \langle \tilde{g} | \tilde{g} \rangle = \langle \tilde{g}' | \tilde{g}' \rangle = 1 \,,$$
$$\left\langle \tilde{f} | \tilde{g} \right\rangle = \left\langle \tilde{f}' | \tilde{g} \right\rangle = \left\langle \tilde{f} | \tilde{g}' \right\rangle = - \left\langle \tilde{f}' | \tilde{g}' \right\rangle = -i\sqrt{2}\frac{\lambda}{1+\lambda^2}$$

with $\lambda \in (\sqrt{2}-1, 1] \Rightarrow |\langle \mathcal{C} \rangle| = \frac{4\sqrt{2}\lambda}{1+\lambda^2} \in (2, 2\sqrt{2}]$

We can gradually increase the global support and local resolution of the chosen wavelet basis to (numerically) find the appropriate wavelet coefficients. We rely on a minimization procedure, given the quadratic nature of the constraints.

Step 2: Bumpification of the Haar wavelet solution

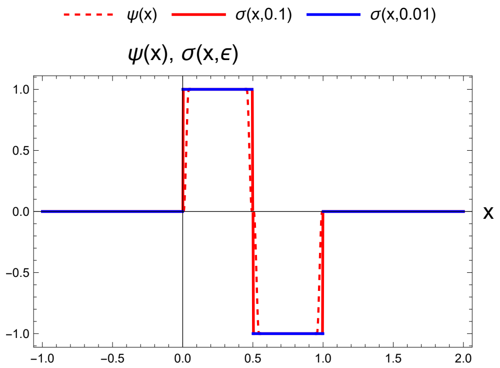


Figure: The mother Haar wavelet and two of its bumpifications. These would-be test functions are not smooth, due to the jumps in the Haar wavelets. Nevertheless, there is a class of smooth bump functions (C^∞ with compact support), which can be used to approximate the Haar wavelets as precisely as we want.

Step 2: Bumpification of the Haar wavelet solution

Consider again the already shown basic Planck-taper window function with support on the interval $[0,1]$. We then introduce the mother bump function with support on $[0,1]$ by

$$\sigma(x,\varepsilon) = \begin{cases} +\sigma_0(2x,\varepsilon), \text{ if } x \in \left(0,\frac{1}{2}\right), \\ -\sigma_0(2x-1,\varepsilon), \text{ if } x \in \left(\frac{1}{2},1\right), \\ 0, \text{ otherwise.} \end{cases}$$

to define $C_0^\infty(\mathbb{R})$ version of the Haar wavelet,

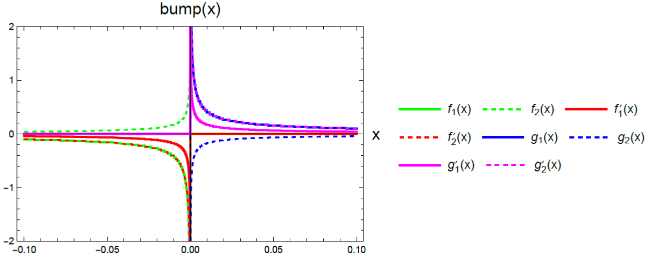
$$\sigma_{n,k}(x,\varepsilon) = 2^{n/2}\sigma(2^n x - k,\varepsilon)$$

with support on the interval $I_{n,k}$ that approximates as precisely as we want $\psi_{n,k}(x)$ per choice of ε. As such, each wavelet solution can be replaced by a bumpified version,

$$f_j(x) = \sum_{n=n_i}^{n_f}\sum_{k=k_i}^{k_f} f_j(n,k)\,\sigma_{n,k}(x,\varepsilon),$$

up to arbitrary precision if ε is chosen small enough.

Step 2: Bumpification of the Haar wavelet solution



We minimize $\mathcal{R} = |\langle\tilde{f}|\tilde{f}\rangle - 1|^2 + |\langle\tilde{f}'|\tilde{f}'\rangle - 1|^2 + |\langle\tilde{g}|\tilde{g}\rangle - 1|^2 + |\langle\tilde{g}'|\tilde{g}'\rangle - 1|^2 + |\langle\tilde{f}|\tilde{g}\rangle + i\sqrt{2}\frac{\lambda}{1+\lambda^2}|^2 + |\langle\tilde{f}|\tilde{g}'\rangle + i\sqrt{2}\frac{\lambda}{1+\lambda^2}|^2 + |\langle\tilde{f}'|\tilde{g}\rangle + i\sqrt{2}\frac{\lambda}{1+\lambda^2}|^2 + |\langle\tilde{f}|\tilde{g}'\rangle - i\sqrt{2}\frac{\lambda}{1+\lambda^2}|^2$. Targeting $\langle\mathcal{C}\rangle \approx 2.82$ for $\lambda = 0.99$ and willing to achieve precision at the percent level, corresponding to $\mathcal{R} = \mathcal{O}(10^{-5})$, we are able to solve the constraints for $\{n_i = -10; n_f = 120; k_i = -5; k_f = -1\}$ for (f, f') and $\{m_i = -10; m_f = 120; \ell_i = 0; \ell_f = 4\}$ for (g, g').

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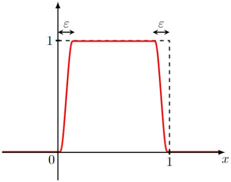
Outlook

The “easy” step: formalisation of the bumpification



As visually expected, it can indeed be rigourously proven that for any $p \in [1,\infty[$, the difference between the window function s^ε and the basic rectangle r is $\forall \varepsilon > 0$ subject to

$$\|r - s^\varepsilon\|_p^p \leq \varepsilon,$$

At the wavelet level, this implies, after some manipulations,

$$\begin{aligned} \|\psi - \sigma^\varepsilon\|_p^p &\leq \varepsilon \\ \|\psi_{n,k} - \sigma^\varepsilon_{n,k}\|_p^p &\leq 2^{-n/2}\varepsilon \end{aligned}$$

This is sufficient to show that a wavelet solution with violation arbitrarily close to $2\sqrt{2}$, can be smoothened into a proper bump solution arbitrary close to $2\sqrt{2}$.

The "hard" step: does a wavelet solution exist?

We must prove the existence of a solution to

$$\langle \widetilde{f} \mid \widetilde{f} \rangle = \langle \widetilde{f}' \mid \widetilde{f}' \rangle = \langle \widetilde{g} \mid \widetilde{g} \rangle = \langle \widetilde{g}' \mid \widetilde{g}' \rangle = 1$$
$$\langle \widetilde{f} \mid \widetilde{g} \rangle = \langle \widetilde{f}' \mid \widetilde{g} \rangle = \langle \widetilde{f} \mid \widetilde{g}' \rangle = -\langle \widetilde{f}' \mid \widetilde{g}' \rangle = -i\frac{\sqrt{2}\lambda}{1+\lambda^2}.$$

where

$$\widetilde{f}_j := \sum_{n=N_0}^{N_1} \sum_{k=-K}^{-1} f_j(n,k)\,\psi_{n,k}, \qquad \widetilde{g}_j := \sum_{n=N_0}^{N_1} \sum_{k=0}^{K-1} g_j(n,k)\,\psi_{n,k},$$
$$\widetilde{f}'_j := \sum_{n=N_0}^{N_1} \sum_{k=-K}^{-1} f'_j(n,k)\,\psi_{n,k}, \qquad \widetilde{g}'_j := \sum_{n=N_0}^{N_1} \sum_{k=0}^{K-1} g'_j(n,k)\,\psi_{n,k},$$

The 3 parameters $N_0 < N_1$ and $K > 1$ set the resolution. Causality is implemented by only using Haar wavelets supported on $[-2^{-N_0}K, -2^{-N_1})$ to represent Bob's test functions (f, f') and only using Haar wavelets supported on $[0, 2^{-N_0}K)$ to represent Alice's test functions (g, g').

Conjecture

For a given $\lambda \in (\sqrt{2}-1, 1)$ *arbitrarily close* to 1, we can find a resolution *sufficiently fine* such that this system of equations has a solution for the wavelet coefficients.

The “hard” step: does a wavelet solution exist?

A posteriori, we noticed some (anti-)symmetry and/or rescaling properties between the various wavelet solutions. We can now build these in a priori. Then the problem can be mapped onto

$$y^T A y = \sum_{(n,k)} \sum_{(m,l)} A_{(n,k),(m,l)}\, y_{n,k}\, y_{m,l} = \frac{2\pi\lambda}{1+\lambda^2} \leq \pi,$$

such that $\|y\|^2 = 1$ with the matrix A defined in terms of the Haar wavelets via

$$A_{(n,-k),(m,l+1)} \equiv -\iint \left(\frac{1}{x+v}\right) \psi_{n,k}(x)\psi_{m,-l-1}(v)\,\mathrm{d}x\mathrm{d}v$$

Due to some index-shift symmetries, A depends only on the difference $N := N_1 - N_0$. So we can focus on $A(N_0 = 0, N_1 = N, K)$. Then, the problem can be rephrased further in terms of the minimal/maximal eigenvalues of A, namely

$$\lambda_{\min} \leq \frac{2\pi\lambda}{1+\lambda^2} \leq \lambda_{\max}.$$

We can show that $\lambda_{\min}$ is well under control and certainly small enough. We thus need to push $\lambda_{\max} \to \pi$.

The “hard” step: does a wavelet solution exist?

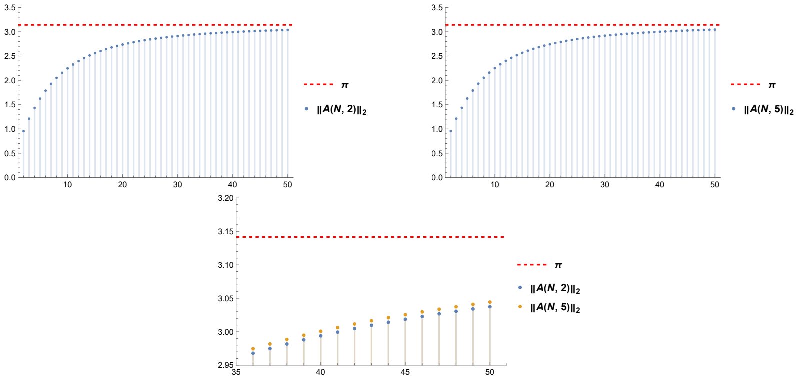


Figure: Numerical evidence in terms of $\lambda_{\max}(A(N,2))$, $\lambda_{\max}(A(N,5))$.

The special case $K = 1$: closed case

I am glad to spare you all underlying details and series manipulations trickery, but suffice to say that for $K = 1$, N arbitrarily large, we are able to show that

$$\begin{aligned}\lambda_{\min} = \lambda_{\max} &= a_0 + 2\sum_{n=1}^{\infty} a_n \\ &= a_0 + 2\left(\sqrt{2} - 1 - (2+\sqrt{2})\ln 2 + 3\ln 3 + \sum_{n=1}^{\infty} \iota_n\right) \approx 3.10\end{aligned}$$

where

$$\begin{aligned}a_n &= 2^{-n/2}\Big(\left(2^{n+1} - 2^n\right)\ln(2) - 2\left(2^{n-1}+1\right)\ln\left(2^{n-1}+1\right) + 3\left(2^n+1\right)\ln\left(2^n+1\right) \\ &\quad - \left(2^{n+1}+1\right)\ln\left(2^{n+1}+1\right)\Big) \\ \iota_n &= \int_0^1 \frac{2^{-n/2}(1-x)}{x+2^n}\,\mathrm{d}x = 2^{-\frac{n}{2}}\left(-\left(\left(2^n+1\right)n\ln 2\right) + \left(2^n+1\right)\ln\left(2^n+1\right) - 1\right)\end{aligned}$$

This corresponds to $|\langle \mathcal{C} \rangle| \approx 2.80$. For the record, $2\sqrt{2} \approx 2.83$.

The special case $K = 1$: closed case

We relied on a corollary of the

fundamental eigenvalue distribution theorem of Szegö[8]

Let $(a_n)_{n\geq 0}$ be an absolutely summable sequence in $\mathbb{C}$ and define an associated sequence of Hermitian Toeplitz (= band) matrices

$$A_n = \begin{bmatrix} a_0 & a_1^* & a_2^* & \cdots & a_n^* \\ a_1 & a_0 & a_1^* & \cdots & a_{n-1}^* \\ a_2 & a_1 & a_0 & \cdots & a_{n-2}^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_0 \end{bmatrix}.$$

The Fourier series $f(t) = a_0 + \sum_{n=1}^{\infty} (a_n + a_n^*) \cos(nt), \qquad t \in [0, 2\pi]$, is real valued and we have $\lim_{n\to\infty} \lambda_{\max}(A_n) = \max f$, $\lim_{n\to\infty} \lambda_{\min}(A_n) = \min f$ where $\lambda_{\max}(A_n)$ increases with n and $\lambda_{\min}(A_n)$ decreases with n.

[8] See f.i. Gray, *Toeplitz and Circulant Matrices: A Review. Foundations and Trends in Communications and Information Theory*, Now Publishers, 2006.

The general case $K > 1$: unfinished business

To get closer to π (or to $2\sqrt{2}$ for the violation), we must increase K[9]. Unfortunately, the matrix A is then no longer Toeplitz, but block Toeplitz.

Despite some nice properties of this A and the known generalization of the Szegö distribution theorem to the block Toeplitz case, it seems we cannot explicitly find the necessary min/max eigenvalue of the corresponding (now matrix-valued) $F(t)$.

This being said, we numerically see it still works. So we have an analytical proof for up to 99% of the Tsirelson bound, the remaining 1% remains conjecture based on numerical evidence:).

[9] Roughly speaking, the "size" of the wavelets.

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Bumpified Haar wavelets. . .

. . . offer an excellent numerical (and analytical) tool to *explicitly* study the Bell-CHSH inequality

$$2 < |\langle \mathcal{C} \rangle| \leq 2\sqrt{2}$$

in free quantum field theory.

We are now armed to enter the terra incognita where the inequality with an interacting QFT can be studied.

Interesting side note: strictly (read: mathematically) speaking, in Algebraic QFT, interactions are not allowed. Roughly speaking, modulo the AQFT axioma's, there is no unitary equivalence between the free and interacting version of a given QFT [$\rightarrow$ Haag's theorem[10].]

Luckily, we are pragmatic physicists, so we ignore this. After all, QFT works and compares quite well with the real ($\neq$ mathematical) experimental world:).

[10] See f.i. Haag, *Local quantum physics: Fields, particles, algebras*, Springer-Verlag, 1992.

Outlook

As a toy model for a $d = 2$ interacting model, we will focus on the Thirring model[11]

$$S = \int d^2x \left(\bar{\psi}(i\partial\!\!\!/)\psi - \frac{g}{2}(\bar{\psi}\gamma_\mu\psi)^2 \right)$$

Quite interesting model, since its Green functions are exactly known![12]

As a consequence, its (positive) Källén-Lehmann spectral function $\rho(\mu)$ is also exactly known. This $\rho(\mu)$ will enter the inner product between (& norm of) the test functions.

A small sacrifice: no more dichotomic operators ($A(f)^2 = 1$) since this is based on the (non-interacting) (anti-)commutation relations, but we can still prove that $\langle A(f)A(f)\rangle = ||f||^2 < \infty$. But these are the relevant quantities entering the $|\langle \mathcal{C}\rangle|$ after all!

[11] Thirring, Annals Phys. **3**, 91 (1958).

[12] Bozkaya et al, J. Phys. A **39**, 11075 (2006) and refs. therein.

Outlook

A few pertinent questions

- Is the maximal violation still attainable when interactions are included? Does the violation, for fixed test functions, change with the coupling constant? Can it run according to the renormalization group scale/energy?
- Of course, in the long run, we are interested in e.g. QED or even QCD, in $d = 4$. Unfortunately, no longer exactly solvable theories, but can we develop a perturbation theory around the free test functions? What happens with the violation when perturbative corrections are added? Any non-perturbative QCD corrections? Interplay with confinement?
- ...

The End.

Thanks!