Queen Mary
University of London

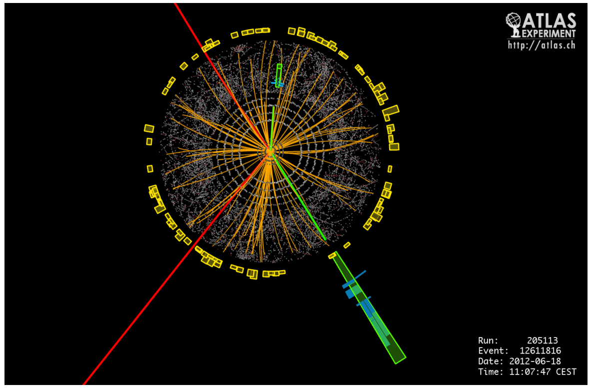
# The Double Copy Theory

## Chris White

**Durham Annual Theory Meeting 2024**

# Introduction

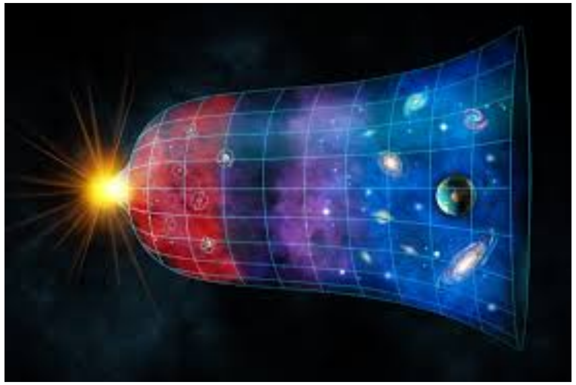
- Fundamental physics is described by (**non-)abelian gauge theories**, and **General Relativity**.



Strong, weak and EM forces

Quantum effects included

Calculations difficult



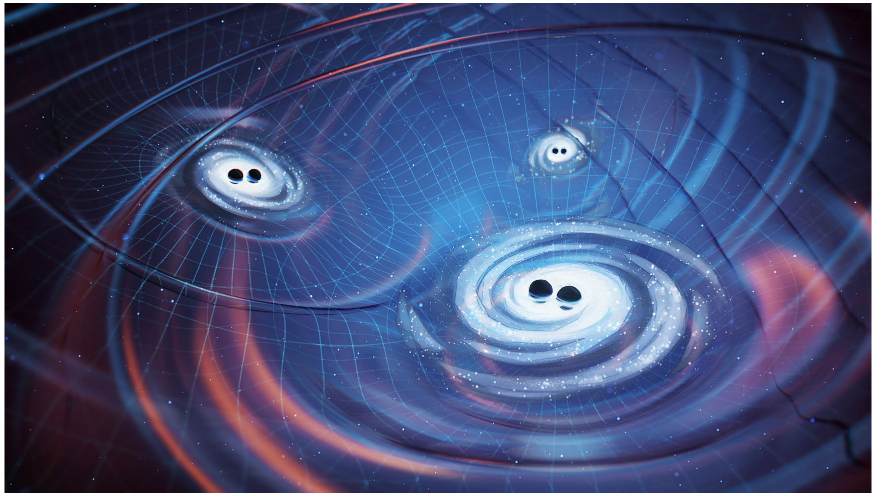
Gravity

Not quantum

Calculations MASSIVELY difficult!

- We know the theories are incomplete on both sides.

# Gauge and gravity theories



- **New calculational tools** are always needed for both gauge theory and gravity.
- For gauge theory, they are needed for **collider physics** experiments.
- In gravity, we need new results for **gravitational wave experiments.**
- We have candidates for quantum gravity (e.g. string theory, loop quantum gravity).
- But there are **unanswered questions**, and new insights to be gained, even in field theory.

Conceptual insights

Practical tools

# The double copy

- Recent years have seen intense activity in quantum field theory:

New ways of thinking about QFT

New calculational tools

Relationships between different theories

Applications to classical physics

- In this talk, I will review something called the **double copy**.
- It started in 2010 as a relationship between **scattering amplitudes** in gauge theories and gravity… (Bern, Carrasco, Johansson)
- …and was inspired by previous work in string theory (Kawai, Lewellen, Tye).
- Since then, the double copy has expanded to include **classical solutions** and beyond.

# Double copy research

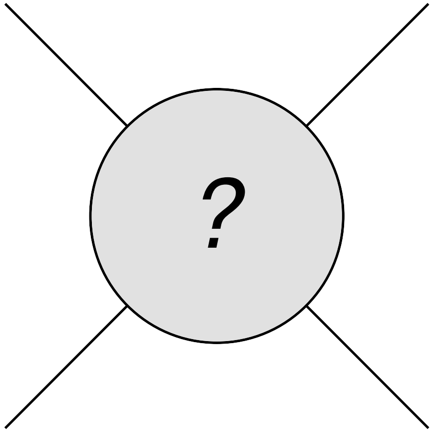
- Practical applications of the double copy include recycling gauge theory results for gravity (e.g. for gravitational waves).
- It also reveals **new mathematical structures** in gauge theory, of interest in their own right.
- Many more (types of) theories are related by similar correspondences.
- Possible links with many different areas:

Phenomenology
Theoretical Physics
Astrophysics
Cosmology
Pure Maths
Optics
Condensed Matter

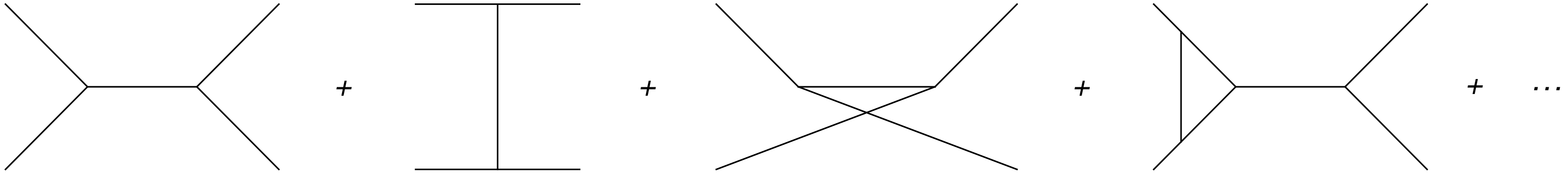
- Now is the perfect time to become involved! 😀

# Scattering amplitudes

- To understand the name **double copy**, let us review its origin.



- In a QFT, the **scattering amplitude** is a number related to particle interaction probabilities.
- Traditionally, we can represent amplitudes as sums of diagrams…
- …with precise **(Feynman) rules** that convert each diagram to algebra.



**VERTEX**
Power of coupling constant / momentum factors

**INTERNAL LINE**
Factors of $1/p^2$, where $p$ is the exchanged 4-momentum

**LOOPS**
Sum (integral) over undetermined momenta

# Gauge theory amplitudes

- An amplitude with *L* loops and *m* external particles will have form:

Coupling

Colour Factor

$$\mathcal{A}_m^{(L)} \sim g^{m-2+2L} \sum_i \int \left( \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \right) \frac{n_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2}.$$

Sum over diagrams

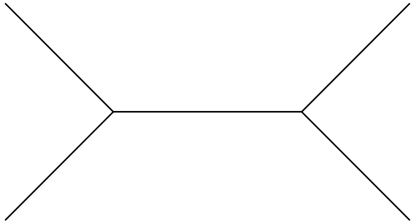
Internal Lines

Loop momenta

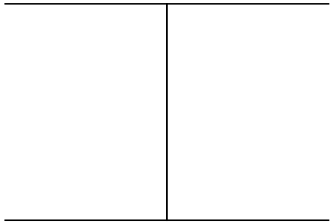
- Each term has a **kinematic numerator** $n_i$, that depends on momenta, polarisations etc.
- The set $\{n_i\}$ is not unique, but will change under **gauge transformations** or **field redefinitions** ("generalised gauge transformations").
- The kinematic numerators can be chosen to have a very special form.

# BCJ Duality

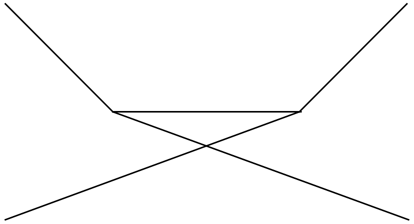
- Colour factors obey certain relations called **Jacobi identities**:



$c_s$



$c_t$



$c_u$

- In this example one can show $c_s + c_t - c_u = 0$ .
- The origin of these identities is the colour **Lie algebra** underlying the theory, and they generalise to higher legs / loops.
- Remarkably, the kinematic numerators $n_i$ can be chosen to obey similar identities.
- This is called **BCJ duality**, and implies there is an abstract **kinematic algebra** that somehow mirrors the colour one! (Bern, Carrasco, Johansson)

# The double copy

- If BCJ duality is manifest in a gauge theory amplitude, something amazing happens.
- Consider replacing the coupling of our gauge theory:

$$g \to \frac{\kappa}{2} = \sqrt{8\pi G_N}$$

Newton's constant

- Also replace colour with a second set of kinematic numerators: $c_i \to \tilde{n}_i$
- The resulting formula

$$\mathcal{M}_m^{(L)} \sim \left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_i \int \left(\prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D}\right) \frac{n_i \tilde{n}_i}{\prod_{\alpha_l} p_{\alpha_l}^2}$$

turns out to be a gravity amplitude !!!

- Which gravity theory you get depends upon the choice of gauge theories.

# Gravity vs. Gauge theory

- To see why this is so remarkable, note that the vertex rule for 3 gluons is:

$$g f^{abc}\left[\eta^{\mu\nu}(q_1-q_2)^{\rho}+\eta^{\mu\rho}(q_2-q_3)^{\nu}+\eta^{\rho\nu}(q_3-q_1)^{\mu}\right]$$

whilst the equivalent rule for gravitons has 170+ terms 🥺

$$\begin{aligned}
&U(q_1,q_2,q_3)_{\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3} = \\
&-\frac{K}{2}\Big[q^2_{(\alpha_1}q^3_{\beta_1)}\left(2\eta_{\alpha_2(\alpha_3}\eta_{\beta_3)\beta_2}-\tfrac{2}{d-2}\eta_{\alpha_2\beta_2}\eta_{\alpha_3\beta_3}\right)\\
&\quad+q^1_{(\alpha_2}q^3_{\beta_2)}\left(2\eta_{\alpha_1(\alpha_3}\eta_{\beta_3)\beta_1}-\tfrac{2}{d-2}\eta_{\alpha_1\beta_1}\eta_{\alpha_3\beta_3}\right)\\
&\quad+q^1_{(\alpha_3}q^2_{\beta_3)}\left(2\eta_{\alpha_1(\alpha_2}\eta_{\beta_2)\beta_1}-\tfrac{2}{d-2}\eta_{\alpha_1\beta_1}\eta_{\alpha_2\beta_2}\right)\\
&\quad+2q^3_{(\alpha_2}\eta_{\beta_2)(\alpha_1}\eta_{\beta_1)(\alpha_3}q^2_{\beta_3)}+2q^1_{(\alpha_3}\eta_{\beta_3)(\alpha_2}\eta_{\beta_2)(\alpha_1}q^3_{\beta_1)}+2q^2_{(\alpha_1}\eta_{\beta_1)(\alpha_3}\eta_{\beta_3)(\alpha_2}q^1_{\beta_2)}\\
&\quad+q^2\cdot q^3\left(\tfrac{2}{d-2}\eta_{\alpha_1(\alpha_2}\eta_{\beta_2)\beta_1}\eta_{\alpha_3\beta_3}+\tfrac{2}{d-2}\eta_{\alpha_1(\alpha_3}\eta_{\beta_3)\beta_1}\eta_{\alpha_2\beta_2}-2\eta_{\alpha_1(\alpha_2}\eta_{\beta_2)(\alpha_3}\eta_{\beta_3)\beta_1}\right)\\
&\quad+q^1\cdot q^3\left(\tfrac{2}{d-2}\eta_{\alpha_2(\alpha_1}\eta_{\beta_1)\beta_2}\eta_{\alpha_3\beta_3}+\tfrac{2}{d-2}\eta_{\alpha_2(\alpha_3}\eta_{\beta_3)\beta_2}\eta_{\alpha_1\beta_1}-2\eta_{\alpha_2(\alpha_1}\eta_{\beta_1)(\alpha_3}\eta_{\beta_3)\beta_2}\right)\\
&\quad+q^1\cdot q^2\left(\tfrac{2}{d-2}\eta_{\alpha_3(\alpha_1}\eta_{\beta_1)\beta_3}\eta_{\alpha_2\beta_2}+\tfrac{2}{d-2}\eta_{\alpha_3(\alpha_2}\eta_{\beta_2)\beta_3}\eta_{\alpha_1\beta_1}-2\eta_{\alpha_3(\alpha_1}\eta_{\beta_1)(\alpha_2}\eta_{\beta_2)\beta_3}\right)\Big]
\end{aligned}$$

- It is not at all obvious *a priori* that amplitudes in these theories should be related!

# Evidence for the double copy

- The double copy remains a conjecture, but with lots of good evidence:
    1. It is derivable from string theory at tree-level (Kawai, Lewellen, Tye). The field theory double copy **goes further** than this, though.
    2. It is tested up to four loops in N=4 SYM theory (Bern, Davies, Dennen, Smirnov$^2$). Also one-loop in **non-supersymmetric** theories.
    3. All-loop evidence in the **soft** and **high energy (Regge)** limits (Oxburgh, White; Akhoury, Sterman, Saotome; Johansson, Sabio Vera, Serna Campillo, Vazquez-Mozo).
    4. Exact Lagrangian understanding for self-dual theories (Monteiro, O'Connell).
    5. Loop-level insights from ambitwistor strings (Geyer, Mason, Monteiro, Tourkine).
    6. Perturbative arguments to all orders (Borsten, Kim, Jurčo, Macrelli, Saemann; Bern, Herrmann, Roiban, Ruf, Zeng).
- It also generalises to other types of theory…

# The zeroth copy

- In our gauge theory amplitude, we can also replace kinematics with colour:

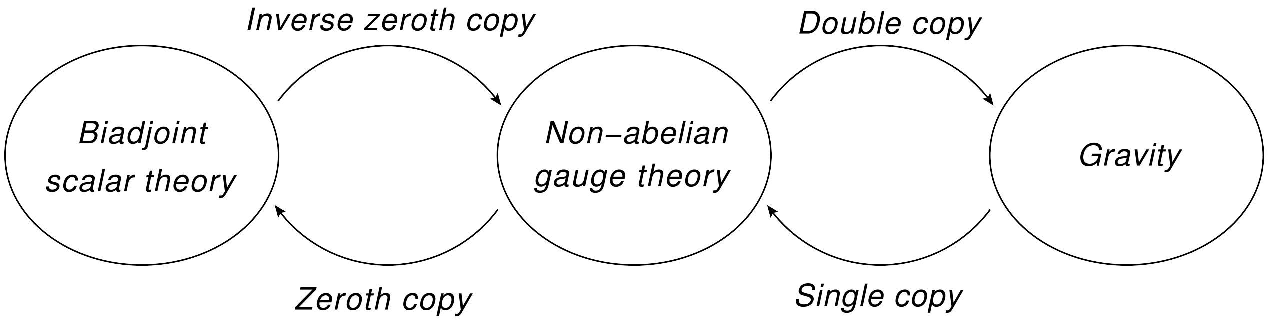
$$\bar{A}_m^{(L)} \sim y^{m-2+2L} \sum_i \int \left( \prod_{l=1}^{L} \right) \frac{d^D p_l}{(2\pi)^D} \frac{c_i \tilde{c}_i}{\prod_{\alpha_i} p_{\alpha_i}^2}.$$

- This is called the **zeroth copy.**
- Turns out to be an amplitude in **biadjoint scalar field theory**:

$$\partial^2 \Phi^{aa'} - y f^{abc} \tilde{f}^{a'b'c'} \Phi^{bb'} \Phi^{cc'} = 0.$$

- Not a physical theory by itself…
- …but properties are somehow inherited by real-world gauge / gravity theories!

# A trinity of theories



*Inverse zeroth copy*

*Double copy*

*Biadjoint scalar theory*

*Non–abelian gauge theory*

*Gravity*

*Zeroth copy*

*Single copy*

- We have seen that amplitudes in various theories are related by **copies***.*
- Current research has two main areas:

| **FUNDAMENTAL** | **APPLIED** |
|---|---|
| Where does the double copy come from? | New (quantum) gravity results. |
| What quantities does it relate? | Astrophysical applications. |
| Is it fully non-perturbative? | Cosmology? |
| How does it link with other theories (e.g. strings)? | Condensed matter / optics? |

# Beyond scattering amplitudes

- If the double copy applies to **complete theories**, we should be able to match up **classical solutions** of gauge / gravity theories.

- In gravity, we can define the **graviton field** via:

Graviton field

Full metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

Minkowski metric

- A **classical double copy** is then a way to match up solutions in biadjoint / gauge / gravity:

$$\Phi^{aa'} \leftrightarrow A^a_\mu \leftrightarrow h_{\mu\nu}$$

  which is somehow relatable to the double copy for amplitudes.

- In general this is only possible order-by-order in perturbation theory.

- Special cases allow for **exact statements**! 😀

# The Kerr-Schild double copy

- The first known classical double copy was the **Kerr-Schild double copy** (Monteiro, O'Connell, White).
- It relies on an infinite family of GR solutions (*Kerr-Schild solutions)* of form

$$h_{\mu\nu} = \phi k_\mu k_\nu, \quad k^2 = 0, \quad k \cdot \partial k^\mu = 0.$$

- Each such solution is characterised by a scalar field $\phi$ and 4-vector $k^\mu$.
- Can prove that stationary Kerr-Schild solutions have corresponding gauge / biadjoint fields:

$$A_\mu^a = c^a \phi k_\mu, \quad \Phi^{aa'} = c^a \tilde{c}^{a'} \phi.$$

- This procedure can be very closely related to the double copy for amplitudes (White, Monteiro, O'Connell).
- The Kerr-Schild family includes some of the most famous objects in GR!

# Example: the Schwarzschild black hole

- The Schwarzschild black hole can be written as

$$h_{\mu\nu} = \frac{\kappa}{2}\phi k_\mu k_\nu, \quad \phi = \frac{M}{4\pi r}, \quad k^\mu = (1, \mathbf{e}_r).$$

- The Kerr-Schild single copy gives an abelian-like gauge field

$$A^\mu = \frac{g c^a \mathbf{T}^a}{4\pi r}(1, \mathbf{e}_r),$$

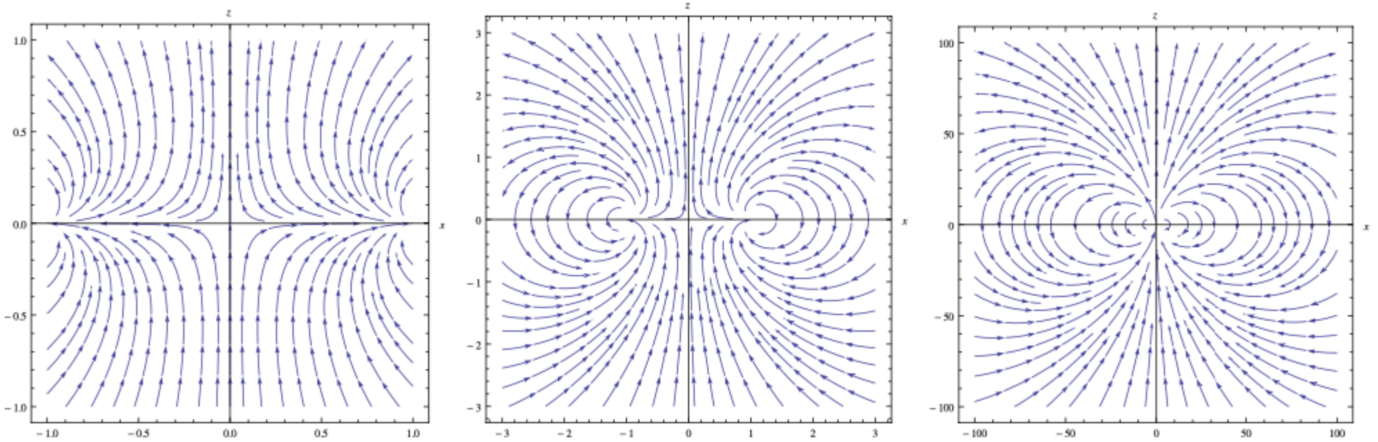
- …which can be gauge-transformed to give:

$$A^\mu = \left(\frac{g c^a \mathbf{T}^a}{4\pi r}, \mathbf{0}\right).$$

The single copy of a black hole is a point charge!

# Example: the Kerr black hole

- We can also apply this to the Kerr (rotating, uncharged) black hole.
- The single copy turns out to give a rotating disk of charge.



- This looks like a bar magnet at large distances, as it should!
- Further examples include accelerating black holes, de Sitter spacetime, multi-Kerr-Schild solutions (e.g. Taub-NUT) etc.

# The Weyl double copy

- A more general classical double copy is the **Weyl double copy** (Luna, Monteiro, Nicholson, O'Connell).
- It uses the so-called **spinorial formalism** of GR, in which tensors are converted to objects with multiple two-component indices $A, \dot{A} \in \{0,1\}$.
- Certain structures — and simplifications — in GR are only manifest in this formalism.
- Electromagnetic solutions are described by the **Maxwell spinor** $\Phi_{AB}$.
- (Vacuum) GR solutions are described by the **Weyl spinor** $\Psi_{ABCD}$.
- Then the **Weyl double copy** gives a relation

$$\Psi_{ABCD} = \frac{\Phi_{(AB}\Phi_{CD)}}{\underset{\text{Scalar Field}}{\phi}}$$

Scalar Field

# The Weyl double copy

- The Weyl double copy is **more general than** the Kerr-Schild approach, but is precisely relatable where they overlap.
- It can be **proven**, where it applies, using twistor theory (White), which also shows it is more general than previously thought! (Chacón, Nagy, White).
- Twistor methods link the Weyl double copy to scattering amplitudes (Guevara; Luna, Moynihan, White), and allow generalisation to other theories (Armstrong-Williams, White).
- A three-dimensional variant called the **Cotton double copy** exists (Carrillo González, Momeni, Rumbutis; Emond, Moynihan), and has a twistor derivation (Carrillo González, Emond, Moynihan, Rumbutis, White).
- One can also include non-trivial source terms, of astrophysical relevance (Easson, Manton, Svesko; Armstrong-Williams, Moynihan, White).

# The convolutional double copy

- A third classical double copy writes arbitrary gravitons as **convolutions** of gauge / scalar fields (Anastasiou, Borsten, Duff, Hughes, Inverso, Nagy):

$$H_{\mu\nu} = A^{I}_{\mu} \star \Phi^{II'} \star \tilde{A}^{I'}_{\nu},$$

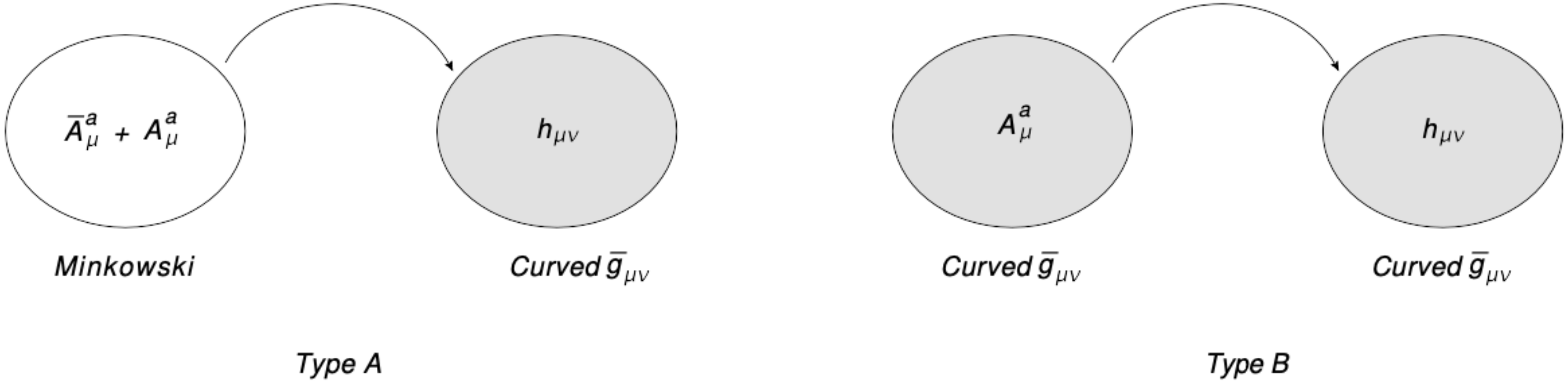
  where

$$[f \star g](x) = \int d^D y f(y) g(x-y).$$

- Motivated by the fact that the amplitude double copy involves products in momentum space.
- A catalogue of exotic (super-)gravity double copies has been obtained.
- This statement also works in arbitrary gauges, by matching up ghosts as well as physical degrees of freedom (Anastasiou, Borsten, Duff, Nagy, Zoccali; Luna, Nagy, White).

# The double copy in curved space

- Can consider classical double copies on curved backgrounds (Bahjat-Abbas, Luna, White; Carrillo González, Penco, Trodden):



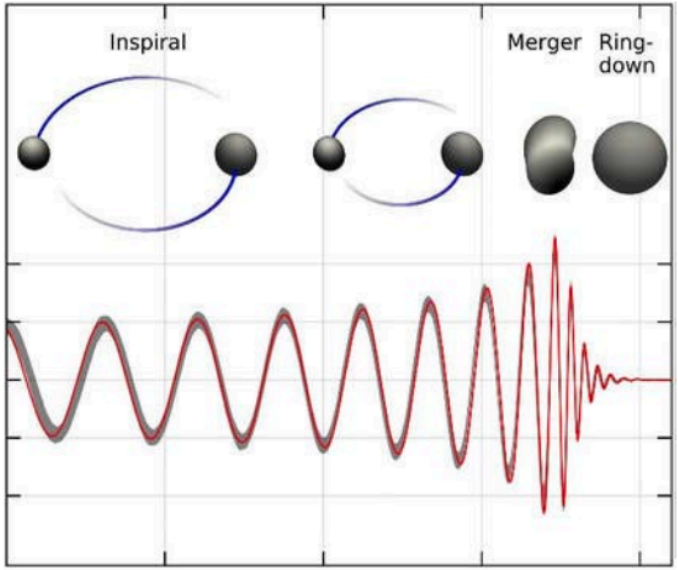
- Also for scattering amplitudes. (Armstrong, Lipstein, Mei; Lipstein, Nagy)
- Has also been explored in ambitwistor string theory (Adamo, Casali, Mason, Nekovar).
- Recent 3D study could make contact with AdS / CFT (Beetar, Carrillo González, Jaitly, Keseman).
- Possible astrophysics / cosmology applications?

# Perturbative classical double copies

- We do not know how to double copy arbitrary solutions exactly.
- But we do not always need to!
- One may consider more complicated solutions, at the price of working order-by-order in perturbation theory (Monteiro, Luna, Nicholson, O'Connell, White; Goldberger, Ridgway; Goldberger, Prabhu, Thompson).
- This is an off-shell generalisation of the BCJ copy for amplitudes.
- The double copy of pure YM theory gives GR plus an axion and a dilaton…
- …and care is needed in removing unwanted degrees of freedom (Johansson, Ochirov).
- Has been used for astrophysical applications.

# Gravitational Waves

- The study of gravitational waves is a hot topic since the recent LIGO discoveries of e.g. black hole mergers.

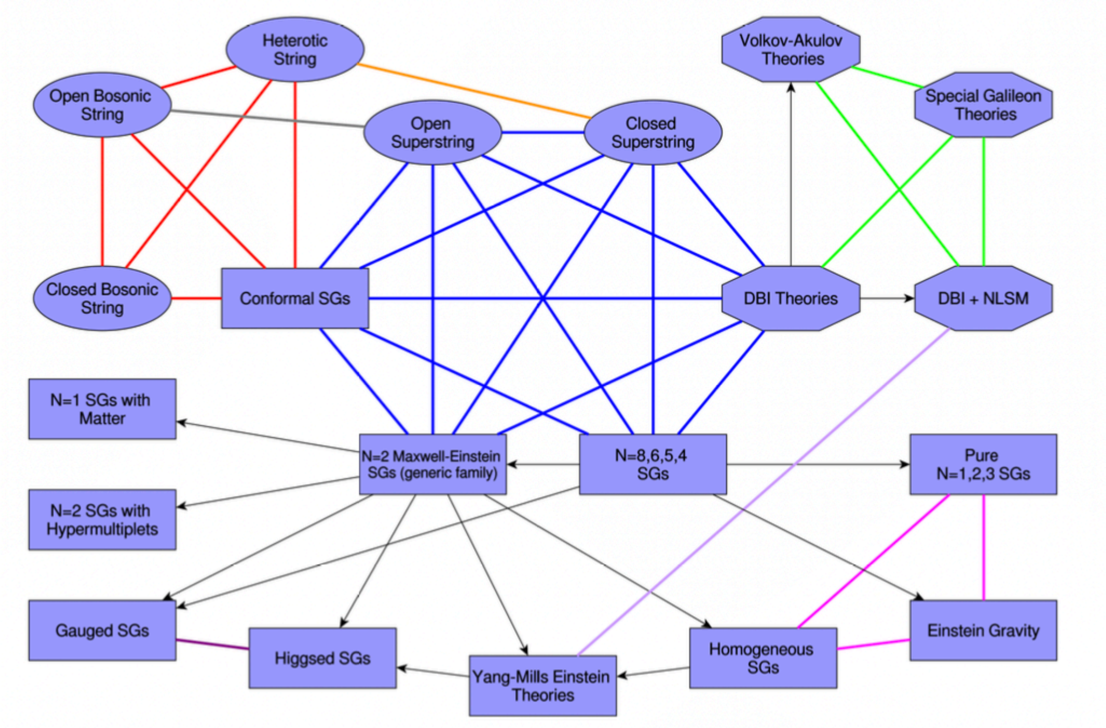


1. *Inspiral*: the black holes orbit closer and closer;
2. *Merger*: they collide and coalesce;
3. *Ring-down*: resulting black hole wobbles and then settles down.

- Experimentalists need precise predictions for the inspiral phase from classical GR.
- The double copy can help! (Kosower, Maybee, O'Connell; Bern, Cheung, Roiban, Shen, Solon, Zeng, Plefka, Steinhoff, Wormsbecher).

# Towards a non-perturbative double copy

- The three theories (biadjoint, gauge, gravity) considered so far are the tip of the iceberg.
- There is a **vast web** of QFTs & string theories related by “double-copies”:



- Tantalising hints of a common structure underlying all of this…
- … that our textbook methods are hiding.
- Is this “thing” non-perturbative?
- How do we find out?

# Strong coupling solutions

- One idea is to find **strong coupling** solutions of biadjoint scalar field theory, and then try and match them up with gauge / gravity objects (solitons e.g. instantons, monopoles).

- Some candidate solutions have been found (Armstrong-Williams, Bahjat-Abbas, de Smet, Stark Muchão, White, Wikeley).

- They look like monopoles and vortices:

$$\Phi^{aa'} = \frac{1}{yr^2}\left[-k\left(\delta^{aa'} - \frac{x^a x^{a'}}{r^2}\right) \pm \sqrt{2k - k^2}\,\frac{\epsilon^{aa'd}x^d}{r}\right].$$

- Not yet clear what to do with them!

- Related questions: how do we double-copy general instantons (Armstrong-Williams, Berman, Chacón, Luna, White, Wikeley)? Any friendly soliton people want to help?

# Symmetries

- Various authors have matched up symmetries between gauge / gravity theories.

- For example, **electromagnetic duality** in gauge theory maps to **Ehlers transformations** in gravity (Alawadhi, Berman, Spence, Veiga; Huang, Kol, O'Connell):

Electric charge

EM Duality

Magnetic charge

Mass

Ehlers

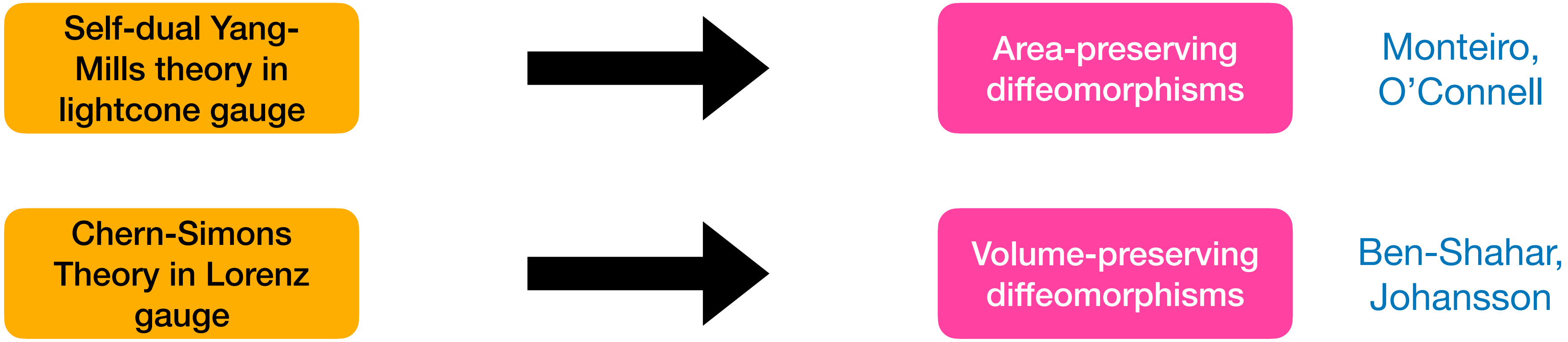
NUT charge

- Finding similar ideas may lead to new solution-generating methods in GR…

- …or reveal new mathematical structures.

# Kinematic algebras

- One new type of mathematical structure that we already know exists is the **kinematic algebra**.
- Earlier we saw that kinematic numerators $n_i$ in amplitudes can be chosen to obey similar Jacobi identities to the colour factors $c_i$.
- This **BCJ duality** suggests there is a **kinematic algebra**, that somehow mirrors the colour algebra.
- For some very special cases, this is known to be a Lie algebra:



Self-dual Yang-Mills theory in lightcone gauge → Area-preserving diffeomorphisms — Monteiro, O'Connell

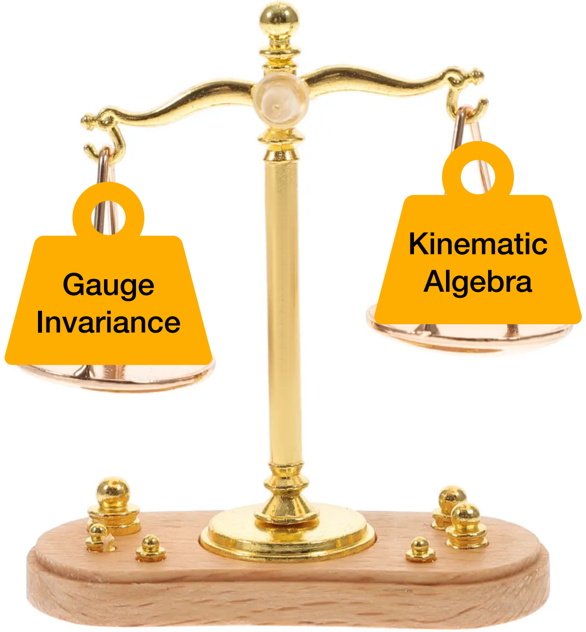
Chern-Simons Theory in Lorenz gauge → Volume-preserving diffeomorphisms — Ben-Shahar, Johansson

# General kinematic algebras

- More generally, kinematic algebras are not expected to be Lie algebras…
- …but more general mathematical structures.
- A particular candidate is a so-called $\mathrm{BV}^{\square}_{\infty}$ algebra, or **homotopy algebra** (Reiterer; then Borsten, Jurco, Kim, Macrelli, Saemann, Wolf; Bonezzi, Chiaffrino, Diaz-Jaramilo, Hohm, Plefka).
- There are higher-order brackets that generalise the Lie bracket of gauge fields.
- Jacobi identities are satisfied only up to terms involving higher-order brackets, leading to an intricate structure of constraints.
- Somehow, this must all reduce to a straightforward Lie algebra in those cases where this is possible.

# Why do we care?

- Kinematic algebras remain mysterious.
- But they are relatively **new**, **exciting** and certainly **fun** 😀!
- New way to think about gauge theories?
- If so, we should explore all ways of gaining intuition about the many open questions.



Which theories have kinematic algebras?

Are previous known cases related?

How do kinematic algebras depend upon the gauge?

When do we get a Lie algebra?

How do we visualise kinematic algebras geometrically?

# Beyond fibre bundles?

- The 20th century gave us a wonderful link between **physics** and **pure mathematics**:

GAUGE FIELDS

CONNECTIONS ON PRINCIPAL FIBRE BUNDLES

- If we make kinematic algebras manifest, gauge invariance takes a back seat.
- Is there some alternative mathematical object, that kinematic algebras act on?
- If so, can gauge theory be applied to open problems in **algebraic geometry**?
- Can this in turn provide the key to the structure underlying all copiable QFTs?

# Kinematic algebras - recent developments

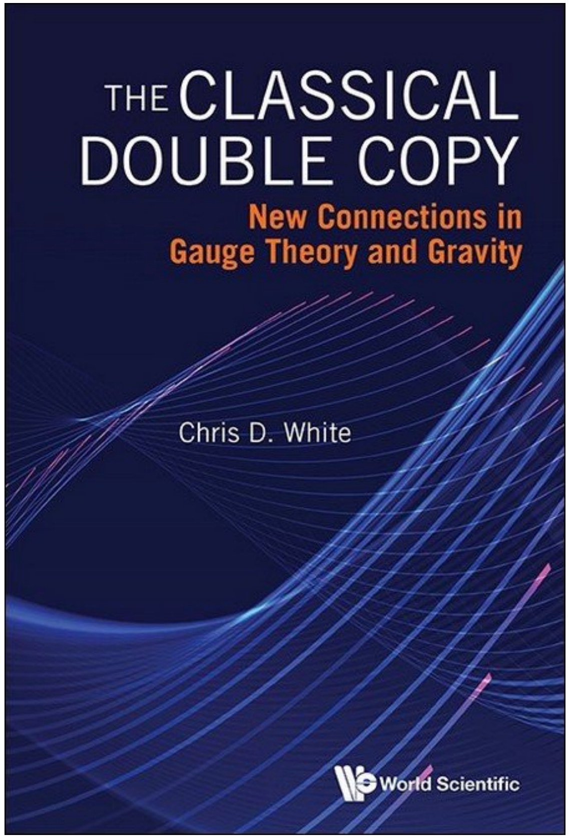
- The $\mathrm{BV}^{\square}_{\infty}$ approach gives a simple Lie algebra for Chern-Simons theory in any gauge (Bonezzi, Chiaffrino, Diaz-Jaramillo, Hohm).
- Also applied to self-dual theory: not a simple Lie algebra in arbitrary gauges! (Bonezzi, Diaz-Jaramillo, Nagy).
- Abelian gauge theories can be used to provide physical / geometric insights (Armstrong-Williams, Nagy, White, Wikeley).
- Hints that kinematic algebras may show up in fluid mechanics (Cheung, Mangan; Tong; Sheikh-Jabbari, Taghiloo, Vahidinia).
- Further work in this area will require **good translation skills** between (very!) pure maths, and high energy physics.

# Conclusions

- The **double copy** is a set of relations between (quantum) field theories.
- It applies to **scattering amplitudes**, **classical solutions**, and possible **non-perturbative** information.
- Both **conceptual** and **practical** applications.
- Many different theories seem to share a common underlying structure…
- …suggesting our traditional QFT methods are not the right ones!
- The double copy creates links with many different (sub)-disciplines.
- The subject is still young - new ideas are possible and welcome!

# Find out more

- There is a recently published monograph (plus more books by others on the way).



THE CLASSICAL DOUBLE COPY

New Connections in Gauge Theory and Gravity

Chris D. White

World Scientific

- My hope is that this is accessible to both astrophysicists and high energy physicists…
- …thus helping us to speak the same language.
- There is also now an annual conference series (**QCD Meets Gravity**) devoted to the double copy and related topics.

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# Thanks for listening!