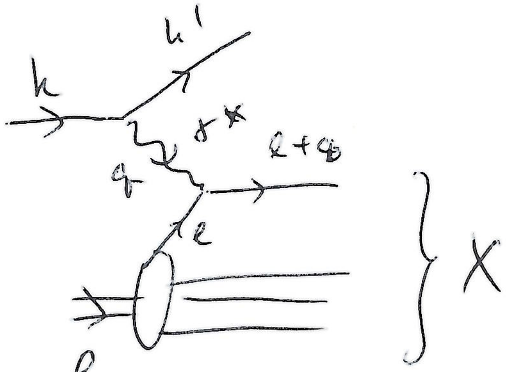
① IPPP Lectures: "Small x and Saturation"

Deep Inelastic Scattering



$Q^2 \equiv -q^2 \geq 0.$

$W^2 \equiv (p+q)^2$ ~ proton + γ* CMS energy

$$x \equiv \frac{Q^2}{2p\cdot q} = \frac{Q^2}{W^2 + Q^2 - M_p^2}$$

Bjorken x variable

use light-cone variables: $v^\pm = \dfrac{v^0 \pm v^3}{\sqrt{2}}$

$$\begin{array}{cccc} & + & - & \perp \\ p^\mu = (& p^+, & \frac{M^2}{2p^+}, & \vec{0}_\perp) \\ q^\mu = (& -\frac{Q^2}{2q^-}, & q^-, & \vec{0}_\perp) \end{array}$$

← frame choice.

q^- is large, Q^2 is large

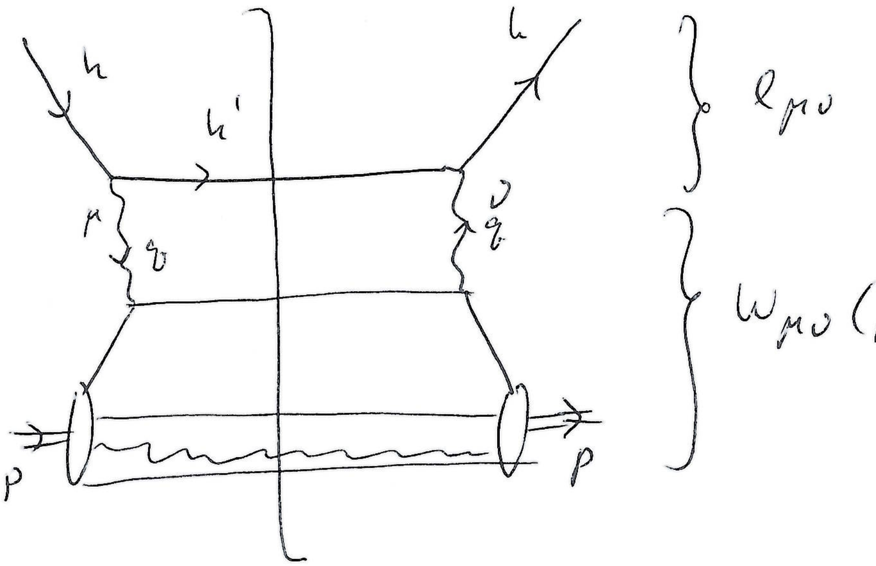
$$0 = (\ell+q)^2 = \ell^2 + \underbrace{2\ell\cdot q}_{2\ell^+q^- - \frac{Q^2}{q^-}\ell^-} + Q^2 \simeq 2\ell^+q^- - Q^2$$

$$\Rightarrow \ell^+ = \frac{Q^2}{2q^-} \Rightarrow \frac{\ell^+}{p^+} = \frac{Q^2}{2p^+q^-} \simeq \frac{Q^2}{2p\cdot q} = x$$

$$\Rightarrow x = \frac{\ell^+}{p^+}$$

~ fraction of the LC proton's momentum carried by struck quark.

Square the handbag diagram:



$\Rightarrow$ can write the cross section as

$$\boxed{\frac{d\sigma}{d^3k'} = \frac{\alpha_{EM}^2}{Q^2 E E'} l_{\mu\nu} W^{\mu\nu}}$$

(rest frame of the proton)

where the hadronic tensor is

$$\boxed{W_{\mu\nu} \equiv \frac{1}{4\pi M_p} \int d^4x \, e^{iq\cdot x} \langle P | j_\mu(x) \, j_\nu(0) | P \rangle}$$

$j_\mu(x)$ = EM current

~ all QCD interactions are in $W_{\mu\nu}$.

$$x = \frac{Q^2}{W^2 + Q^2 - M_p^2} \underset{\substack{\uparrow \\ W^2 \gg Q^2, M_p^2}}{\simeq} \frac{Q^2}{W^2} \Rightarrow$$

High energy W^2
$\Updownarrow$
Low x

Dipole Picture of DIS

$$W_{\mu\nu} = \frac{1}{4\pi M_p} \int d^4x \, e^{i q\cdot x} \langle P | j_\mu(x) j_\nu(0) | P \rangle$$

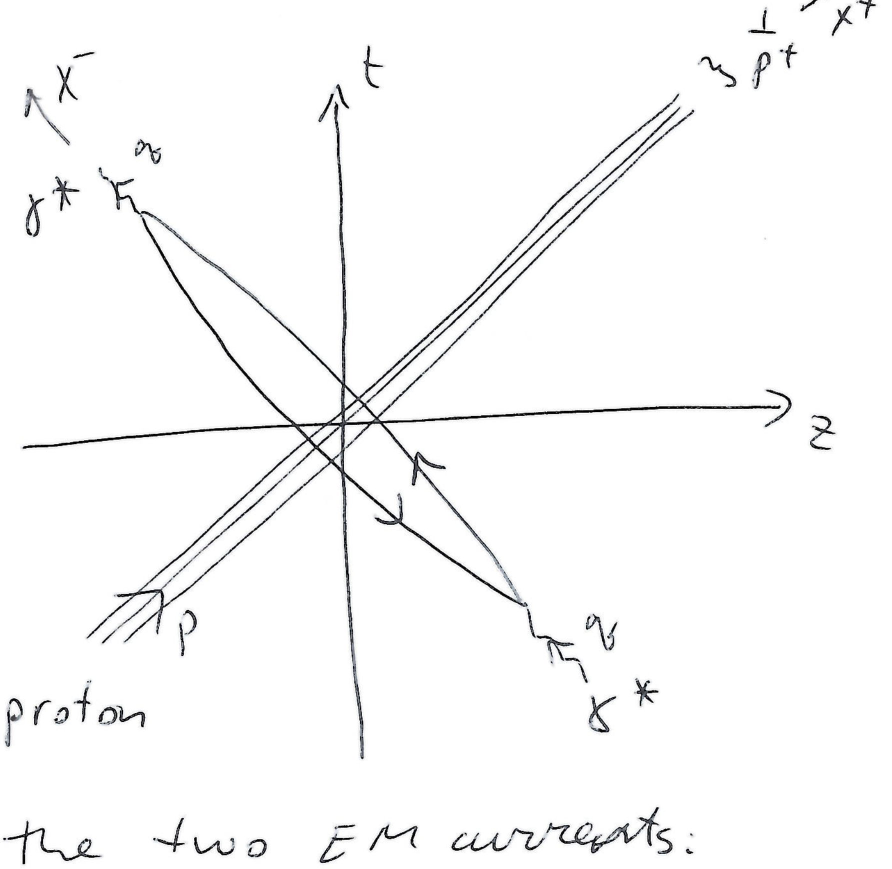
as $q^\mu = \left(-\frac{Q^2}{2q^-}, q^-, \vec{0}_\perp \right)$ $\Rightarrow$

$$\Rightarrow W_{\mu\nu} = \frac{1}{4\pi M_p} \int d^4x \, e^{i\left(q^- x^+ - \frac{Q^2}{2q^-} x^-\right)} \langle P | j_\mu(x) j_\nu(0) | P \rangle$$

Let's visualize the process in space-time:

proton extent in x^- direction is:

$\sim r_p \frac{m}{p^+} \simeq \frac{1}{p^+}$.



Separation between the two EM currents:

$x^- \approx \frac{2q^-}{Q^2}$ $\Rightarrow$ $\frac{2q^-}{Q^2} \gtrless \frac{1}{p^+}$ $\Rightarrow 1 \gtrless \frac{Q^2}{2p^+q^-} = x$

$\uparrow$ uncertainty principle

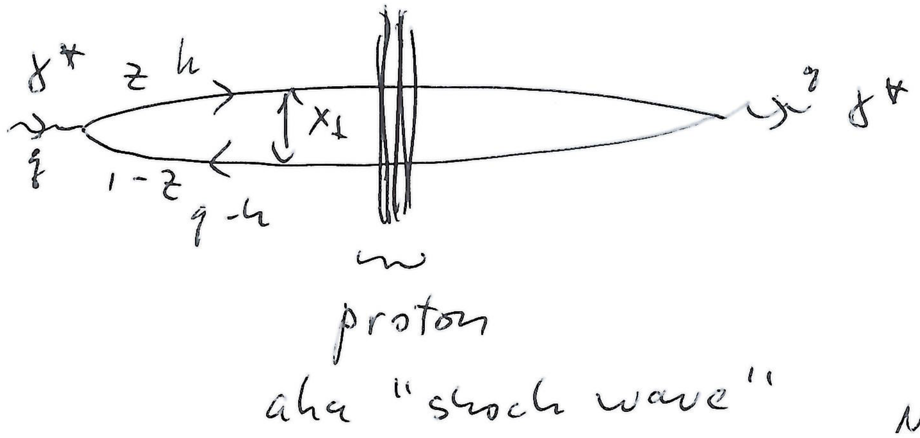
$\Rightarrow$ if $x \ll 1 \Rightarrow \boxed{\frac{2q^-}{Q^2} \gg \frac{1}{p^+}}$

④

$\Rightarrow \gamma^*$ splits into a $q\bar{q}$ pair long before interacting with the proton, and, for forward amplitude, they $\underbrace{}_{(q\bar{q})}$ merge back into γ^* long after the interaction.

($W_{\mu\nu} = 2\,\mathrm{Im}(i T_{\mu\nu})$, $T_{\mu\nu}$ = same as $W_{\mu\nu}$ but with T-product)

$\Rightarrow$



Gribov 1970

Bjorken & Kogut 1973

Frankfurt, Strikman 1988

Mueller 1990

Nikolaev & Zakharov 1991

total DIS cross section is

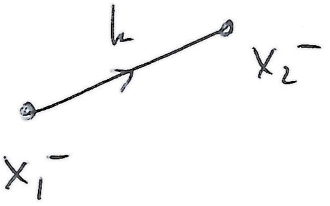
$$\boxed{\sigma^{\gamma^* p} = \int \frac{d^2 x_\perp}{4\pi} \int_0^1 \frac{dz}{z(1-z)} \left| \Psi^{\gamma^* \to q\bar{q}}(\vec{x}_\perp, z) \right|^2 \cdot \sigma^{q\bar{q}N}(x_\perp, s)}$$

$z = \dfrac{k^-}{q^-}$ ~ light-cone momentum fraction
$0 < z < 1$

$x_\perp$ = transverse size of the dipole.

$\psi^{\gamma^* \to q\bar{q}}(x_\perp, z)$ = virtual photon's LC wave function.

Take a scalar propagator: Fourier-transform into x^- space.



$$\int_{-\infty}^{\infty} \frac{dk^+}{2\pi} e^{-ik^+(x_2^- - x_1^-)} \frac{i}{k^2 - m^2 + i\varepsilon} =$$

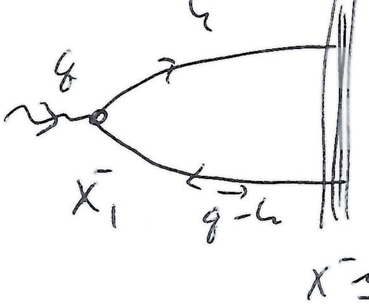
$$= \theta(x_2^- - x_1^-) \frac{1}{2k^-} \theta(k^-) e^{-i \frac{k_\perp^2 + m^2}{2k^-}(x_2^- - x_1^-)}$$

$$- \theta(x_1^- - x_2^-) \frac{1}{2k^-} \theta(-k^-) e^{-i \frac{k_\perp^2 + m^2}{2k^-}(x_2^- - x_1^-)}$$

$$= e^{-i \frac{k_\perp^2 + m^2}{2k^-}(x_2^- - x_1^-)} \frac{1}{2k^-} \Big[\theta(x_2^- - x_1^-)\, \theta(k^-) - \theta(x_1^- - x_2^-) \cdot \theta(-k^-)\Big]$$

~ different x^- orderings come in with different contributions; particle is on mass shell.

know that $x_1^- < 0$ (restricted)!



$$\int_{-\infty}^{0} dx_1^- \, e^{i\left(\frac{k_\perp^2 + m^2}{2k^-} + \frac{(\vec{q}_\perp - \vec{k}_\perp)^2 + m^2}{2(q^- - k^-)}\right)x^-} \cdot e^{-i\left(\frac{-Q^2}{2q^-}\right)x^-} \cdot e^{\varepsilon x_1^-}$$

← external phase ← reg.

$$= \frac{-i}{\underbrace{\frac{k_\perp^2 + m^2}{2k^-} + \frac{(\vec{q}_\perp - \vec{k}_\perp)^2 + m^2}{2(q^- - k^-)} + \frac{Q^2}{2q^-} - i\varepsilon}_{\text{energy denominator}}}$$

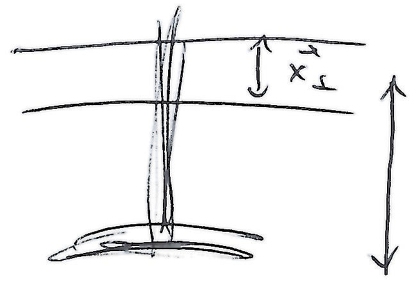
Particles are on mass shell, but the "+" momentum component is not conserved.

$\Rightarrow \psi^{\gamma^* \to q\bar{q}}(x_\perp, z)$ is well known (at LO & NLO).

$\sigma^{q\bar{q}N} =$ total cross section of a dipole scattering on the target.

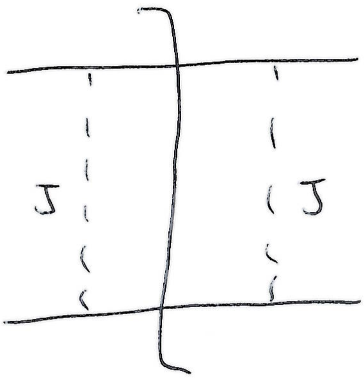
$N(\vec{x}_\perp, \vec{b}_\perp, s) =$ (Im part of) dipole forward scattering amplitude

$$\boxed{\sigma^{q\bar{q}N} = 2 \int d^2 b_\perp \, N(\vec{x}_\perp, \vec{b}_\perp, s)}$$ Optical Theorem



$\vec{b}_\perp \sim$ impact parameter.

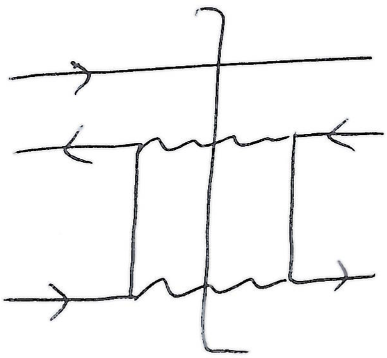
How does the dipole interact with the proton?



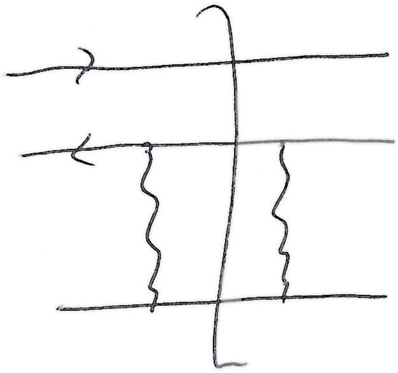
$\Rightarrow$ $\boxed{\sigma \sim s^{2(J-1)}}$ ~ high-s limit

J = spin of exchanged particle

$\Rightarrow$



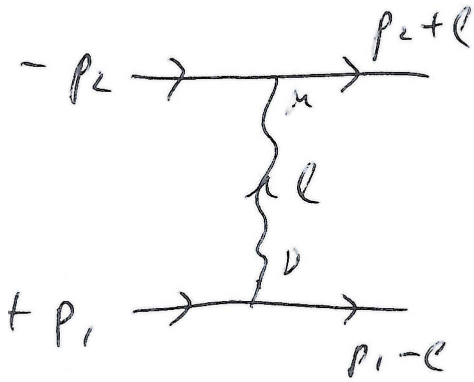
$$\sigma \sim s^{2\left(\frac{1}{2}-1\right)} \sim \frac{1}{s} \sim \text{decreases with } s$$



$$\sigma \sim s^{2(1-1)} = s^0 = 1$$

$\Rightarrow$ σ = const with energy

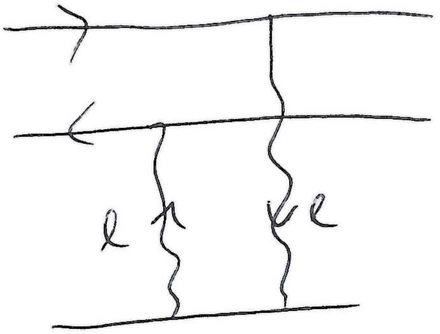
$\Rightarrow$ gluon exchange dominates.



$$0=(p_2+\ell)^2 \simeq 2p_2^-\ell^+ \Rightarrow \ell^+ \simeq 0$$

$$0=(p_1-\ell)^2 \simeq -2p_1^+\ell^- \Rightarrow \ell^- \simeq 0$$

$$\Rightarrow \frac{-i g_{\mu\nu}}{\ell^2} \simeq \frac{i}{\ell_\perp^2} g_{\mu\nu} \sim \text{Coulomb (Glauber) gluon.}$$



$+ \dots \quad \propto \int_\Lambda \frac{d^2 l_\perp}{(l_\perp^2)^2}\left(2 - e^{i\vec{l}_\perp\cdot\vec{x}_\perp} - e^{-i\vec{l}_\perp\cdot\vec{x}_\perp}\right)$

$$\propto x_\perp^2 \ln\left(\frac{1}{x_\perp \Lambda}\right)$$

$$\Rightarrow \sigma^{q\bar{q}N} = \frac{\pi \alpha_s^2 C_F}{N_c} x_\perp^2 \ln\frac{1}{x_\perp^2 \Lambda^2} = \frac{\alpha_s \pi^2}{N_c} x_\perp^2 \underbrace{x G_N}_{x \text{ times gluon PDF}}$$

more careful calculation with proton = quark model.

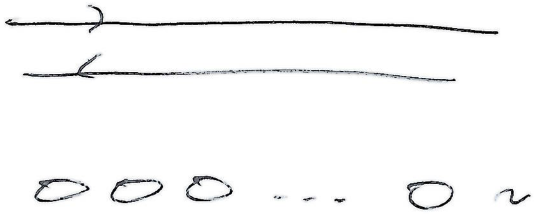
nucleus

$$\frac{d\sigma^{q\bar{q}A}}{d^2 b} = T(\vec{b}_\perp)\,\sigma^{q\bar{q}N} = 2N(\vec{x}_\perp, \vec{b}_\perp, s)$$

$$T(\vec{b}) \equiv \int_{-\infty}^{\infty} dz\, \rho(\vec{b}_\perp, z)$$ ~ nuclear profile function

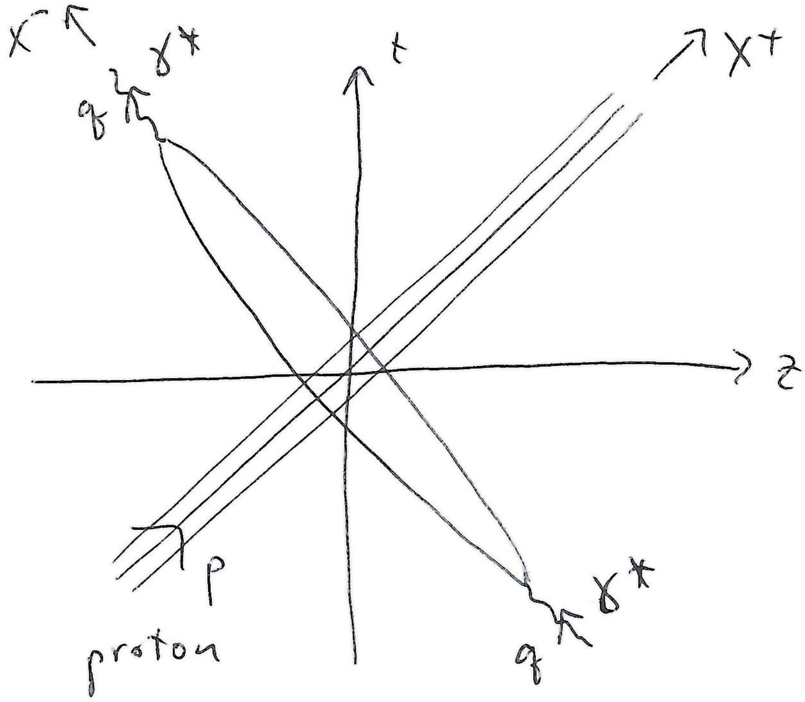
ρ = # density of nucleons, approximately constant.

$T(\vec{b}) \propto A^{1/3}$

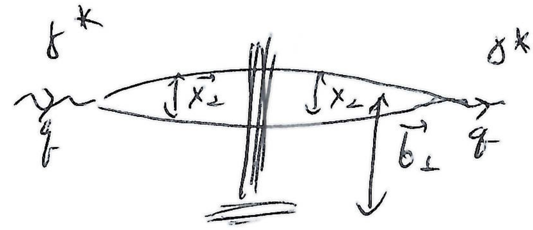


~ many ($\sim A^{1/3}$) nucleons at fixed $\vec{b}_\perp$.

Last time Dipole Picture of DIS



~ DIS at small x



$$\boxed{\sigma^{\gamma^* p} = \int \frac{d^2x_\perp}{2\pi} d^2b_\perp \int_0^1 \frac{dz}{z(1-z)} \left| \Psi^{\gamma^*\to q\bar{q}}(\vec{x}_\perp, z)\right|^2 N(\vec{x}_\perp, \vec{b}_\perp, s)}$$

s = CMS energy of $\gamma^* + p$ system

$\Psi^{\gamma^* \to q\bar{q}}$ ~ LC wavefunction, well-known

$$N(\vec{x}_\perp, \vec{b}_\perp, s) = \frac{\alpha_s \pi^2}{2N_c} T(\vec{b}_\perp)\, x_\perp^2\, xG_N\left(x, \frac{1}{x_\perp^2}\right)$$

$$= \frac{\alpha_s^2 \pi C_F}{N_c} T(\vec{b}_\perp)\, x_\perp^2 \ln\left(\frac{1}{x_\perp \Lambda}\right).$$

$$T(\vec{b}_\perp) = \int_{-\infty}^{\infty} dz\, \rho(\vec{b}_\perp, z) \quad \Rightarrow \quad T(\vec{b}_\perp) \sim A^{1/3}$$

$\sim$ # nucleons at a given impact parameter

density of nucleons

$$\Rightarrow \quad N \propto x_\perp^2 \Lambda^2 \cdot A^{1/3}$$

$\sim$ can get large for large A and $x_\perp$.

$$\Rightarrow \quad N(\vec{x}_\perp, \vec{b}_\perp, s) = \frac{1}{2} T(\vec{b}_\perp)\, \sigma^{q\bar{q}N}$$

$\Rightarrow$ for a 2-gluon exchange

$$N(\vec{x}_\perp, \vec{b}_\perp, s) = \frac{\alpha_s \pi^2}{2N_c} T(\vec{b}_\perp)\, x_\perp^2\, x G_N(x, 1/x_\perp^2)$$

with $x G_N(x, Q^2) = \frac{\alpha_s C_F}{\pi} \ln \frac{Q^2}{\Lambda^2}$

for proton = quark.

Unitarity & Black Disk Limit

$$|\psi_f\rangle = \hat{S}\,|\psi_i\rangle = |\psi_i\rangle + \underbrace{(\hat{S} - \mathbb{1})}_{i\hat{T}}|\psi_i\rangle$$

total cross section:

$$\sigma_{tot} \propto \left|(\hat{S} - \mathbb{1})|\psi_i\rangle\right|^2 = 2 - S - S^*$$

$$S = \langle \psi_i |\, \hat{S}\, | \psi_i \rangle, \quad \underbrace{\hat{S}^\dagger \hat{S} = \hat{S}\hat{S}^\dagger = \mathbb{1}}_{\text{unitarity}}$$

Elastic cross section:

$$\sigma_{el} \sim \left| \langle \psi_i | (\hat{S} - \mathbb{1}) | \psi_i \rangle \right|^2 = |1 - S|^2$$

inelastic cross section:

$$\sigma_{inel} = \sigma_{tot} - \sigma_{el} \propto 1 - |S|^2 .$$

We write:

$$\boxed{\begin{aligned} \sigma_{tot} &= 2 \int d^2 b_\perp \left[1 - \operatorname{Re} S \right] \\ \sigma_{el} &= \int d^2 b_\perp \left| 1 - S(\vec{b}_\perp) \right|^2 \\ \sigma_{inel} &= \int d^2 b_\perp \left[1 - |S(\vec{b}_\perp)|^2 \right] \end{aligned}}$$

$$1 = \langle \psi_i | \hat{S}^{+} \hat{S} | \psi_i \rangle = \sum_X \langle \psi_i | \hat{S}^{+} | X \rangle \langle X | \hat{S} | \psi_i \rangle$$

$$\geq |S|^2 \Rightarrow \boxed{|S| \leq 1}$$ $\sim$ unitarity.

If $S = -1 \Rightarrow \sigma_{tot} = 4\pi R^2 = \sigma_{el}$, $\sigma_{inel} = 0$

$\sim$ low-energy limit.

High energy: require that $\sigma_{inel} \geq \sigma_{el}$

$$\Rightarrow \text{Re}\, S \geq 0 \quad \Rightarrow \quad \sigma_{tot} = 2\int d^2b_\perp \left[1 - \text{Re}\, S\right] \leq 2\underbrace{\int d^2b}_{2\pi R^2}$$

$$\Rightarrow \boxed{\sigma_{tot} \leq 2\pi R^2}$$ ~ black disk limit

$\sigma_{el} = \sigma_{inel} = \pi R^2$ ~ in the black disk regime

(Note that σ_{el} is 50% of the total cross section.)

We had $\sigma^{q\bar{q}A} = 2\int d^2b_\perp \, N(\vec{x}_\perp, \vec{b}_\perp, s)$

$$\Rightarrow \boxed{N = 1 - \text{Re}\, S}$$ ~ Im part of the forward T-matrix element

$\Rightarrow$ if $\text{Re}\, S \geq 0 \Rightarrow \boxed{N \leq 1}$ ~ unitarity constraint.

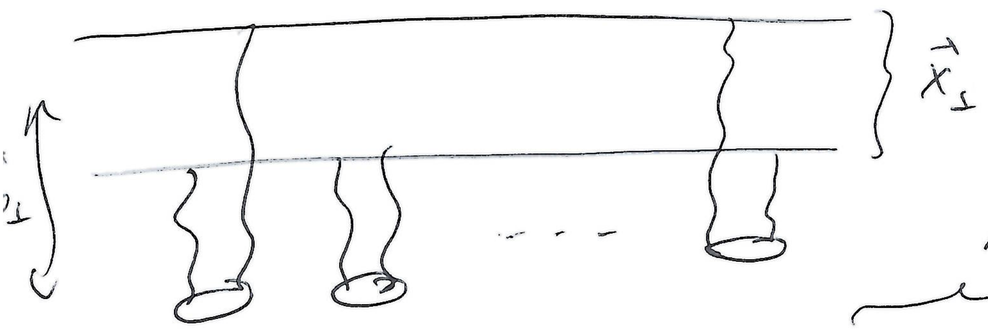
$$N^{2\text{-gluons}}(\vec{x}_\perp, \vec{b}_\perp, s) = \frac{\alpha_s \pi^2}{2N_c} T(\vec{b}_\perp)\, x_\perp^2 \times G_N\!\left(x, \frac{1}{x_\perp^2}\right)$$

$\Rightarrow N^{2\text{-gluons}} \propto A^{1/3} x_\perp^2$ ~ grows with A and with $x_\perp$

$\Rightarrow$ N 2-gluons can violate unitarity ($\Rightarrow$ bad!).

Gribov-Glauber-Mueller Picture

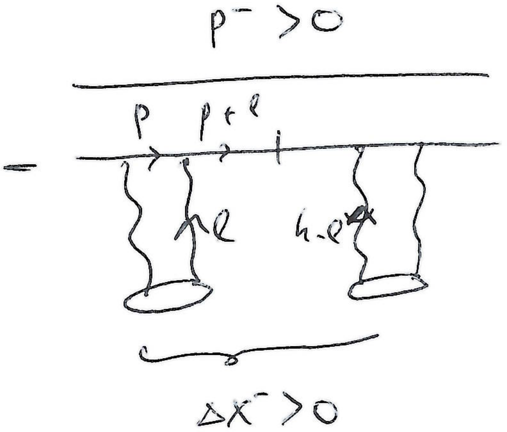
If the interaction with 1 nucleon becomes strong $\Rightarrow$ need to account for multiple interactions:



$$N(\vec{x}_\perp, \vec{b}_\perp, s) = 1 - \underbrace{e^{-\frac{1}{2} \overbrace{T(\vec{b})\, \sigma^{q\bar{q}N}}^{N_{2G} = N\ 2\text{-gluons}}}}_{\substack{\text{multiple exchanges} \\ \text{simply exponentiate}}}$$

$$\Rightarrow \boxed{N(\vec{x}_\perp, \vec{b}_\perp, s) = 1 - e^{-\frac{\alpha_s \pi^2}{2N_c} x_\perp^2 \, x G_N\left(x, \frac{1}{x_\perp^2}\right)}}$$

GGM formula (Mueller, 1990)



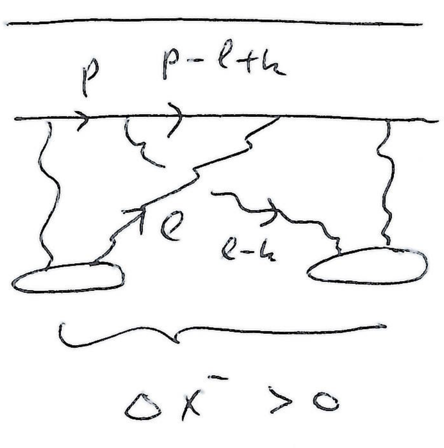
$$\Rightarrow \propto \int_{-\infty}^{\infty} \frac{dl^+}{2\pi} e^{-il^+ \Delta x^-} \frac{i}{(p+l)^2 + i\varepsilon}$$

$$\approx \int_{-\infty}^{\infty} \frac{dl^+}{2\pi} e^{-il^+ \Delta x^-} \frac{i}{2p^- l^+ + i\varepsilon} =$$

$$= \frac{1}{2p^-} \quad \Rightarrow \text{pick the pole,}$$

putting $(p+l)^2 = 0 \Rightarrow$ intermediate quark goes on mass shell.

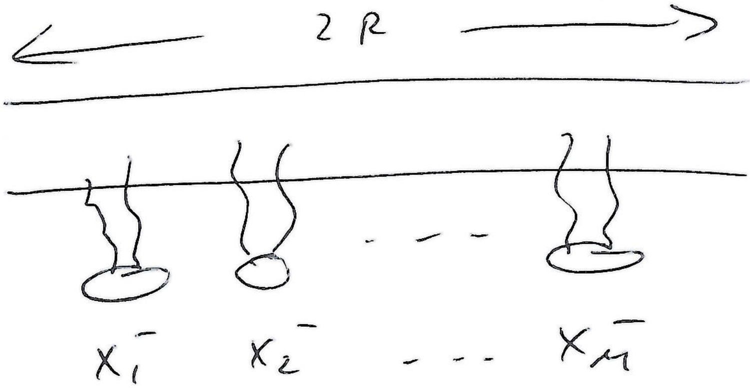
<u>Out of order</u>:



$$\propto \int_{-\infty}^{\infty} \frac{dl^+}{2\pi} e^{-il^+ \Delta x^-} \frac{i}{(p-l+k)^2 + i\varepsilon}$$

$$= \int_{-\infty}^{\infty} \frac{dl^+}{2\pi} \underset{\substack{\uparrow \\ \Delta x^- > 0 \\ \Downarrow \\ \text{close in lower half-plane}}}{e^{-il^+ \Delta x^-}} \frac{i}{\underbrace{-2p^- l^+ + \ldots + i\varepsilon}_{\text{pole in upper half-plane}}} = 0.$$

For M scatterings: $N_{LO} = \frac{1}{2} T(\vec{b}) \sigma^{q\bar{q}N}$ 12''



$= \cancel{(N_{LO})^M}$

$$= \int_{-R}^{R} dx_1^- \int_{x_1^-}^{R} dx_2^- \cdots \int_{x_{M-1}^-}^{R} dx_M^- \; \rho(\vec{b}, x_1^-)\, \rho(\vec{b}, x_2^-) \cdots$$

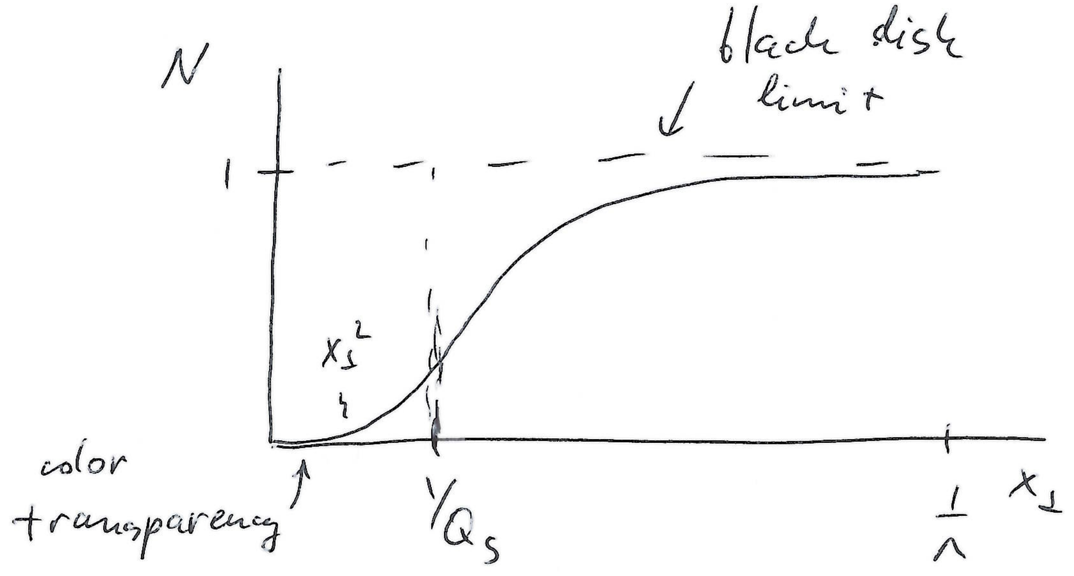
$$\cdots \rho(\vec{b}, x_M^-) \cdot \left(\frac{-1}{2} \sigma^{q\bar{q}N}\right)^M = \frac{1}{M!} \left[\int_{-R}^{R} dx^- \rho(\vec{b}, x^-)\right]^M \cdot$$

$$\cdot \left(\frac{1}{2}(-) \sigma^{q\bar{q}N}\right)^M = \frac{1}{M!} \left[-\frac{1}{2} \sigma^{q\bar{q}N} T(\vec{b})\right]^M =$$

$$= \frac{1}{M!} \left[-N_{LO}\right]^M \Rightarrow \sum_{M=0}^{\infty} \frac{1}{M!} \left[-N_{LO}\right]^M = e^{-N_{LO}} = S'$$

$$\Rightarrow S' = 1 - N \Rightarrow N = 1 - e^{-N_{LO}}.$$

$$N(x_\perp, \vec{b}_\perp, s) = 1 - e^{-\frac{\alpha_s^2 C_F \pi}{2N_c} T(\vec{b}_\perp)\, x_\perp^2 \ln \frac{1}{x_\perp^2 \Lambda^2}}$$



Note that now $N \le 1$ $\Rightarrow$ unitarity is not violated, even for large A and $x_\perp$ (as long as $x_\perp \ll \frac{1}{\Lambda}$).

$x_\perp \sim$ small $\Rightarrow N \sim x_\perp^2$ (color transparency)

$x_\perp \sim$ large $\Rightarrow N \simeq 1$ (black disk limit)

$\Rightarrow$ transition between the two regimes happens at $x_\perp \simeq \frac{1}{Q_s}$ with $Q_s^2 = \frac{4\pi\alpha_s^2 C_F}{N_c} T(\vec{b}_\perp)$

such that $N(\vec{x}_\perp, \vec{b}_\perp) = 1 - e^{-\frac{1}{4} x_\perp^2 Q_s^2 \ln\left(\frac{1}{x_\perp \Lambda}\right)}$

saturation scale

$$Q_s^2 \propto T(\vec{b}_\perp) \propto A^{1/3} \Rightarrow \boxed{Q_s^2 \propto A^{1/3}}$$

$\Rightarrow$ for large nuclei, $Q_s^2 \gg \Lambda^2 \Rightarrow$

transition to saturation regime is

perturbative. $\Rightarrow$ large nuclei are perturbative at high energies!

Unitarity is not violated due to

this saturation regime.

Small-x Evolution

The above calculation does not include

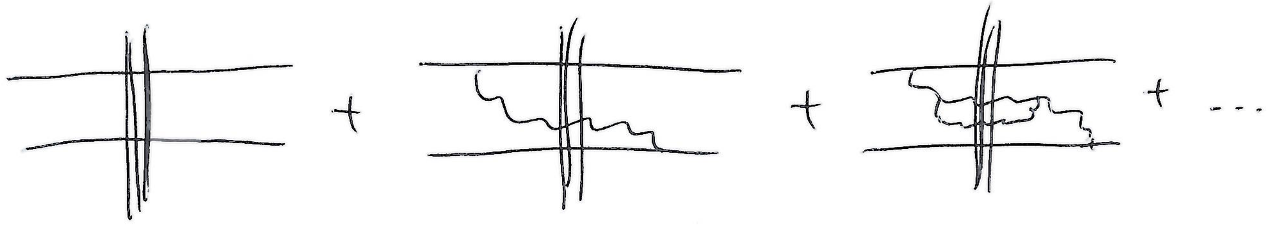
any energy dependence in the resulting

cross section. Energy dependence comes in

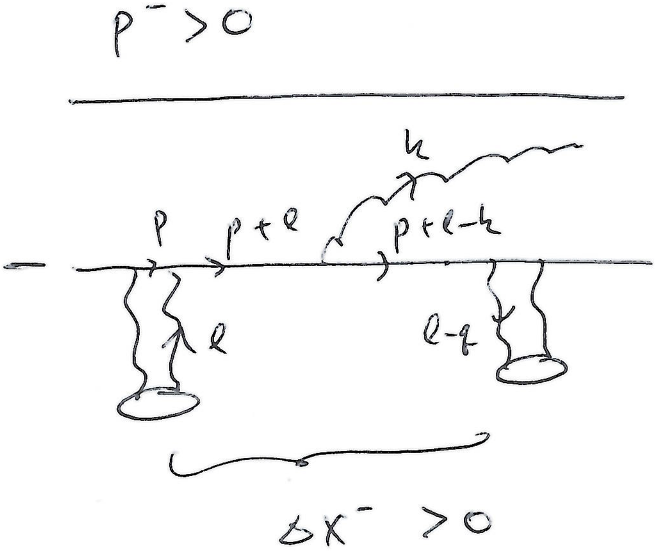
through quantum corrections, which bring

in powers of $\alpha_s \ln \frac{1}{x}$. Those are given by

the long-lived s-channel gluons:



There are no emissions from inside the shock wave: those are energy suppressed.



$$\propto \int\limits_{-\infty}^{\infty} \frac{d\ell^+}{2\pi}\, e^{-i\ell^+\Delta x^-} \frac{1}{(p+\ell)^2+i\varepsilon} \cdot \frac{1}{(p+\ell-k)^2+i\varepsilon} \approx$$

$$\approx \int\limits_{-\infty}^{\infty} \frac{d\ell^+}{2\pi}\, e^{-i\ell^+\Delta x^-} \frac{1}{2p^-\ell^+ + i\varepsilon} \; \frac{1}{2p^-(\ell^+-k^+)+\ldots+i\varepsilon}$$

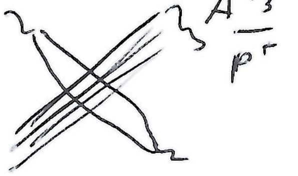
$$\approx -i\,\frac{1}{2p^-} \cdot \left[\frac{1}{-2p^-k^+} + \frac{1}{2p^-k^+}\, e^{-ik^+\Delta x^-}\right] =$$

$$= -i\,\frac{1}{2p^-} \cdot \frac{1}{2p^-k^+} \underbrace{\left[e^{-ik^+\Delta x^-} - 1\right]}_{\approx -ik^+\Delta x^- \simeq -i\frac{k^+}{p^+} \ll 1.} \quad \text{as } \Delta x^- \simeq \frac{1}{p^+}$$

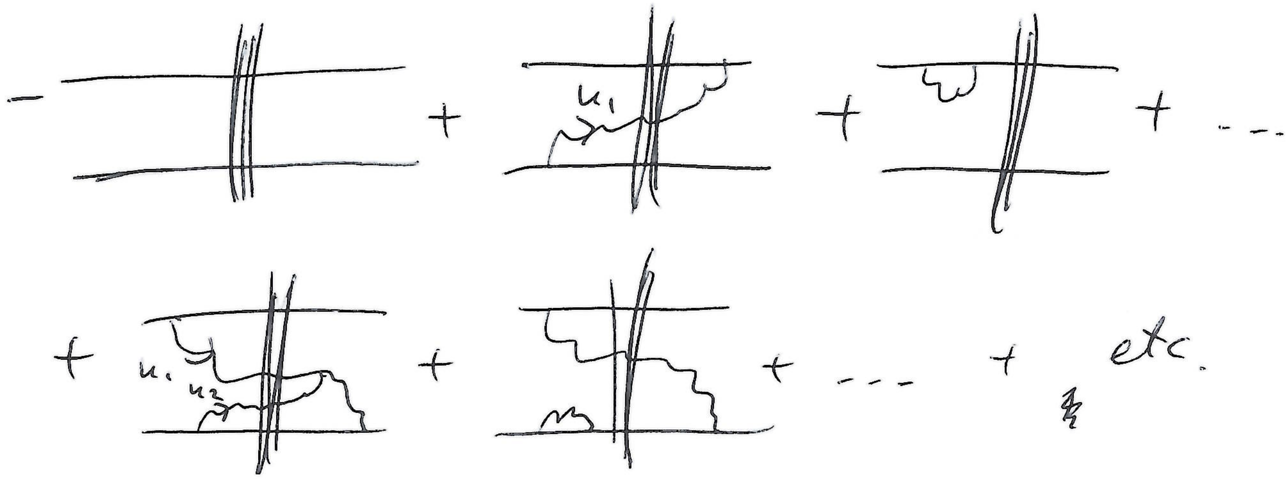
in fact, since you have $\propto A^{1/3}$ nucleons,

one gets up to $\frac{k^+}{p^+} A^{1/3} = x\, A^{1/3} \ll 1 \Rightarrow x \ll A^{-1/3}$

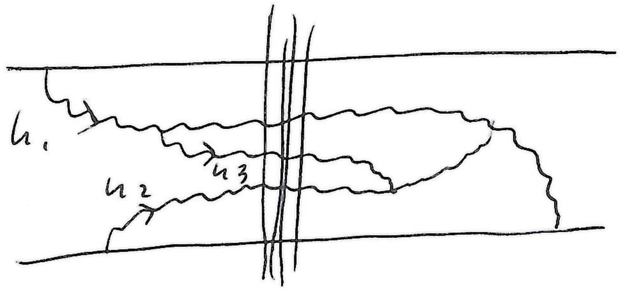
c.f. DIS estimate: $\frac{2q^-}{Q^2} \gg \frac{1}{p^+} A^{1/3}$ ← now for nucleus $\Rightarrow 1 \gg x A^{1/3}$ ↑ same



$\Rightarrow$ We need to sum up an ∞ cascade of long-lived gluons:



To give us powers of $\alpha_s \ln \frac{1}{x}$, the gluons "−" momenta have to be ordered,



$$k_1^- \gg k_2^- \gg k_3^- \gg \dots$$

Their transverse momenta are comparable,

$$k_{1\perp} \sim k_{2\perp} \sim \dots \sim k_{N\perp} \sim \dots$$

$\Rightarrow$ life-times are ordered:

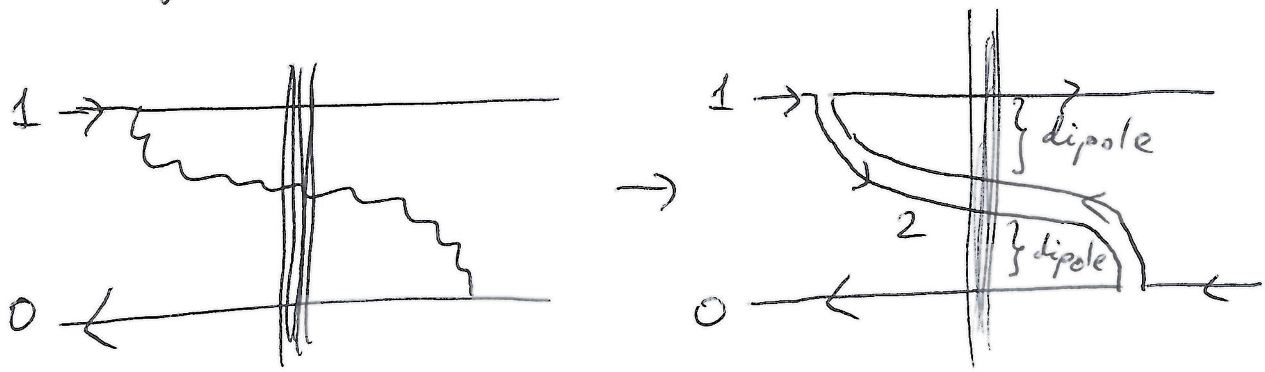
$$\boxed{\frac{2k_1^-}{k_{1\perp}^2} \gg \frac{2k_2^-}{k_{2\perp}^2} \gg \frac{2k_3^-}{k_{3\perp}^2} \gg \dots .}$$

$\Rightarrow$ still, hard to resum a gluon cascade due to color factors 't Hooft 1977

Large-N_c limit: 

$$N_c \otimes \bar{N}_c = 1 \oplus (N_c^2 - 1) \approx N_c^2 - 1 .$$

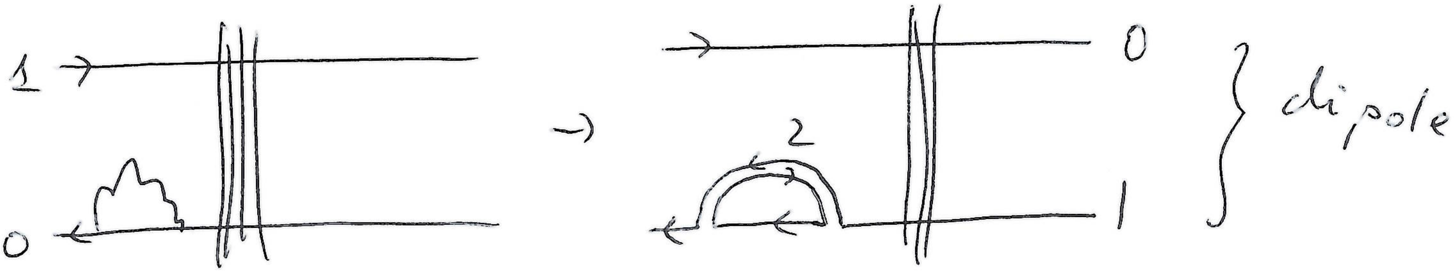
Replace the gluon by a $q\bar{q}$ pair. Only planar diagrams contribute:



Mueller's dipole model   Mueller 1993

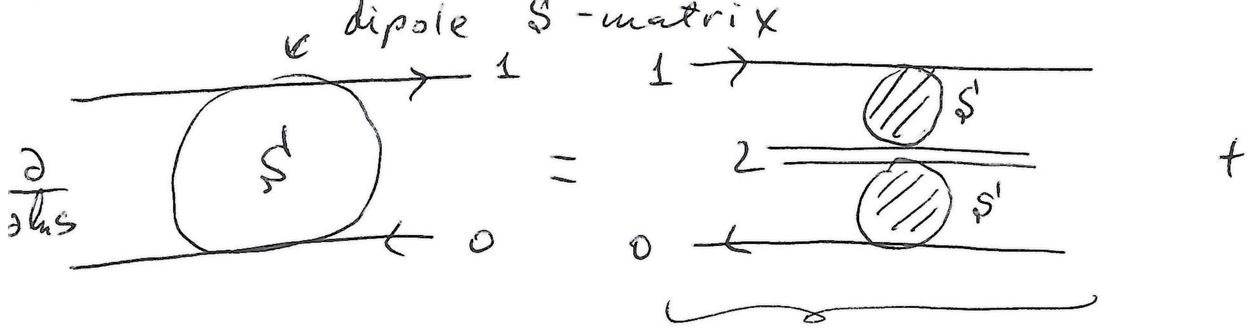
In one step of evolution a dipole may split into two dipoles: $01 \to 02 + 21$ (above).

There are also virtual corrections:

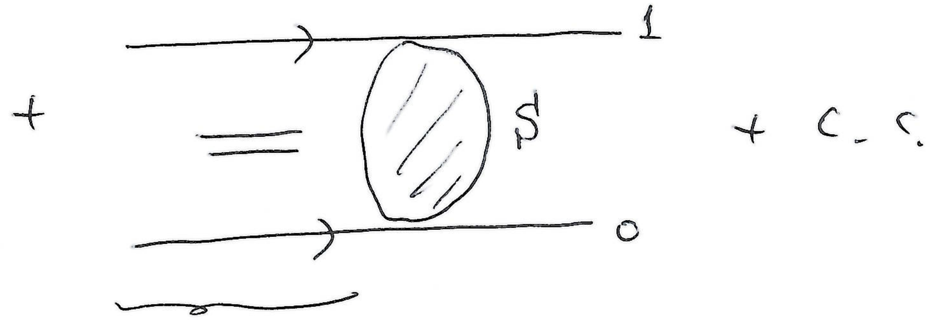


~ still have only 1 dipole going through the shock wave.

We are ready to write down a diagrammatic representation of resummation:



all diagrams with the gluon crossing the shock wave



virtual diagrams

Performing a diagrammatic calculation, we get ($Y = \ln s/\Lambda^2$)

$\vec{x}_{10} = \vec{x}_1 - \vec{x}_0$

$x_{10} = |\vec{x}_{10}|$, etc.

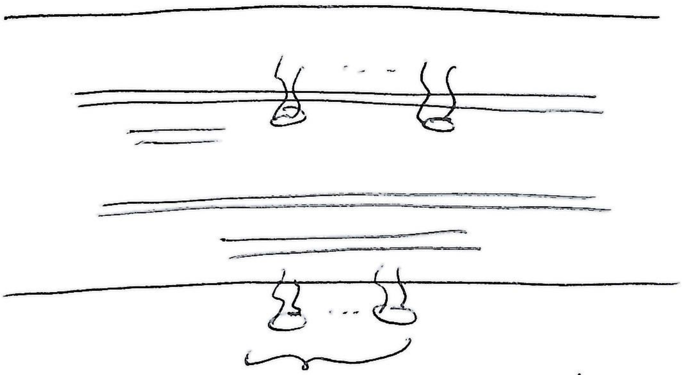
$$\boxed{\frac{\partial}{\partial Y} S_{10}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \, \frac{x_{10}^2}{x_{21}^2 x_{20}^2} \left[S_{21}(Y) \, S_{20}(Y) - S_{10}(Y) \right]}$$

$S_{10} = 1 - N_{10} \Rightarrow$

$$\boxed{\frac{\partial}{\partial Y} N_{10}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \, \frac{x_{10}^2}{x_{21}^2 x_{20}^2} \Big[\underbrace{N_{21}(Y) + N_{20}(Y) - N_{10}(Y)}_{\text{BFKL}} - N_{21}(Y) \, N_{20}(Y) \Big]}$$

Balitsky '96; YK '99

Initial condition $\underbrace{N_{10}(Y=Y_0) = 1 - e^{-\frac{1}{4} x_\perp^2 Q_{s0}^2 \ln\frac{1}{x_\perp \Lambda}}}_{\text{GGM f-la.}}$



each dipole interacts with the target via multiple rescatterings. $\sim (\alpha_s^2 A^{1/3})^{\cdots}$ powers

Evolution: resums powers of $\alpha_s N_c Y \sim \alpha_s N_c \ln\frac{1}{x}$.

initial conditions resum powers of $\alpha_s^2 A^{1/3}$.

$\sim$ Beyond large-N_c: JIMWLK functional differential / integral equation.

(small correction to N compared to BK, $< 0.1\%$)

BK solution $\sim$ see slides.

$Q_s(Y) \propto e^{\Delta Y}$ $\sim$ saturation scale <u>grows</u> with energy (with $x \to 0$).

$\Rightarrow$ $\boxed{Q_s^2(x) \propto A^{1/3} \left(\frac{1}{x}\right)^{\Delta}}$ $\Rightarrow$ the higher the energy (the smaller the x) and/or the larger the nucleus, the more perturbative interactions get.