

Spectroscopy of lattice gauge theories from spectral densities

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Swansea

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Based on work [[arXiv:2405.01388](https://arxiv.org/abs/2405.01388)]

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Overview

- Lattice setup.
- Benchmarks for our findings.
- **What** is spectral density?
- **How** do we study it?
- Numerical results.

Lattice setup

- We consider a $\text{Sp}(4)$ gauge theory with $N_f = 2$ (dynamical) fermions in the fundamental representation and $N_{\text{as}} = 3$ in the 2-index antisymmetric one. → [[Phys.Rev.D106 \(2022\) 1, 014501](#)]
- We write the Euclidean action, discretised in four dimensions, as the sum of the gauge S_g and fermion S_f actions,

$$S \equiv S_g + S_f ,$$

where

$$S_g \equiv \beta \sum_x \sum_{\mu < \nu} \left(1 - \frac{1}{2N} \text{Re } \mathcal{P}_{\mu\nu}(x) \right) ,$$

$$S_f \equiv a^4 \sum_{j=1}^{N_f} \sum_x \bar{Q}^j(x) D_m^{(f)} Q^j(x) + a^4 \sum_{j=1}^{N_{\text{as}}} \sum_x \bar{\Psi}^j(x) D_m^{(\text{as})} \Psi^j(x) ,$$

- We perform simulation by using (rational) hybrid Monte-Carlo simulations (RHMC)

$$Z = \int \mathcal{D}U \mathcal{D}Q \mathcal{D}\bar{Q} \mathcal{D}\Psi \mathcal{D}\bar{\Psi} e^{-S[U, Q, \bar{Q}, \Psi, \bar{\Psi}]}$$

LSDensities, new python library [\[https://github.com/LupoA/lsdensities\]](https://github.com/LupoA/lsdensities)

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README
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✎
☰

LSDensities: Lattice Spectral Densities

Lsdensities is a Python library for the calculation of smeared spectral densities from lattice correlators.

Solutions can be obtained with the [Hansen Lupo Tantalo](#) method and [Bayesian inference with Gaussian Processes](#), or combinations of the two.

This library is based on [mpmath](#) for performing the high-precision arithmetic operations that are necessary for the solution of the inverse problem.

About

Smeared spectral densities from lattice correlators

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Benchmarks for our findings

- Comparisons with spectral density findings will be done using technologies already used in the literature:

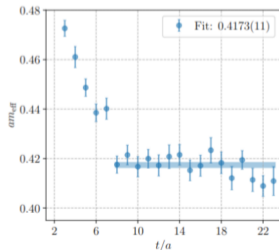
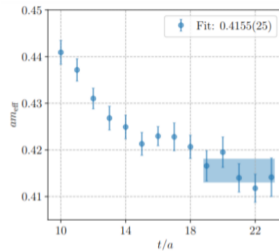
- **Effective mass plateaus** to isolate ground states

$$C(t) = \langle O(t) \bar{O}(0) \rangle \xrightarrow{t \rightarrow \infty} K \cdot e^{-M_0 t} \Rightarrow am_{\text{eff}} = -\ln \left[\frac{C(t+1)}{C(t)} \right]$$

- **Generalised Eigenvalue Problem (GEVP)** to isolate excited states

$$C(t_2) v_n(t_2, t_1) = \lambda_n(t_2, t_1) C(t_1) v_n(t_2, t_1) \rightarrow \lambda_n(t_2, t_1)$$

where $C(t)$ is a matrix of correlation functions having the same spectrum.



What is spectral density

■ What is spectral density?

→ At positive Euclidean times $t \geq 0$ the previous correlator can be rewritten as

$$C(t) = \int_0^\infty dE \rho_L(E) e^{-tE}$$

and we defined

$$\rho_L(E) = \langle 0 | O(0) \delta(E - H_L) \bar{O}(0) | 0 \rangle_L$$

■ Several applications:

- Spectroscopy [[arXiv:2212.08019](#)].
 - Case study: $\text{Sp}(4)$ theory with $N_f = 2$, $N_{\text{as}} = 3$ dynamical fermions.
- Study of inclusive decay rates [[arXiv:2111.12774](#)].
- Study of sphaleron rate (and maybe deconfinement?) [[arXiv:2309.13327](#)].

Spectral density extraction

To extract $\rho_L(E)$ from $C(t)$:

- Having a finite volume Hamiltonian H_L , we will have

$$\rho_L(E) = \sum_n w_n(L) \delta(E - E_n(L))$$

which is mostly lost in the continuum limit, where above the multi-particle threshold the spectral density becomes continuous.

→ We smear the spectral densities using a smearing kernel $\Delta_\sigma(E, \omega)$

$$\hat{\rho}_\sigma(\omega) = \int_0^\infty dE \Delta_\sigma(E, \omega) \rho_L(E)$$

- We need to perform an inverse Laplace-transform which is **ill-posed**.

Ill-posed problem

- The problem is ill-posed. This can be seen by expanding

$$\bar{\Delta}_\sigma(E, \omega) = \sum_{t=0}^{t_{\max}} g_t(\omega) e^{-(t+1)E}$$

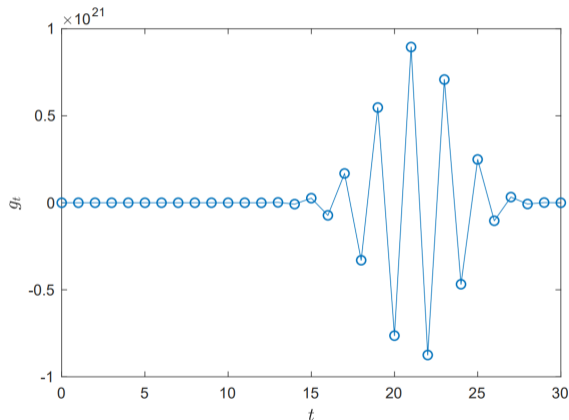
(therefore

$$\hat{\rho}(\omega) = \sum_{t=0}^{t_{\max}} g_t(\omega) C(t+1)$$

) and finding the coefficients $g_t(\omega)$ by minimizing

$$A[\vec{g}] = \int_0^\infty dE |\Delta_\sigma(E, \omega) - \bar{\Delta}_\sigma(E, \omega)|^2$$

- Therefore, if $C(t) = \bar{C}(t) + \delta(C(t))$ and the uncertainty on the spectral density $\delta(C(t)) \times g_t(E)$ will be uncontrolled.



[arXiv:1903.06476]

- └ Mass extractions and spectral density
 - └ **HOW** do we study spectral density?

Spectral density algorithm

We will reconstruct spectral densities using a modified Backus-Gilbert method: **HLT method** (Hansen-Tantalo-Lupo) [1].

To determine the vector of coefficients $\vec{g} = \vec{g}(E)$ for the spectral reconstruction, we minimize the functional

$$W[\vec{g}] = \frac{A[\vec{g}]}{A[0]} + \lambda \frac{B[\vec{g}]}{B_{\text{norm}}} \quad , \quad \lambda \in (0, \infty)$$

where $B_{\text{norm}} = C^2(1)/E^2$ (lattice spacing $a = 1$, for convenience)

$$A[\vec{g}] = \int_0^\infty dE e^{\alpha E} |\bar{\Delta}_\sigma(E, \omega) - \Delta_\sigma(E, \omega)|^2$$

$$B[\vec{g}] = \sum_{\tau, \tau'} g_\tau \text{Cov}_{\tau\tau'}[C] g_{\tau'}$$

For each energy we reconstruct the spectral density

$$\hat{\rho}(E) = \sum_t g_t(E) C(t)$$

Smearing kernels

In order to check the quality of reconstruction, we also check that at each energy the reconstruction of the kernels we use:

$$\bar{\Delta}_\sigma(E, \omega) = \sum_{t=0}^{t_{\max}} g_t(\omega) e^{-(t+1)E}$$

We use as target kernels:

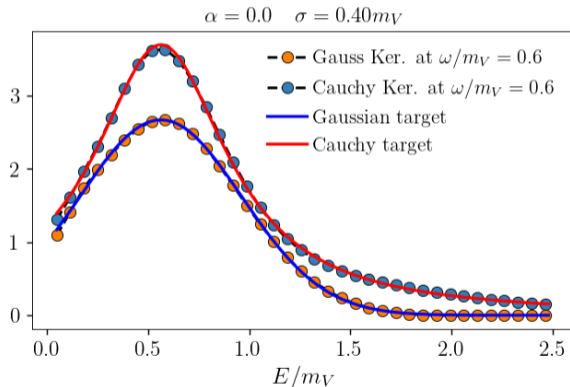
- *Gaussian kernel:*

$$\bar{\Delta}_\sigma^{(1)}(E, \omega) = e^{-\frac{(E-\omega)^2}{2\sigma^2}} / Z(\omega)$$

with $Z(\omega) = \int_0^\infty dE e^{-\frac{(E-\omega)^2}{2\sigma^2}}$.

- *Cauchy kernel:*

$$\bar{\Delta}_\sigma^{(2)}(E, \omega) = \frac{\sigma}{[(E - \omega)^2 + \sigma^2]}$$



- └ Mass extractions and spectral density
- └ **HOW** do we study spectral density?

Spectral density reconstruction systematic errors

- Minimize

$$W[\vec{g}] = \frac{A[\vec{g}]}{A[0]} + \lambda \frac{B[\vec{g}]}{B_{\text{norm}}}$$

while varying α and λ .

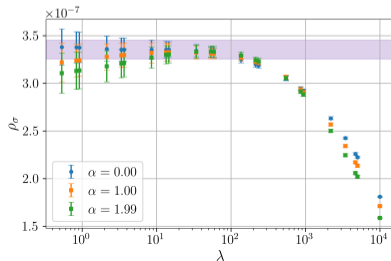
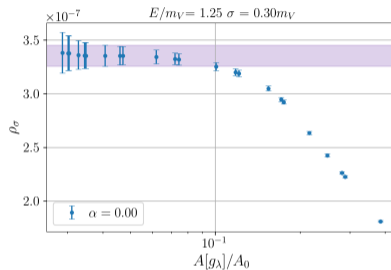
- The first component of systematic error for each of these values $\hat{\rho}(E)$, will be estimated as

$$\sigma_{1, \text{sys}}(\hat{\rho}(E)) = |\hat{\rho}_{\lambda_*}(E) - \hat{\rho}_{\lambda_*/10}(E)|$$

where λ_* was found through the plateaus procedure described above.

- The second component of systematic error for each of the values $\hat{\rho}(E)$, will be estimated as

$$\sigma_{2, \text{sys}}(\hat{\rho}(E)) = |\hat{\rho}_{\lambda_*, \alpha_2}(E) - \hat{\rho}_{\lambda_*, \alpha_1}(E)|$$



Spectral density fits

Given this procedure, we can perform fits of the spectral density, minimizing the functional [2]

$$\chi^2 = \sum_{E, E'} \left(f_{\sigma}^{(k)}(E) - \hat{\rho}_{\sigma}(E) \right) \text{Cov}_{EE'}^{-1}[\hat{\rho}_{\sigma}] \left(f_{\sigma}^{(k)}(E') - \hat{\rho}_{\sigma}(E') \right)$$

where we fit the spectral densities as:

- Sum of Gaussians

$$f_{\sigma}^{(k)}(E) = \sum_{n=1}^k \mathcal{A}_n \Delta_{\sigma}^{(1)}(E - E_n)$$

- Sum of Cauchy functions

$$f_{\sigma}^{(k)}(E) = \sum_{n=1}^k \mathcal{A}_n \Delta_{\sigma}^{(2)}(E - E_n)$$

(remember that $\rho_L(E) = \sum_n w_n(L) \delta(E - E_n(L))$ and $\hat{\rho}_{\sigma}(\omega) = \int_0^{\infty} dE \Delta_{\sigma}(E, \omega) \rho(E)$)

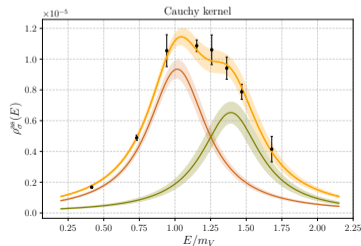
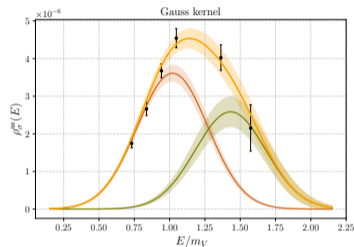
- └ Mass extractions and spectral density
- └ **HOW** do we study spectral density?

Energy levels fitting: cross checks

- We perform several cross checks, for example we fit using both a Gaussian and Cauchy kernel

$$\sigma_{1, \text{sys}}(aE_n) = |aE_{n, \text{Gauss}} - aE_{n, \text{Cauchy}}|$$

and we evaluate the difference between the same energy state, determined using the two kernels.



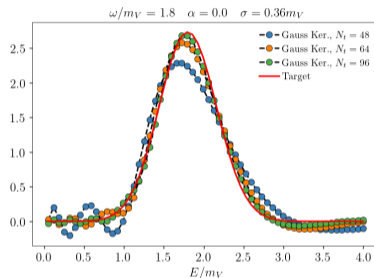
Improving the results: enlonging time extent

- We can increase values of N_t .

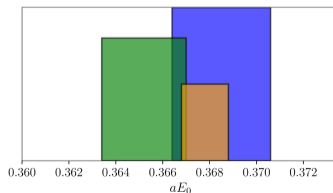
→ Increase basis to expand Kernel and spectral density, more accurate reconstruction.

$$\bar{\Delta}_\sigma(E, \omega) = \sum_{t=0}^{t_{\max}} g_t(\omega) e^{-(t+1)E}$$

(where $t_{\max} < T$).



■ $N_t = 48$ ■ $N_t = 64$ ■ $N_t = 96$



Numerical results: comparison with GEVPs

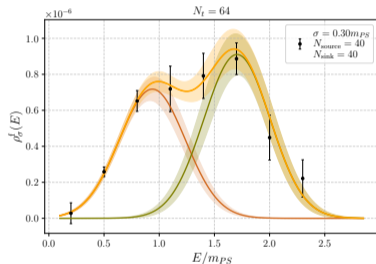
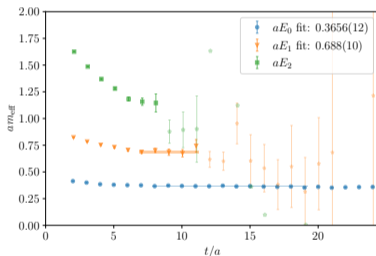
- We compare the GEVPs from several channels to check the excited states.

Channel	Interpolating operator
PS (F/AS)	$\bar{F}^i \gamma_5 F^j$
V (F/AS)	$\bar{F}^i \gamma_\mu F^j$
T (F/AS)	$\bar{F}^i \gamma_0 \gamma_\mu F^j$
AV (F/AS)	$\bar{F}^i \gamma_5 \gamma_\mu F^j$
AT (F/AS)	$\bar{F}^i \gamma_5 \gamma_0 \gamma_\mu F^j$
S (F/AS)	$\bar{F}^i F^j$

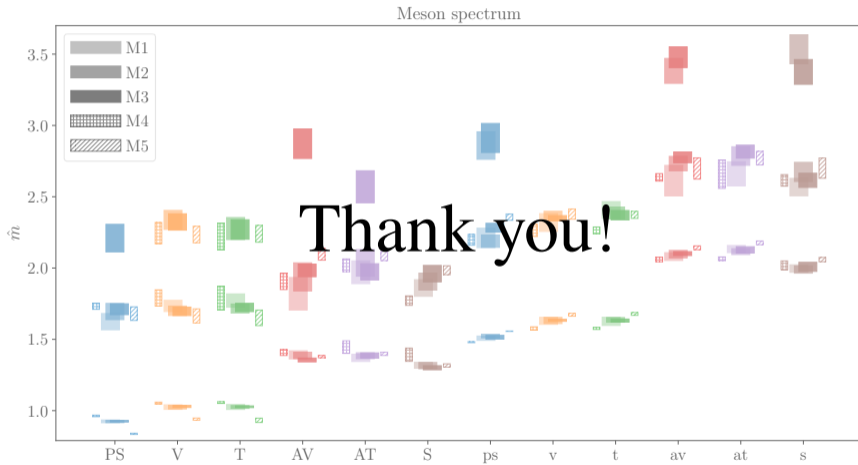
where i, j are flavour indices and $F = Q, \Psi$.

- They come out to be compatible within statistical uncertainty.

→ Example:



Channel	aE_0 (HLT)	aE_0 (GEVP)	aE_1 (HLT)	aE_1 (GEVP)
PS (f)	0.3652(11)	0.3656(12)	0.670(10)	0.688(10)



Bibliography

- 1 M. Hansen, A. Lupo and N. Tantalo, *Phys. Rev. D* **99** (2019) no.9, 094508 doi:10.1103/PhysRevD.99.094508 [arXiv:1903.06476 [hep-lat]].
- 2 A. Lupo, L. Del Debbio, M. Panero and N. Tantalo, *PoS LATTICE2022* (2023), 215 doi:10.22323/1.430.0215 [arXiv:2212.08019 [hep-lat]].
- 3 M. Albanese et al. [APE], *Phys. Lett. B* 192, 163-169 (1987) doi:10.1016/0370-2693(87)91160-9
- 4 S. Gusken, *Nucl. Phys. B Proc. Suppl.* 17, 361-364 (1990) doi:10.1016/0920-5632(90)90273-W

Backup slides: $Sp(2N)$ Lie group

We denote as $Sp(2N)$ the subgroup of $SU(2N)$ preserving the norm induced by the antisymmetric matrix Ω ,

$$\Omega = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix},$$

where 1_N is the $N \times N$ identity matrix. This definition can be converted into a constraint on the group element U

$$U\Omega U^T = \Omega.$$

Due to unitarity, the previous condition can be also written as

$$U\Omega = \Omega U^*,$$

which implies the following block structure

$$U = \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix},$$

Backup slides: Wilson-Dirac operators on the lattice

The massive Wilson-Dirac operators are defined as

$$D_m^{(f)} Q^j(x) \equiv (4/a + m_0^f) Q^j(x) - \frac{1}{2a} \sum_{\mu} \left\{ (1 - \gamma_{\mu}) U_{\mu}^{(f)}(x) Q^j(x + \hat{\mu}) + (1 + \gamma_{\mu}) U_{\mu}^{(f), \dagger}(x - \hat{\mu}) Q^j(x - \hat{\mu}) \right\},$$

and

$$D_m^{(\text{as})} \Psi^j(x) \equiv (4/a + m_0^{\text{as}}) \Psi^j(x) - \frac{1}{2a} \sum_{\mu} \left\{ (1 - \gamma_{\mu}) U_{\mu}^{(\text{as})}(x) \Psi^j(x + \hat{\mu}) + (1 + \gamma_{\mu}) U_{\mu}^{(\text{as}), \dagger}(x - \hat{\mu}) \Psi^j(x - \hat{\mu}) \right\},$$

Backup slides: antisymmetric links definition

The link variables $U_{\mu}^{(\text{as})}(x)$ are defined as follows:

$$U_{\mu, (ab)(cd)}^{(\text{as})} = \left(e^{(ab)T} U_{\mu}^{(\text{f})} e^{(cd)} U_{\mu}^{(\text{f})T} \right),$$

where $e^{(ab)}$ are the elements of an orthonormal basis in the $(N(2N-1)-1)$ -dimensional space of $2N \times 2N$ antisymmetric and Ω -traceless matrices, and the multi-indices (ab) run over the values $1 \leq a < b \leq 2N$.

The entry ij of each element of the basis is defined as follows. For $b \neq N+a$,

$$e_{ij}^{(ab)} \equiv \frac{1}{\sqrt{2}} (\delta_{aj} \delta_{bi} - \delta_{ai} \delta_{bj}),$$

while for $b = N+a$ and $2 \leq a \leq N$,

$$e_{i, i+N}^{(ab)} = -e_{i+N, i}^{(ab)} \equiv \begin{cases} \frac{1}{\sqrt{2a(a-1)}}, & \text{for } i < a, \\ \frac{1-a}{\sqrt{2a(a-1)}}, & \text{for } i = a. \end{cases}$$

Backup slides: RHMC, rational hybrid Monte-Carlo

- The (R)HMC algorithms generate a Markov chain of gauge configurations distributed as required by the lattice action.
- Bosonic degrees of freedom ϕ and ϕ^\dagger , known as pseudofermions, are introduced replacing a generic number n_f of fermions.
- Powers of the determinant of the hermitian Dirac operator, $Q_m^R = \gamma_5 D_m^R$, in representation R can then be expressed as

$$(\det D_m^R)^{n_f} = (\det Q_m^R)^{n_f} = \int \mathcal{D}\phi \mathcal{D}\phi^\dagger e^{-a^4 \sum_x \phi^\dagger(x) (Q_m^2)^{-n_f/2} \phi(x)},$$

- For odd values of n_f , the rational approximation is used to compute odd powers of the determinant above, resulting in the RHMC.

Backup slides: RHMC, rational hybrid Monte-Carlo (2)

- The fictitious hamiltonian is

$$H = \frac{1}{2} \sum_{x, \mu, a} \pi^a(x, \mu) \pi^a(x, \mu) + H_g + H_f ,$$

- The molecular dynamics (MD) evolution in fictitious time τ is dictated by

$$\frac{dU_\mu(x)}{d\tau} = \pi(x, \mu) U_\mu(x) , \quad \frac{d\pi(x, \mu)}{d\tau} = F(x, \mu) ,$$

where $F(x, \mu)$, known as the HMC force.

- Numerical integration of the MD equations thus leads to a new configuration of the gauge field, which is then accepted or rejected according to a Metropolis test.

Backup slides: formulas for spectral density reconstruction

■ A0.mp and A0E.mp

$$A_0(\omega) \equiv A[0](\omega) = \int_{E_0}^{\infty} dE e^{\alpha E} \Delta_{\sigma}(E, \omega)^2 = \frac{e^{\frac{\alpha^2 \sigma^2}{4} + \alpha \omega} \left(\operatorname{erf} \left(\frac{\alpha \sigma^2 + 2\omega - 2e_0}{2\sigma} \right) + 1 \right)}{4\sqrt{\pi}\sigma}$$

■ ft.mp

$$\begin{aligned} f_t(\omega) &= \int_{E_0}^{\infty} dE \Delta_{\sigma}(E, \omega) b_T(t, E) e^{\alpha E} \\ &= \frac{1}{2} \left\{ e^{\frac{1}{2}(\alpha+t-T)(\sigma^2(\alpha+t-T)+2\omega)} \left(\operatorname{erf} \left(\frac{\sigma^2(\alpha+t-T) + \omega - e_0}{\sqrt{2}\sigma} \right) + 1 \right) \right. \\ &\quad \left. + e^{\frac{1}{2}(\alpha-t)(\sigma^2(\alpha-t)+2\omega)} \operatorname{erfc} \left(\frac{\sigma^2(t-\alpha) - \omega + e_0}{\sqrt{2}\sigma} \right) \right\} \end{aligned}$$

Backup slides: formulas for spectral density reconstruction (2)

- In the code, we express $f_t(\omega)$ by means of the following function called `generalised_ft`:

$$\tilde{f}_t(\omega) = e^{\frac{1}{2}(\alpha-t)(\sigma^2(\alpha-t)+2\omega)} \operatorname{erfc}\left(\frac{\sigma^2(t-\alpha)-\omega+e_0}{\sqrt{2}\sigma}\right),$$

so that we can write $f_t(\omega) = \frac{\tilde{f}_t(\omega) + \tilde{f}_{T-t}(\omega)}{2}$

- `Smatrix_mp`

$$S_{tr} = \frac{e^{E_0(\alpha-r-t-2)}}{t+r+2-\alpha} + \frac{e^{E_0(\alpha+r+t+2-2T)}}{2T-t-r-2-\alpha} + \frac{e^{E_0(\alpha+r-t-T)}}{T+t-r-\alpha} + \frac{e^{E_0(\alpha-r+t-T)}}{T-t+r-\alpha}$$

We also have

$$B_{tr} = \operatorname{Cov}_{tr}.$$

$B_{\text{norm}} = C(1)$ can be used to make $B[g]$ dimensionless.

- The minimisation then amounts to solve the following linear system

$$\vec{g} = \left(S + \frac{\lambda A_0(\omega)}{(1-\lambda)(\omega)} B \right)^{-1} \vec{f}.$$

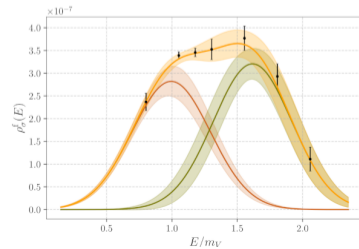
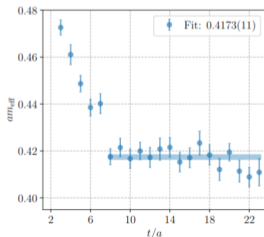
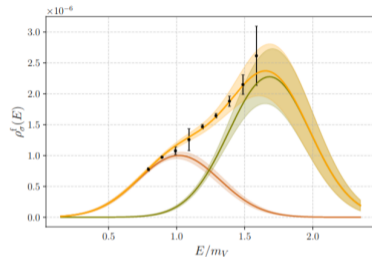
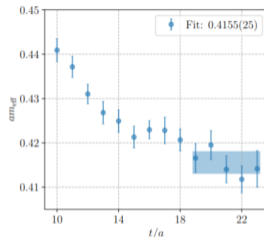
Backup slides: Numerical results: using Wuppertal and APE smearings

- We use APE [3] and Wuppertal smearing [4] to increase the overlap between the operators and the ground state

$$C(t) = \sum_n \frac{\langle 0|O(0)|n\rangle\langle n|\bar{O}(0)|0\rangle}{2E_n} e^{-tE}$$

- Similarly we get

$$\hat{\rho}_\sigma(E) = \sum_n \frac{\langle 0|O(0)|n\rangle\langle n|\bar{O}(0)|0\rangle}{2E_n} \times \Delta_\sigma(E - E_n(L))$$



Backup slides: Wuppertal and APE smearings formulas

- **Wuppertal smearing** acts on fermion fields increasing the overlap of ground state.

$$q^{(n+1)}(x) = \frac{1}{1+2d\varepsilon} \left[q^{(n)}(x) + \varepsilon \sum_{\mu=\pm 1}^{\pm d} U_{\mu}(x) q^{(n)}(x + \hat{\mu}) \right]$$

- **APE smearing** averages out UV fluctuations of the gauge fields.

$$U_{\mu}^{(n+1)}(x) = P \left\{ (1 - \alpha) U_{\mu}^{(n)}(x) + \frac{\alpha}{6} S_{\mu}^{(n)}(x) \right\}, \quad S_{\mu}(x) = \sum_{\pm \nu \neq \mu} U_{\nu}(x) U_{\mu}(x + \hat{\nu}) U_{\nu}^{\dagger}(x + \hat{\mu})$$

Backup slides: varying Wuppertal and APE smearings

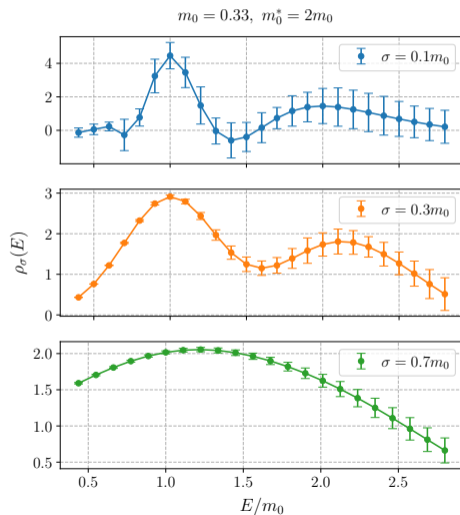
Mean amplitudes ratios				
ϵ^{APE}	$\epsilon_f^{\text{Wuppertal}}$	N_{source}	N_{sink}	$\mathcal{A}_2/\mathcal{A}_1$
0.4	0.18	80	20	1.32(19)
0.4	0.18	80	40	1.15(11)
0.4	0.18	80	80	0.75(15)
0.4	0.18	40	80	1.24(18)
0.4	0.18	20	80	1.80(28)
0.4	0.24	90	30	1.01(20)
0.4	0.4	170	170	0.63(11)
0.4	0.05	20	20	2.28(27)
0.0	0.18	80	40	1.27(11)

Table: Amplitudes ratio between the two-gaussian fits, for different levels of sink and source Wuppertal smearing and APE smearing.

Backup slides: choosing smearing radius

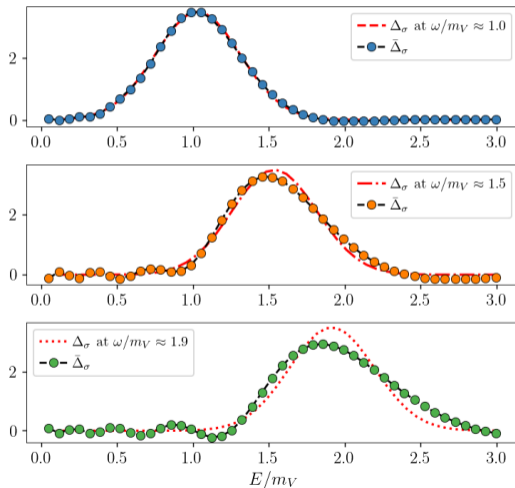
■ Choosing the correct smearing radius is delicate:

- Too large choice make the fitting procedure difficult.
- Too small one results in unreliable and even useless reconstruction.



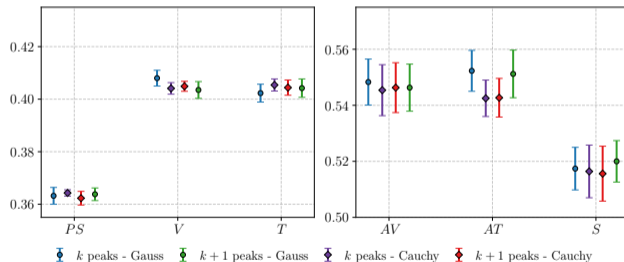
Backup slides: finiteness of information

- As the quantity of physical information in the input correlators is finite, the reconstruction will happen up to finite energies
 - The higher energies will become progressively less reliable.
 - Higher systematic effects entities.



Backup slides: Numerical results: systematic errors evaluation

- The systematic errors due to excited states contaminations and change in kernel appears to be under controlled.

 aE_0 fits, fundamental sector, ensemble M2

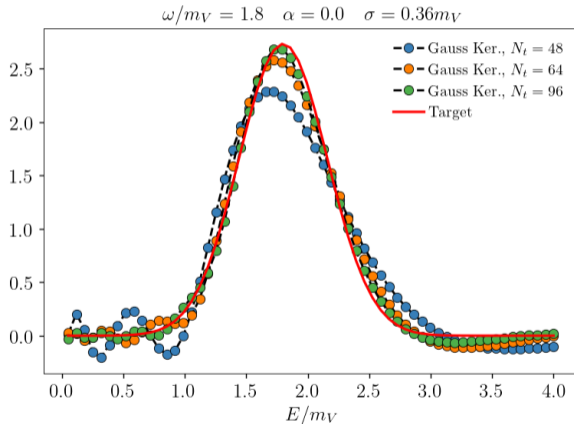
Backup slides: Improving the results: enlonging time extent

- We can increase values of N_t .

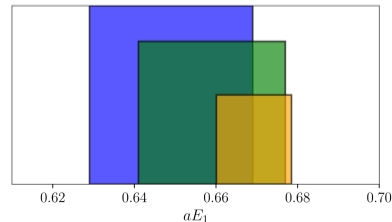
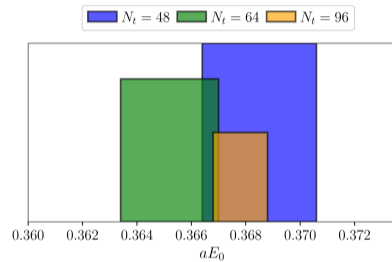
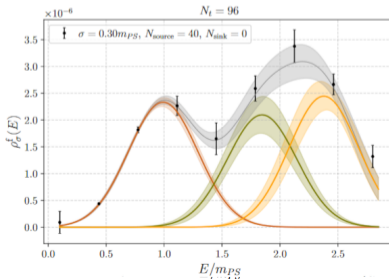
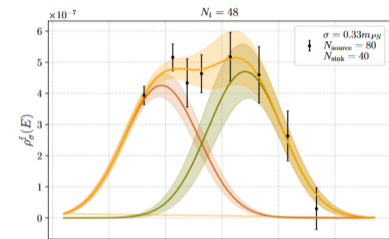
→ Increase basis to expand Kernel and spectral density, more accurate reconstruction.

$$\bar{\Delta}_\sigma(E, \omega) = \sum_{t=0}^{t_{\max}} g_t(\omega) e^{-(t+1)E}$$

(where $t_{\max} < T$).



Backup slides: Improving the results: elongating time extent (2)



Backup slide: Benchmarks for our findings

- Comparisons with spectral density findings will be done using technologies already used in the literature:

- **Effective mass plateaus** to isolate ground states

$$C(t) = \langle \mathcal{O}(t) \bar{\mathcal{O}}(0) \rangle \xrightarrow{t \rightarrow \infty} K \cdot e^{-M_0 t} \Rightarrow am_{\text{eff}} = -\ln \left[\frac{C(t+1)}{C(t)} \right]$$

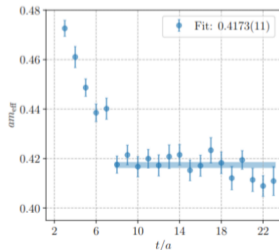
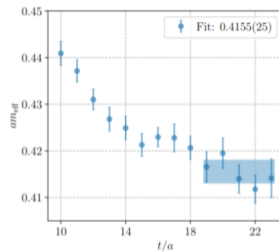
- **Generalised Eigenvalue Problem (GEVP)** to isolate excited states

$$C(t_2) v_n(t_2, t_1) = \lambda_n(t_2, t_1) C(t_1) v_n(t_2, t_1) \rightarrow \lambda_n(t_2, t_1)$$

where $C(t)$ is a matrix of correlation functions having the same spectrum.

- We will also use additional tools:

- **Wuppertal smearing**, acting on fermion fields.
 - **APE smearing**, acting on gauge links.



Backup slides: Ensembles

Label	β	am_0^f	am_0^{as}	$N_t \times N_s^3$
M1	6.5	-1.01	-0.71	48×20^3
M2	6.5	-1.01	-0.71	64×20^3
M3	6.5	-1.01	-0.71	96×20^3
M4	6.5	-1.01	-0.70	64×20^3
M5	6.5	-1.01	-0.72	64×32^3

Table: Ensembles generated for and analysed. The inverse coupling is denoted as β and the fundamental and antisymmetric bare fermion masses by am_0^f and am_0^{as} , respectively. The lattice volume is $N_t N_s^3 a^4$.



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Backup slides: Outline

- We know how to reconstruct spectral densities \rightarrow HLT algorithm.
- Systematics evaluation for $\rho(E)$ reconstruction can be done (α , λ variation).
- Fits of the finite volume spectral densities \rightarrow Spectroscopy of gauge theories.
- Evaluation of systematics for energy levels $aE_n \rightarrow$ Different smearing kernels.
- Spectroscopy results can be improved as we consider larger lattices.

Backup slide: Energy levels fitting: cross checks

- We perform several cross checks, for example we fit using both a Gaussian and Cauchy kernel

$$\sigma_{1, \text{sys}}(aE_n) = |aE_{n, \text{Gauss}} - aE_{n, \text{Cauchy}}|$$

and we evaluate the difference between the same energy state, determined using the two kernels.

- Difference between the two and three Gaussian (or Cauchy) functions

$$\sigma_{2, \text{sys}}(aE_n) = |aE_{n, k=3} - aE_{n, k=2}|$$

