Bell inequality for gluons

(but also a nice Bell operator for many qutrit states that appear in colliders)

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Quantum tests in colliders workshop, Merton College, Oxford
03.10.2024

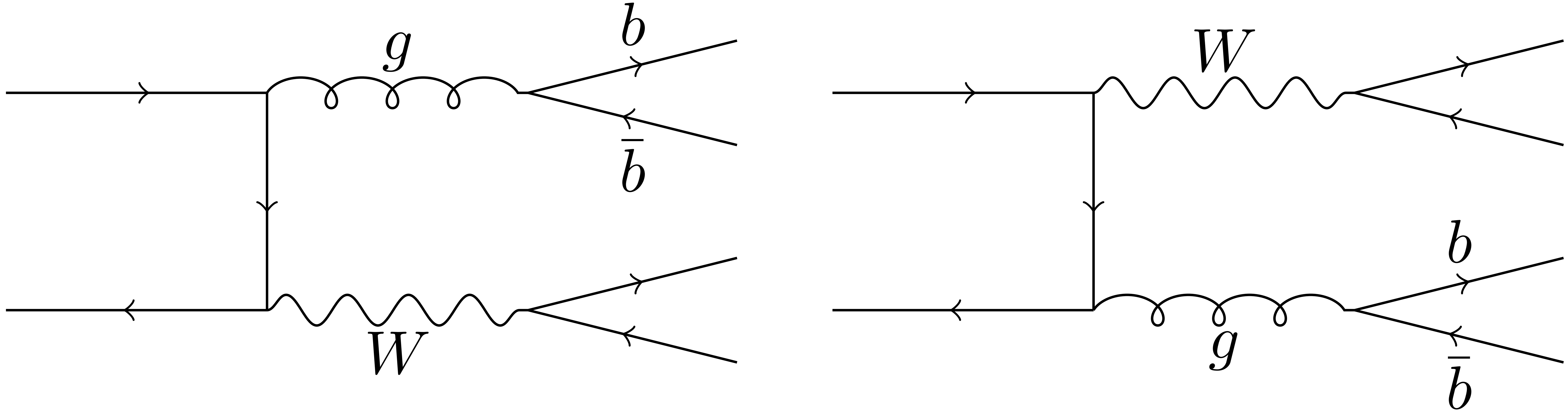
Based on RG, 2410.XXXX

It was all a dream...

I work primarily on vector boson + jets processes.

In some cases, I noticed that there is a “clean” intermediate gluon state there

Example: $pp \to W + (g \to b\bar{b})$ (at LO)



I am dreaming of doing some “quantum thing” with the spins involved here

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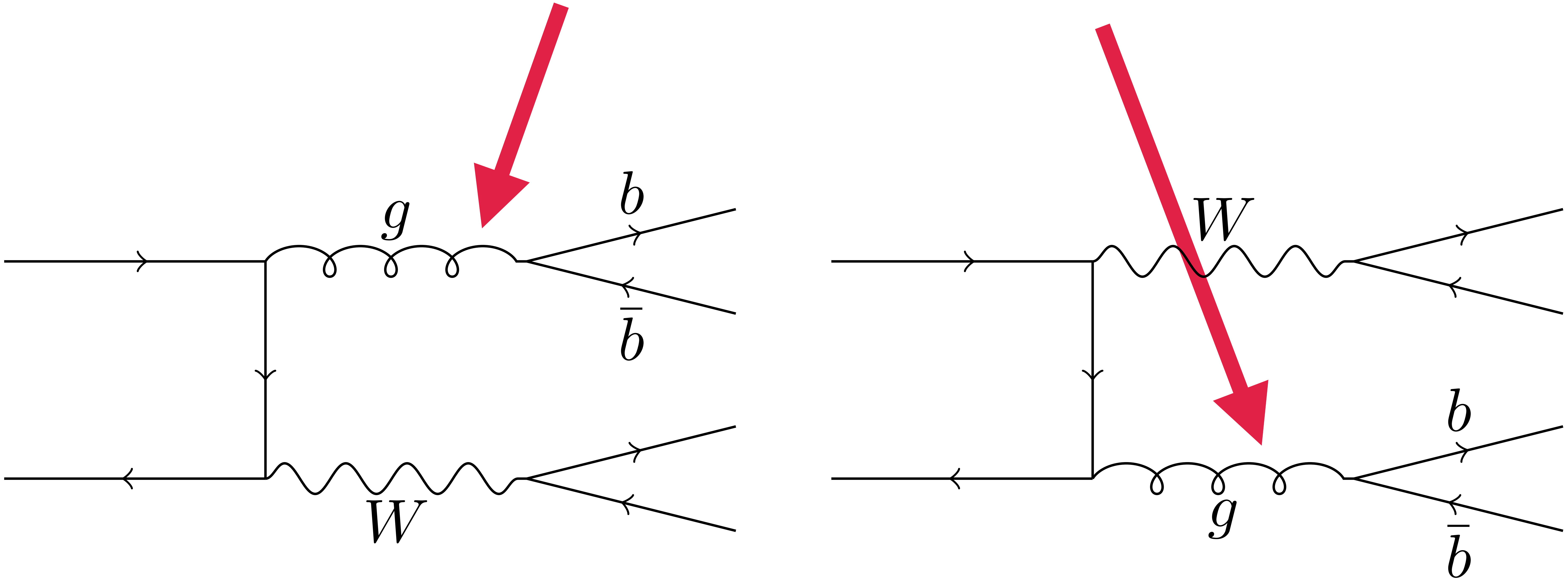
The gluon here must be off-shell, and has transverse and longitudinal polarizations



$\Rightarrow$ This is a system of two qutrits!

It was all a dream...

Can we use CGLMP?

$$\mathscr{B}^{xy}_{CGLMP} = -\frac{2}{\sqrt{3}}\left(\hat{S}_x \otimes \hat{S}_x + \hat{S}_y \otimes \hat{S}_y\right) + \lambda_4 \otimes \lambda_4 + \lambda_5 \otimes \lambda_5$$

(λ_4, λ_5 = Gell Mann matrices)

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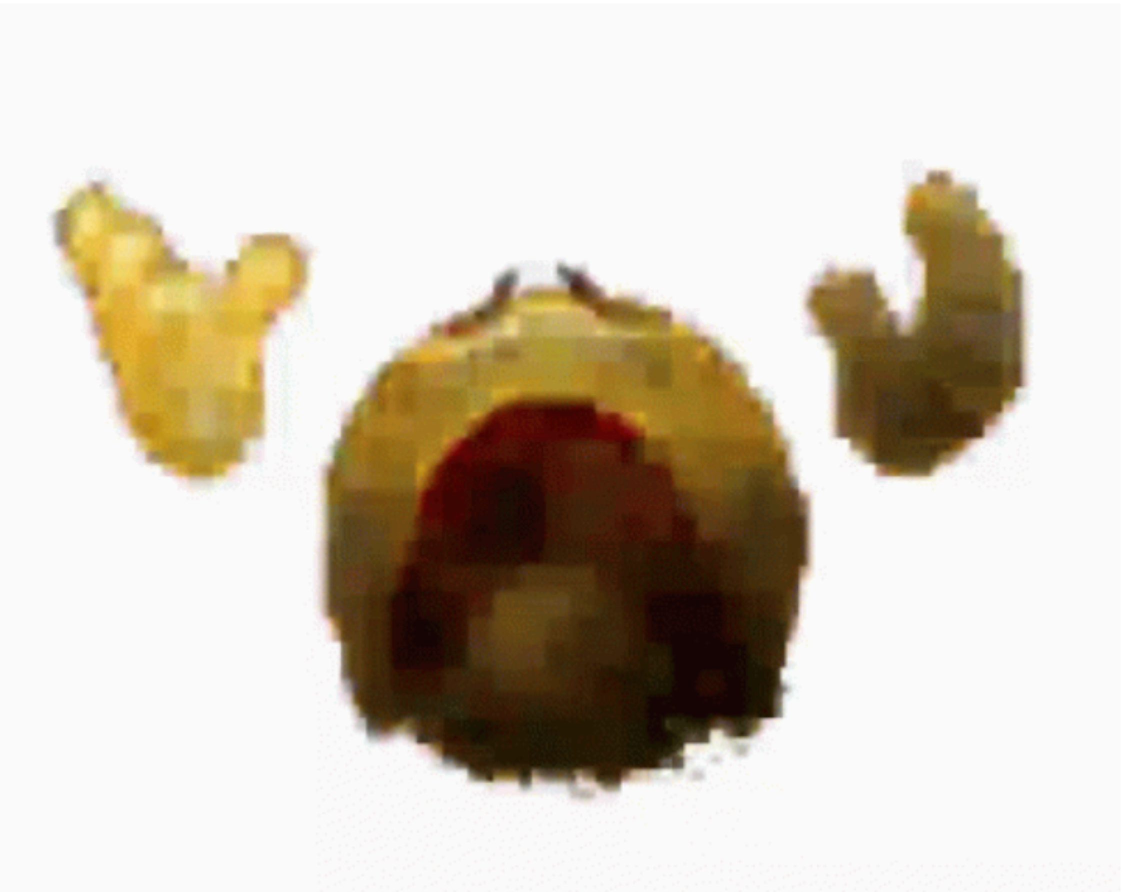
NO! I can't calculate $\langle \hat{S}_i \rangle$ of the gluon from the distribution of $b\bar{b}$!



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But I *could* calculate this...

$$\lambda_4 = \hat{S}_x^2 - \hat{S}_y^2$$

$$\lambda_5 = \{\hat{S}_x, \hat{S}_y\}$$

What *can* we do?

It turns out, that the gluon splitting will give us access to the parity invariant component of the density matrix

$$\rho \supset \sum_{i,j=1}^{3} b_{ij}\hat{S}_{\{ij\}} \otimes \hat{1}_3 + \sum_{i,j=1}^{3} b'_{ij}\hat{1}_3 \otimes \hat{S}_{\{ij\}} + \sum_{i,j,k,l=1}^{3} \beta_{ijkl}\hat{S}_{\{ij\}} \otimes \hat{S}_{\{kl\}}$$

$(\hat{S}_{\{ij\}} = \{\hat{S}_i, \hat{S}_j\})$

(alternative basis: only the $L = 2$ components of the $T_{L,M}$ expansion)

we could also add terms here that are not parity invariant on the *W* side, but I choose to ask the more general question

Can we make a Bell inequality for qutrits based only on this component?

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It would have to contain measurements of only ***squares*** of spin operators…

An older anwser and my proposition

P. Caban (0804.2997): Consider the CHSH Bell operator (I call it LP - Linear Polarizer)

$$\hat{\mathscr{B}}^{\mathrm{LP}} = \hat{O}^{\mathrm{LP}}_{\alpha} \otimes (\hat{O}^{\mathrm{LP}}_{\beta} - \hat{O}^{\mathrm{LP}}_{\beta'}) + \hat{O}^{\mathrm{LP}}_{\alpha'} \otimes (\hat{O}^{\mathrm{LP}}_{\beta} + \hat{O}^{\mathrm{LP}}_{\beta'}), \text{ where}$$

$$\hat{O}^{\mathrm{LP}}_{\alpha} = \left(\cos(\alpha)\hat{S}_x + \sin(\alpha)\hat{S}_y\right)^2 - \left(-\sin(\alpha)\hat{S}_x + \cos(\alpha)\hat{S}_y\right)^2 = \cos(2\alpha)\lambda_4 + \sin(2\alpha)\lambda_5$$

it is studied for a scalar state of the vector bosons - we test it in our systems, for reference.

I also propose the "Spin Squared (SS-)Bell inequality" with different operators:

$$\hat{\mathscr{B}}^{\mathrm{SS}} = \hat{O}^{\mathrm{SS}}_{\alpha} \otimes (\hat{O}^{\mathrm{SS}}_{\beta} - \hat{O}^{\mathrm{SS}}_{\beta'}) + \hat{O}^{\mathrm{SS}}_{\alpha'} \otimes (\hat{O}^{\mathrm{SS}}_{\beta} + \hat{O}^{\mathrm{SS}}_{\beta'}), \text{ with}$$

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the difference

In practice...

LP-Bell operator

$$\hat{\mathscr{B}}^{\mathrm{LP}} = \hat{O}^{\mathrm{LP}}_{\alpha} \otimes (\hat{O}^{\mathrm{LP}}_{\beta} - \hat{O}^{\mathrm{LP}}_{\beta'}) + \hat{O}^{\mathrm{LP}}_{\alpha'} \otimes (\hat{O}^{\mathrm{LP}}_{\beta} + \hat{O}^{\mathrm{LP}}_{\beta'}), \text{ where}$$

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1 when spin in the α drection (+ or -), -1 when in the $\alpha + \pi/2$ direction, 0 when not transverse

SS-Bell operator

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1 when spin in the α direction (+ or -), -1 when in the $|0\rangle$ state along the α direction, never 0

Acting with these Bell operators on a general density matrix ρ, we have

$$\mathrm{tr}\left(\hat{\mathscr{B}}^{\mathrm{LP}}\rho\right) = \vec{a}^{T}K(\vec{b} - \vec{b}') + \vec{a}'^{T}K(\vec{b} + \vec{b}')$$

$$\mathrm{tr}\left(\hat{\mathscr{B}}^{\mathrm{SS}}\rho\right) = \vec{a}^{T}K(\vec{b} - \vec{b}') + \vec{a}'^{T}K(\vec{b} + \vec{b}') + 2\vec{\delta}_1^{T}\vec{a}' + 2\vec{\delta}_2^{T}\vec{b} + 2\gamma,$$ **where**

$$\vec{a} = (\cos(2\alpha), \sin(2\alpha)), \quad \vec{b} = (\cos(2\beta), \sin(2\beta))$$
$$\vec{a}' = (\cos(2\alpha'), \sin(2\alpha')), \quad \vec{b}' = (\cos(2\beta'), \sin(2\beta'))$$

$$K_{ij} = \begin{pmatrix} \mathrm{tr}\left(\lambda_4 \otimes \lambda_4 \rho\right) & \mathrm{tr}\left(\lambda_4 \otimes \lambda_5 \rho\right) \\ \mathrm{tr}\left(\lambda_5 \otimes \lambda_4 \rho\right) & \mathrm{tr}\left(\lambda_5 \otimes \lambda_5 \rho\right) \end{pmatrix}$$

$$\vec{\delta}_1 = \left(\mathrm{tr}\left(\lambda_4 \otimes \left(\hat{1}_3 - \hat{S}_z^2\right)\rho\right), \mathrm{tr}\left(\lambda_5 \otimes \left(\hat{1}_3 - \hat{S}_z^2\right)\rho\right)\right)$$
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→ **Settings of the measurements (we get to optimise these)**

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→ **Things dependent on the quantum state**

We define Bell observables, as

$$\mathcal{I}_2^{\mathrm{LP}} = \max_{\vec{a},\vec{a}',\vec{b},\vec{b}'} \left| \vec{a}^T K(\vec{b} - \vec{b}') + \vec{a}'^T K(\vec{b} + \vec{b}') \right| = \sqrt{\mathrm{tr}(K^2)}$$

$$\mathcal{I}_2^{\mathrm{SS}} = \max_{\vec{a},\vec{a}',\vec{b},\vec{b}'} \left| \vec{a}^T K(\vec{b} - \vec{b}') + \vec{a}'^T K(\vec{b} + \vec{b}') + 2\vec{\delta}_1^T \vec{a}' + 2\vec{\delta}_2^T \vec{b} + 2\gamma \right|$$

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Simplified maximization from Horodecki "Violating Bell inequality by mixed spin-1/2 states: necessary and sufficient condition"

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I strongly belive that there is not an analytical way to maximize this in general - resort to numerics

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Bell inequalities:

$\mathcal{I}_2^{\mathrm{LP}} \leq 2$

$\mathcal{I}_2^{\mathrm{SS}} \leq 2$

Example states

At high energies in the central region, a diboson system such as Wg or ZZ forms a spin-2 state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|{+}{+}\rangle - |{-}{-}\rangle \right)$$

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for this, we get $\mathcal{I}_2^{\mathrm{SS}} = \mathcal{I}_2^{\mathrm{LP}} = 2\sqrt{2}$ ***- maximal violation.***

Example states

At total energies near threshold, in the central region, a diboson system with equal boson masses forms a spin-1 state along the beam:

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(b = spin states along the **beam,** not along the particles line of motion)

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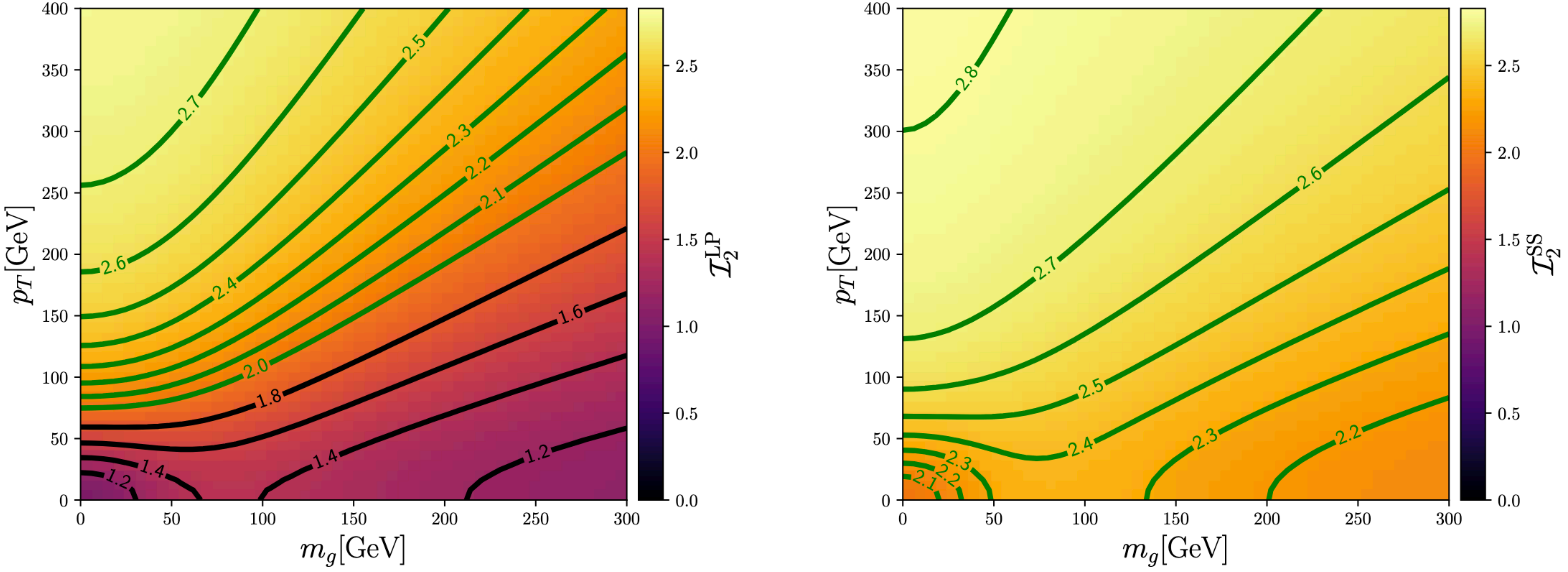
$$\mathcal{I}_2^{\mathrm{LP}} = 2\sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{2} < 2$$

$$\mathcal{I}_2^{\mathrm{SS}} = \frac{1}{2} \max_{\alpha,\alpha',\beta,\beta'} \left| \cos(2(\alpha+\beta)) - \cos(2(\alpha+\beta')) + \cos(2(\alpha'+\beta)) + \cos(2(\alpha'+\beta')) + \cos(2\alpha') + \cos(2\beta) \right| \approx 2.36 > 2$$

Results: Wg

We take the central region, $\theta^* = \pi/2$, for a generic p_T and m_g



When $p_T \to 0$ GeV and $m_g \to 0$ GeV, the state factorises
(but the LO picture not reliable there in this case)

Results: Wg

We investigate general θ^* for $m_g \ll p_T, m_W$

$pp \to W + (g \to b\bar{b})$, simulation in MadGraph

I estimate that we can see around ~ 6000 events in Run $2+3$ in ATLAS/CMS, when on top of fiducial cuts + detection efficiencies, we apply the cuts

$$p_{Tg} > 20\ \text{GeV},\ \cos(\theta^*) < 0.2, m_g > 20\ \text{GeV}$$

This gives:

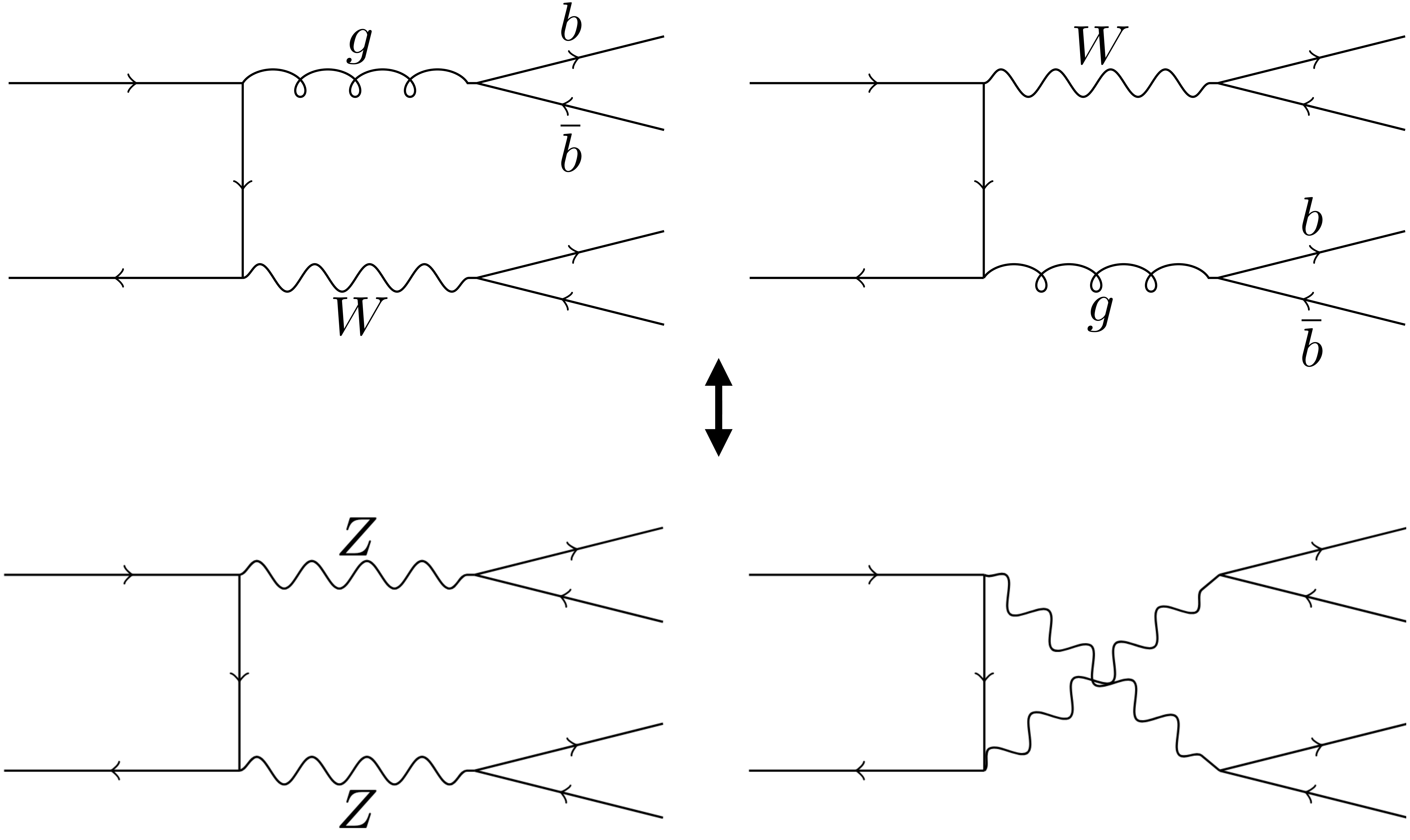
$$\mathcal{I}_3^{\text{SS}} = 2.43 \pm 0.33\ (1.3\sigma)$$

For HL-LHC, we thus expect ~ 60000 events, with gives

$$\mathcal{I}_3^{\text{SS}} = 2.43 \pm 0.10\ (4.3\sigma)$$

(statistical error comes from quantum state tomography from decays)

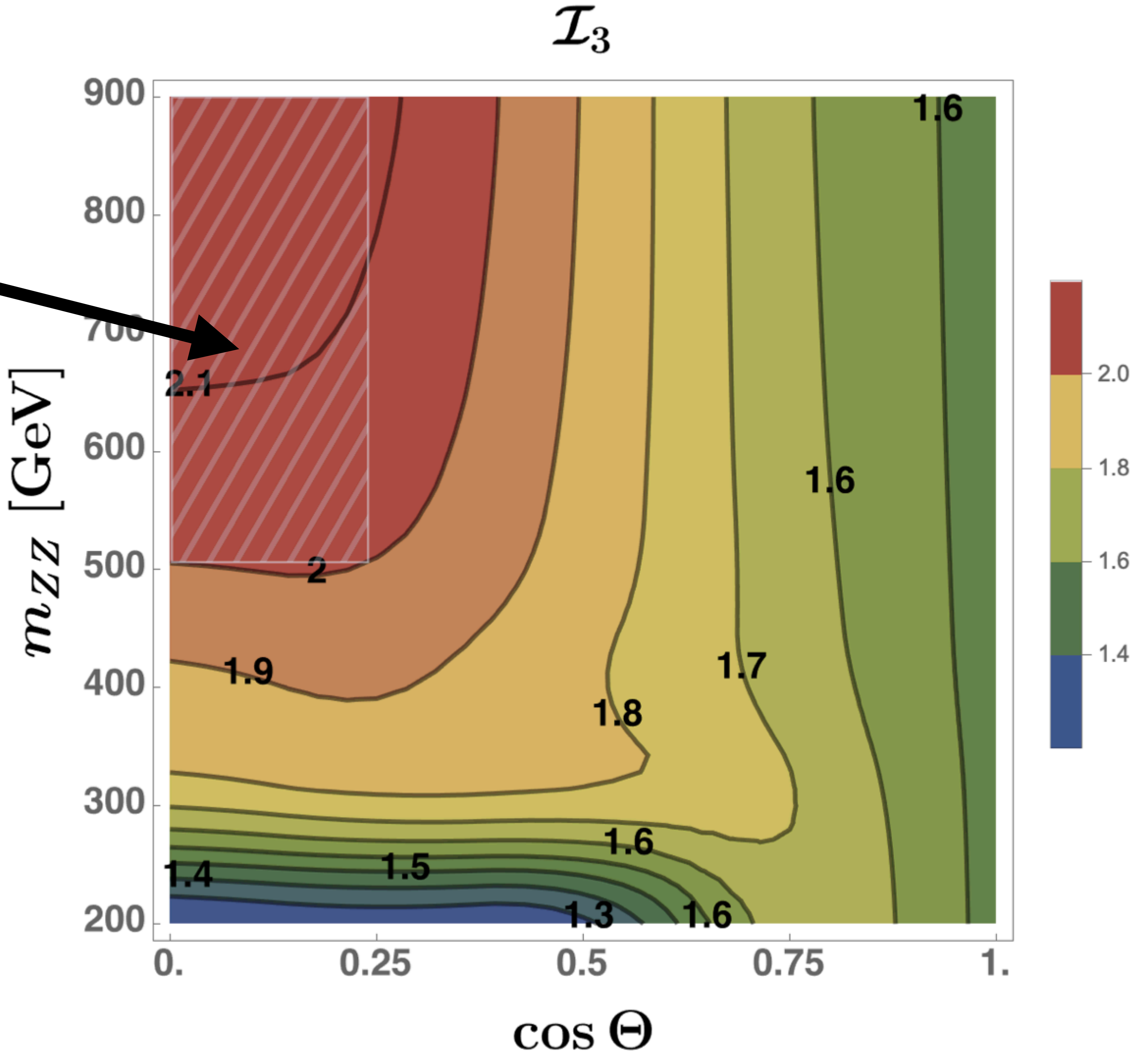
Application to ZZ



Application to ZZ: previous work

Using the CGLMP inequality (Fabbrichesi et al., 2302.00683)

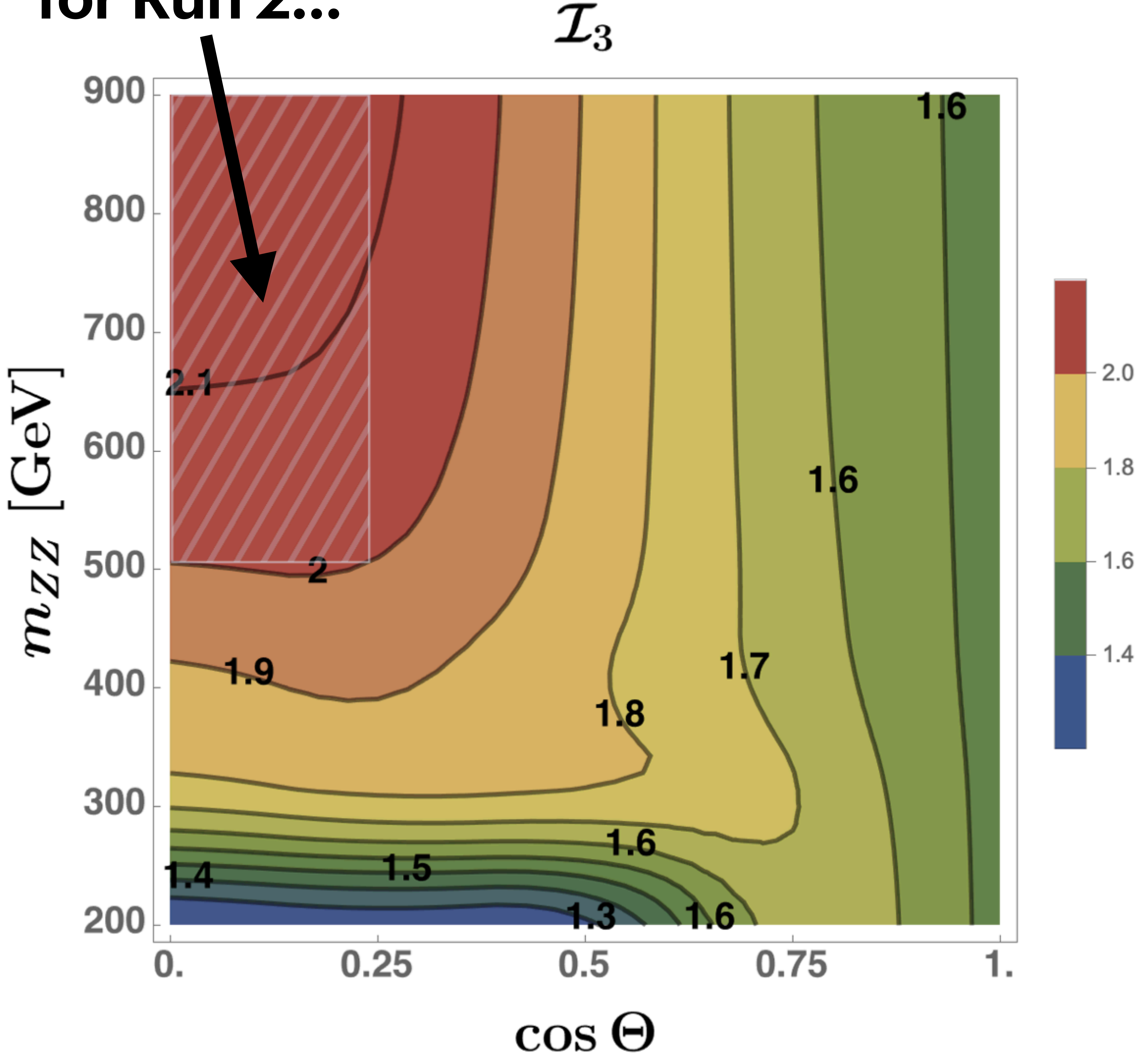
~ 4 **events in this bin for Run 2…**



Application to ZZ: comparison

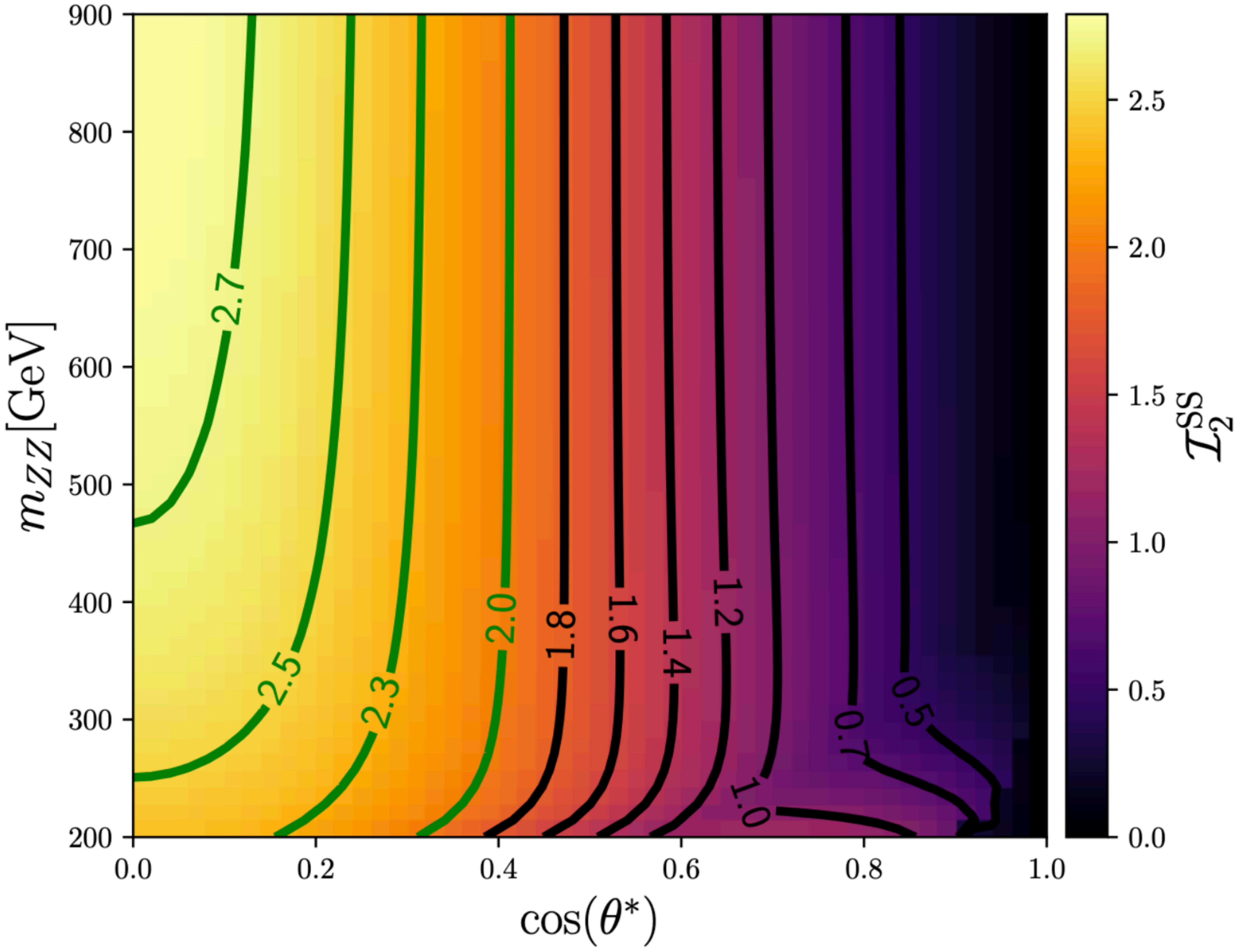
~ 4 events in this bin for Run 2…

CGLMP inequality



SS-Bell inequality

Used amplitudes from Aoude et al., 2307.09675



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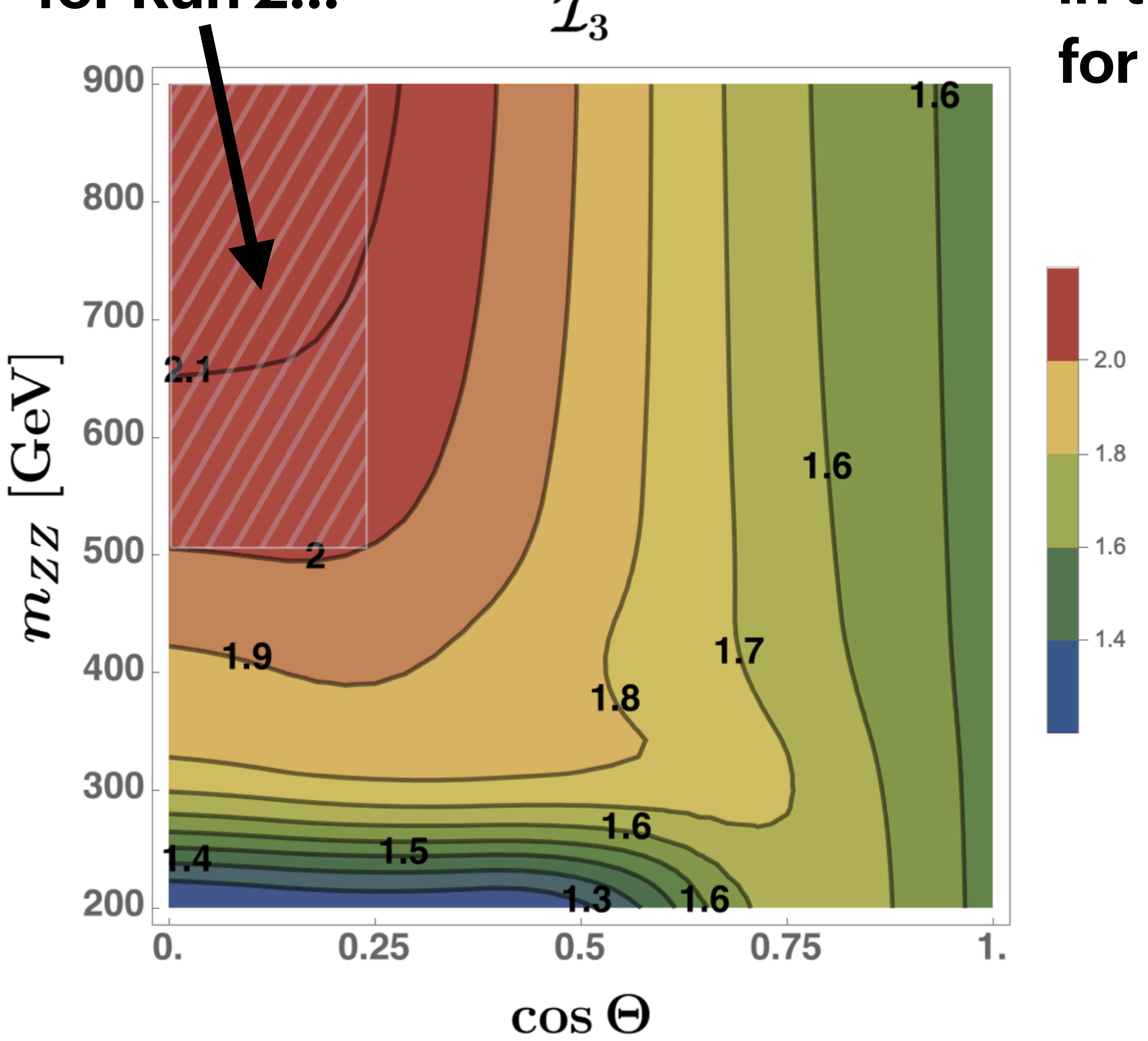
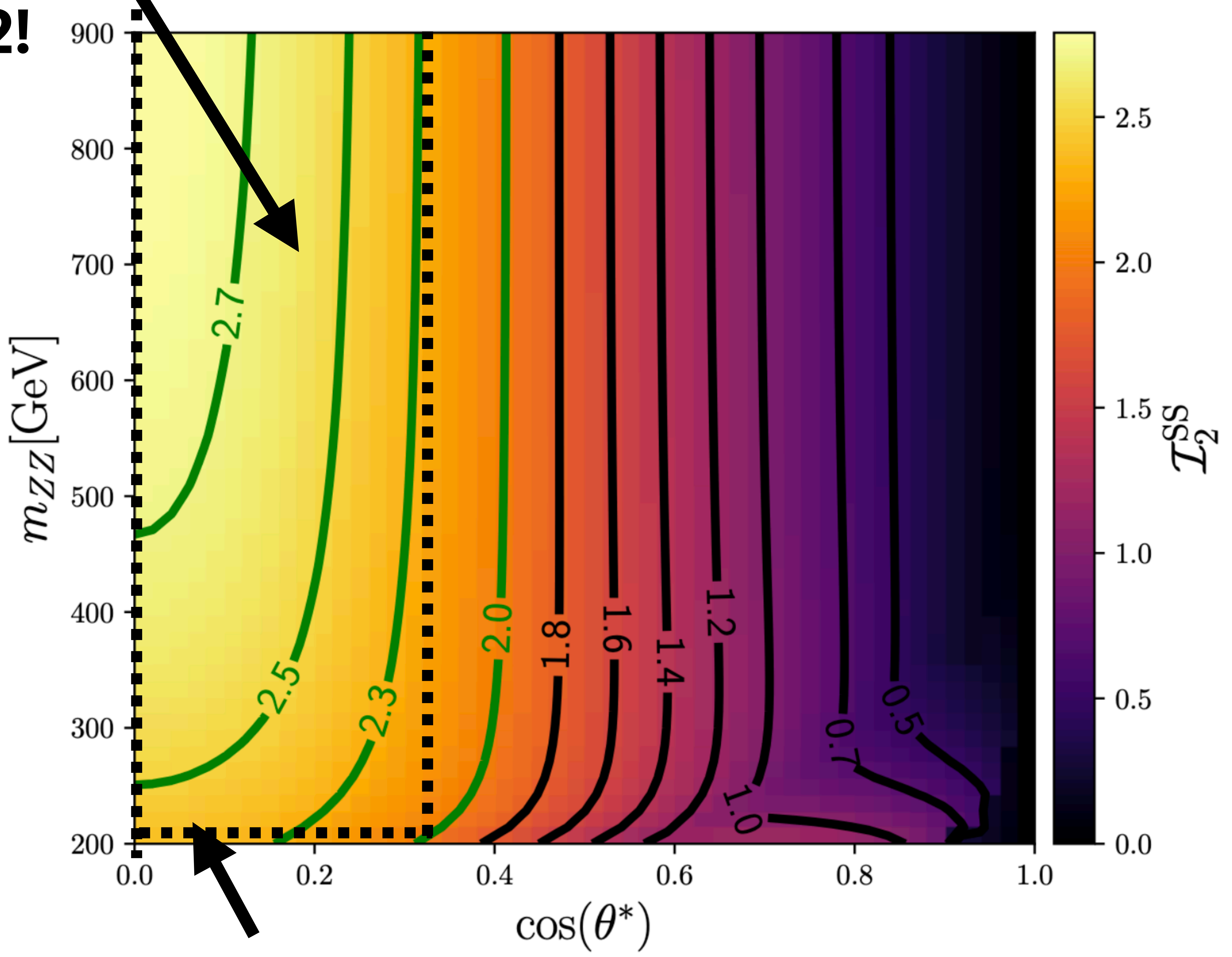
CGLMP inequality



~ 400 **events in this bin for Run 2!**

SS-Bell inequality



(but its a bit silly to do Bell inequalities very close to threshold)

Summary

- I have an inequality that seems to be violated by the types of two-qutrit states that we see in practice.
- It was not constructed for that purpose! By construction, it is meant to apply to decays via non-parity violating couplings, e.g., all other decays of gauge bosons
- This lets us expand "quantum stuff" to processes without weak decays; I tried $pp \to W + b\bar{b}$, and kept my dream alive.

Future directions: the problem of space-like vs. time-like separations,
higher-order corrections,
generalization to $\hat{O}_{\vec{n}} = 2\left(\vec{n} \cdot \hat{\vec{S}}\right)^2 - \hat{1}_3$
other processes (than this and WZ/WW). Any ideas?

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Thank You!

Backup: gluon tomography

$$K_{ij} = \langle g_i(\theta_1, \phi_1, \zeta_1) g_j(\theta_2, \zeta_2) \rangle, \tag{33}$$
$$\delta_{1i} = \langle g_i(\theta_1, \phi_1, \zeta_1) r(\theta_2, \zeta_2) \rangle, \tag{34}$$
$$\delta_{2i} = \langle g_i(\theta_2, \phi_2, \zeta_2) r(\theta_1, \zeta_1) \rangle, \tag{35}$$
$$\gamma = \langle r(\theta_1, \zeta_1) r(\theta_2, \zeta_2) \rangle, \tag{36}$$

where

$$g_1(\theta, \phi, \zeta) = 5\zeta \sin^2(\theta) \cos(2\phi), \tag{37}$$
$$g_2(\theta, \phi, \zeta) = 5\zeta \sin^2(\theta) \sin(2\phi), \tag{38}$$
$$r(\theta, \zeta) = 5\zeta \sin^2(\theta) + 1 - 4\zeta, \tag{39}$$

and ζ denotes a factor necessary to account for the mass of final-state fermions, particularly in the case of $g \to b\bar{b}$. For a generic particle of mass M splitting into two fermions of masses m_x, m_y, we obtain

$$\zeta = \frac{2M^2 + (m_x + m_y)^2}{2(M^2 - (m_x + m_y)^2)}. \tag{40}$$

Backup: “significance” for ZZ

- For Run-2+3, there is no way to do reliable tomography with ~800 events
- For HL-LHC, I get 2.46 ± 0.17 (2.5σ) with a $\cos(\theta^*) < 0.3$ cut