UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia Galileo Galilei

INFN PADOVA

Solving the Strong CP Problem in String-Inspired Theories with Modular Invariance

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Based on **2505.20395** with **F. Feruglio, A. Marrone** and **A. Strumia**
also **2305.08908** with **FF** and **AS** and **2406.01689** with **FF, AS** and **M. Parriciatu**

PASCOS 2025
Durham, UK, 21 July 2025

30th International Symposium on Particles, Strings and Cosmology

Finanziato dall'Unione europea
NextGenerationEU

Ministero dell'Università e della Ricerca

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PIANO NAZIONALE DI RIPRESA E RESILIENZA

Outline

1. Strong CP problem
2. Existing solutions
3. Modular invariance and global supersymmetry
4. Example model
5. Conclusions

The strong CP problem

$$\mathscr{L}_{\text{QCD}} = \overline{q} i \not{D} q - \left[\overline{q_R} M_q q_L + \text{h.c.} \right] - \frac{1}{4 g_3^2} G^a_{\mu\nu} G^{a,\mu\nu} + \frac{\theta_{\text{QCD}}}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a,\mu\nu}$$

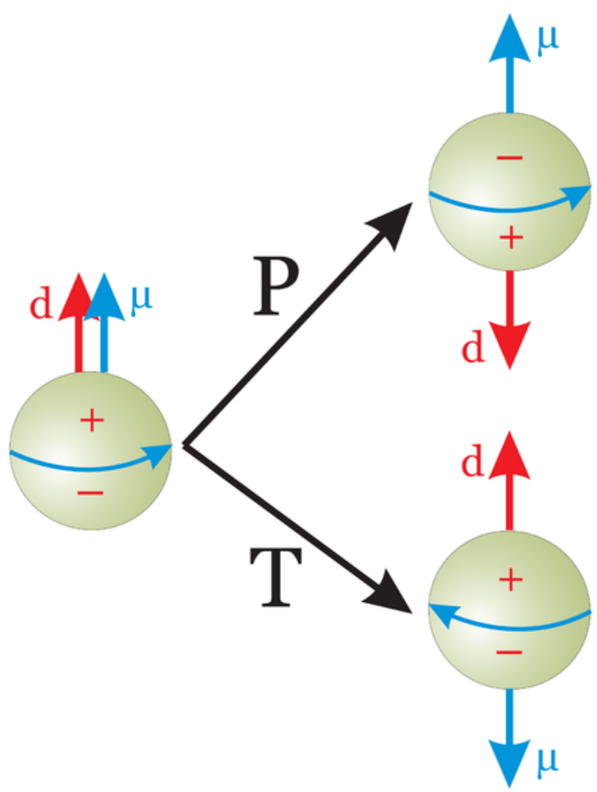
$$\boxed{\bar{\theta} = \theta_{\text{QCD}} + \arg \det M_q}$$ CPV parameter

The strong CP problem

$$\mathscr{L}_{\rm QCD} = \bar{q} i \not{D} q - \left[\overline{q_R} M_q q_L + \text{h.c.} \right] - \frac{1}{4 g_3^2} G^a_{\mu\nu} G^{a,\mu\nu} + \frac{\theta_{\rm QCD}}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a,\mu\nu}$$

$$\bar{\theta} = \theta_{\rm QCD} + \arg\det M_q$$ CPV parameter

Neutron EDM d



$d = 2.4 \times 10^{-16}\, \bar{\theta}\, e \cdot \text{cm}$ Pospelov, Ritz, hep-ph/9908508v4

$|d| \leq 1.8 \times 10^{-26}\, e \cdot \text{cm}$ (90% C.L.) Abel et al., 2001.11966

$$|\bar{\theta}| \lesssim 10^{-10}$$

Why so small???

The strong CP problem

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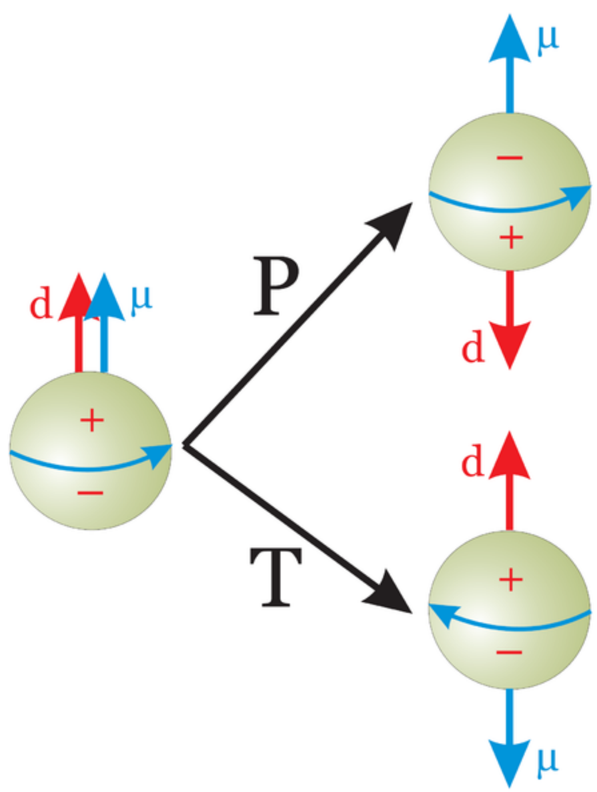

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Why so small???

… and the CPV phase in the CKM matrix $\delta_{\text{CKM}} \approx 1.2$

Solution 1: the Axion

Promote $\bar{\theta}$ to a dynamical scalar field a, the axion, which washes out CP violation in QCD

$$\mathscr{L}_a = \frac{1}{2}(\partial_\mu a)^2 + \frac{1}{32\pi^2}\frac{a}{f_a}G\tilde{G} + \dots$$

$$\bar{\theta} = \frac{\langle a \rangle}{f_a} \qquad \text{with} \qquad \langle a \rangle = 0$$

New global $\mathrm{U}(1)_{\mathrm{PQ}}$

Peccei, Quinn, PRL **38** (1977) 1440; PRD **16** (1977) 1791

- spontaneously broken => the axion is a NGB
- anomalous under QCD $(\partial_\mu J^\mu_{\mathrm{PQ}} \propto G\tilde{G})$ => the axion is a pNGB

Quality problem

- Corrections of order $(f_a/M_{\mathrm{Pl}})^\#$ from higher-dimensional operators
- $\mathrm{U}(1)_{\mathrm{PQ}}$ should be an accidental symmetry in a complete model

Solution 2: CP (P) is symmetry of UV

- CP (P) is a symmetry of the UV
- It is broken spontaneously in such a way that $\bar{\theta} = 0$ and $\delta_{\mathrm{CKM}} = \mathcal{O}(1)$

Nelson—Barr models

Nelson, PLB **136** (1984) 387; Barr, PRL **53** (1984) 329

New heavy vector-like quarks Q and scalars η with CPV complex VEVs $\langle \eta \rangle$

$$(\overline{q_R}\ \overline{Q_R})\, M_q \begin{pmatrix} q_L \\ Q_L \end{pmatrix} = (\overline{q_R}\ \overline{Q_R}) \begin{pmatrix} y \mathrm{v}_H & y' \langle \eta \rangle \\ 0 & \mu \end{pmatrix} \begin{pmatrix} q_L \\ Q_L \end{pmatrix}$$

- CP is a symmetry => $\theta_{\mathrm{QCD}} = 0$ and the couplings (y, y', μ) are real
- $\det M_q = y \mathrm{v}_H \mu$ is real (and positive) => $\arg \det M_q = 0$
- Effective light quark mass matrix depends on multiple $\langle \eta \rangle$ => $\delta_{\mathrm{CKM}} \neq 0$

Additional matter, tuning, loop corrections…

Dine, Draper, 1506.05433

Our solution

Supersymmetry + Modular invariance + CP

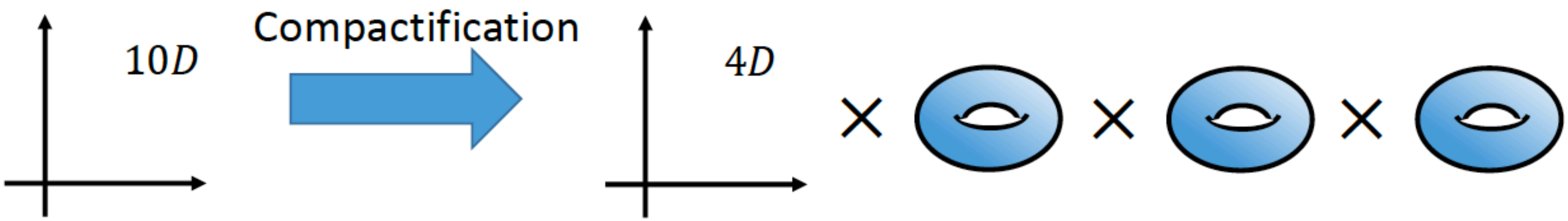
$$\Downarrow$$

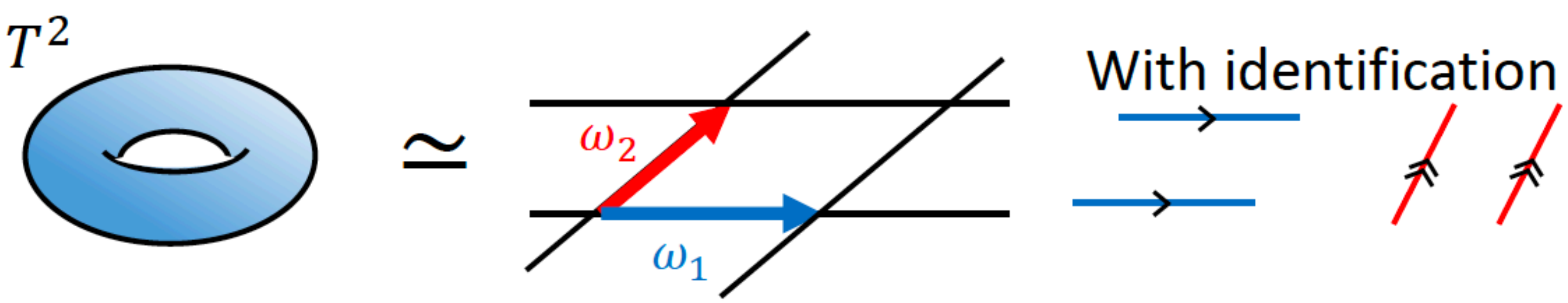
$$\boxed{\bar{\theta} = 0}$$

Modular invariance

String theory requires extra dimensions

Images: Takuya H. Tatsuishi





$$\tau = \frac{\omega_2}{\omega_1}, \quad \mathrm{Im}\,(\tau) > 0$$

modulus

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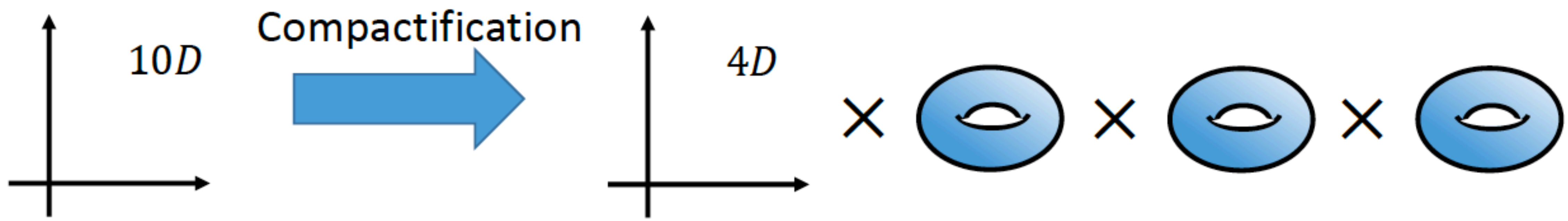

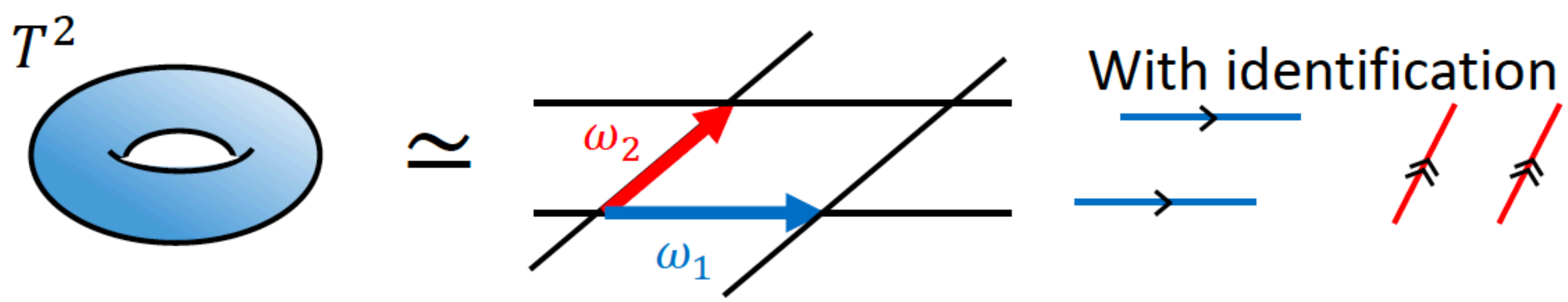

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modulus

Lattice is invariant under basis transformations

$$\begin{cases} \omega_2 \to \omega_2' = a\omega_2 + b\omega_1 \\ \omega_1 \to \omega_1' = c\omega_2 + d\omega_1 \end{cases} \qquad a, b, c, d \in \mathbb{Z} \qquad ad - bc = 1$$

$$\boxed{\tau \to \tau' = \frac{a\tau + b}{c\tau + d}}$$

modular transformations

τ and τ' describe the same torus

Modular group

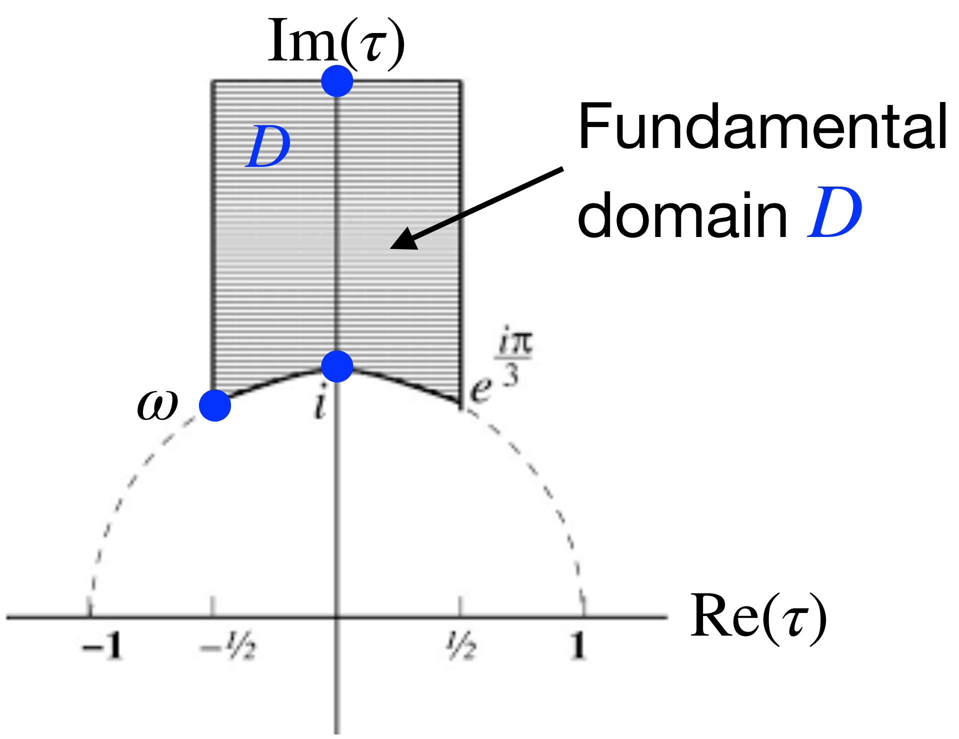
Homogeneous modular group

$$\Gamma = \langle S,\ T \mid S^4 = (ST)^3 = I \rangle \cong \mathrm{SL}(2,\mathbb{Z})$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\tau \xrightarrow{S} -\frac{1}{\tau} \qquad \tau \xrightarrow{T} \tau + 1$$

duality   discrete shift symmetry



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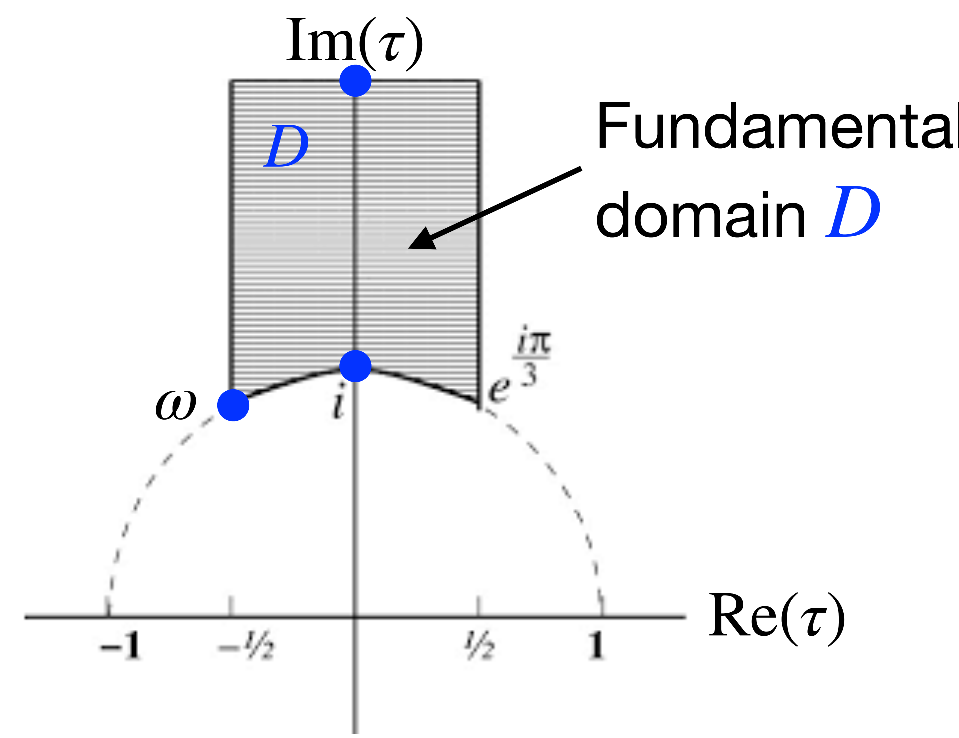

Fixed points

- $\tau = i: \quad i \xrightarrow{S} -\frac{1}{i} = i \quad \Rightarrow \quad Z_4^S$
- $\tau = \omega \equiv e^{\frac{2\pi i}{3}}: \quad \omega \xrightarrow{ST} -\frac{1}{\omega + 1} = \omega \quad \Rightarrow \quad Z_3^{ST} \times Z_2^{S^2}$
- $\tau = i\infty: \quad i\infty \xrightarrow{T} i\infty + 1 = i\infty \quad \Rightarrow \quad Z^T \times Z_2^{S^2}$

Modular forms

Holomorphic functions on $\mathscr{H} = \{\tau \in \mathbb{C} : \ \mathrm{Im}(\tau) > 0\}$ transforming under Γ as

$$f\left(\gamma\tau\right) = \left(c\tau + d\right)^k f(\tau)\,, \qquad \gamma \in \Gamma$$

k is weight, a non-negative even integer

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Normalised Eisenstein series

$$E_k(\tau) = \frac{1}{2\zeta(k)} \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^k}$$

Each modular form of weight k can be written as a polynomial in E_4 and E_6

$$f(\tau) = \sum_{a,b\geq 0} c_{ab}\, E_4^a(\tau)\, E_6^b(\tau) \qquad \text{with} \qquad 4a + 6b = k$$

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Modular weight k	0	2	4	6	8	10	12	14
Modular forms	1	–	E_4	E_6	$E_8 = E_4^2$	$E_{10} = E_4 E_6$	E_4^3, E_6^2	$E_{14} = E_4^2 E_6$

Supersymmetry

$\mathcal{N} = 1$ global SUSY action

$$\mathscr{L} = \int \mathrm{d}^2\theta\, \mathrm{d}^2\overline{\theta}\; K\left(e^{2V}\Phi, \Phi^\dagger\right) + \left[\int \mathrm{d}^2\theta\; W(\Phi) + \frac{1}{16} \int \mathrm{d}^2\theta\; f_a \mathscr{W}^a \mathscr{W}^a + \text{h.c.} \right]$$

Kähler potential K
(kinetic terms,
gauge interactions)

Superpotential W
(Yukawa interactions)

Gauge kinetic functions f_a

$$f_3 = \frac{1}{g_3^2} - i\frac{\theta_{\mathrm{QCD}}}{8\pi^2}$$

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$$\bar{\theta} = -\,8\pi^2\,\mathrm{Im}\,f_3 \;+ \arg\det\left(Y^u Y^d\right) = \arg\left[e^{-8\pi^2 f_3}\det\left(Y^u Y^d\right)\right]$$

- $M_q = M_u \oplus M_d = \mathrm{v}_u\, Y^u \oplus \mathrm{v}_d\, Y^d$ with $\mathrm{v}_{u,d} = \langle H_{u,d}\rangle$ real
- $\bar{\theta}$ does not depend on K Hiller, Schmaltz, hep-ph/0105254

Modular-invariant SUSY theories

Under modular transformations $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

Ferrara et al., PLB **225** (1989) 363

Ferrara et al., PLB **233** (1989) 147

Feruglio, 1706.08749

$$\begin{cases} \tau \to \dfrac{a\tau + b}{c\tau + d} \\ S \to S \\ \Phi \to \left(c\tau + d\right)^{-k_\Phi} \Phi \\ V \to V \end{cases}$$

- τ: modulus is promoted to a (dimensionless) superfield
- S: dilaton
- Φ: matter supermultiplets — k_Φ is modular weight of Φ
- V: vector supermultiplets

Modular symmetry acts non-linearly on field space

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Modular invariance of the action requires

$$\begin{cases} K(\tau, \Phi, \tau^\dagger, \Phi^\dagger) \to K(\tau, \Phi, \tau^\dagger, \Phi^\dagger) + f_K(\tau, \Phi) + \overline{f_K}\left(\tau^\dagger, \Phi^\dagger\right) \\ W(\tau, \Phi) \to W(\tau, \Phi) \\ f_a(S, \tau) \to f_a(S, \tau) + \frac{1}{8\pi^2} \sum_\Phi 2T_a(\Phi)\, k_\Phi \ln(c\tau + d) \end{cases}$$

$T_a(\Phi)$ is Dynkin index of Φ gauge rep

anomaly term

Konishi, Shizuya, Nuovo Cim. A **90** (1985) 111
Arkani-Hamed, Murayama, hep-th/9707133

Modular-invariant SUSY theories

Minimal Kähler potential

$$K = -h^2 \ln\left(-i\tau + i\tau^\dagger\right) + \sum_{\Phi} \frac{\Phi^\dagger e^{2V} \Phi}{\left(-i\tau + i\tau^\dagger\right)^{k_\Phi}}$$

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Superpotential

$$W = U_i^c\, Y_{ij}^u(\tau)\, Q_j\, H_u + D_i^c\, Y_{ij}^d(\tau)\, Q_j\, H_d + E_i^c\, Y_{ij}^e(\tau)\, L_j\, H_d + \frac{1}{2\Lambda_L} L_i\, C_{ij}^\nu(\tau)\, L_j\, H_u\, H_u$$

τ-dependent Yukawa couplings (and Majorana mass)

$$Y_{ij}^u(\tau) \to \left(c\tau + d\right)^{k_{Y_{ij}^u}}\, Y_{ij}^u(\tau) \qquad \text{with} \qquad k_{Y_{ij}^u} = k_{U_i^c} + k_{Q_j} + k_{H_u}$$

(similarly for $Y^d,\ Y^e,\ C^\nu$)

are modular forms!

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are modular forms!

$$\det\left(Y^u Y^d\right) \to \left(c\tau + d\right)^{k_{\det}} \det\left(Y^u Y^d\right)$$

$$k_{\det} = \sum_{i=1}^{3}\left(2k_{Q_i} + k_{U_i^c} + k_{D_i^c}\right) + 3\left(k_{H_u} + k_{H_d}\right)$$

Modular-invariant SUSY theories

Gauge kinetic functions

$$e^{-8\pi^2 f_a(S,\tau)} \to \left(c\tau + d\right)^{k_{f_a}} e^{-8\pi^2 f_a(S,\tau)} \qquad \text{with} \qquad k_{f_a} = -\sum_{\Phi} 2T_a(\Phi)\, k_\Phi$$

In particular,

$$k_{f_3} = -\sum_{i=1}^{3} \left(2k_{Q_i} + k_{U_i^c} + k_{D_i^c}\right)$$

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$\bar{\theta}$ becomes a field-dependent quantity

$$\bar{\theta} = \arg A(S,\tau)$$

$$A(S,\tau) \equiv e^{-8\pi^2 f_3(S,\tau)} \det\left[Y^u(\tau) Y^d(\tau)\right]$$

$$A(S,\tau) \to \left(c\tau + d\right)^{k_A} A(S,\tau) \qquad \text{with} \qquad k_A = 3\left(k_{H_u} + k_{H_d}\right)$$

Conditions for $\bar{\theta} = 0$

1. Higgs weights $k_{H_u} + k_{H_d} = 0$ $\Rightarrow$ $A(S, \tau)$ is modular-invariant

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3. $\langle \tau \rangle$ is the only source of CP violation $\langle \mathrm{Im} S \rangle = 0$ ⇨ $A(S, \tau)$ is a real constant (assumed to be positive)

$$\bar{\theta} = \arg A(S, \tau) = 0$$

independently of the vacuum $\langle \tau \rangle$

Structure of $A(S, \tau)$

$$A(S,\tau) \equiv e^{-8\pi^2 f_3(S,\tau)} \det\left[Y^u(\tau) Y^d(\tau)\right]$$

$$k_A = k_{f_3} + k_{\det} = 0 \quad \Rightarrow \quad k_{f_3} = -\, k_{\det}$$

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- $k_{f_3} = k_{\det} = 0$ (and no singularities in either gauge or Yukawa factors)
  $\theta_{QCD} = 0$ while $f_3 = \frac{1}{g_3^2}$ and $\det\left(Y^u Y^d\right)$ are real τ-independent constants

[Solution proposed in Feruglio, Strumia, **AT**, 2305.08908]

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  [Solution proposed in Feruglio, Strumia, **AT**, 2305.08908]

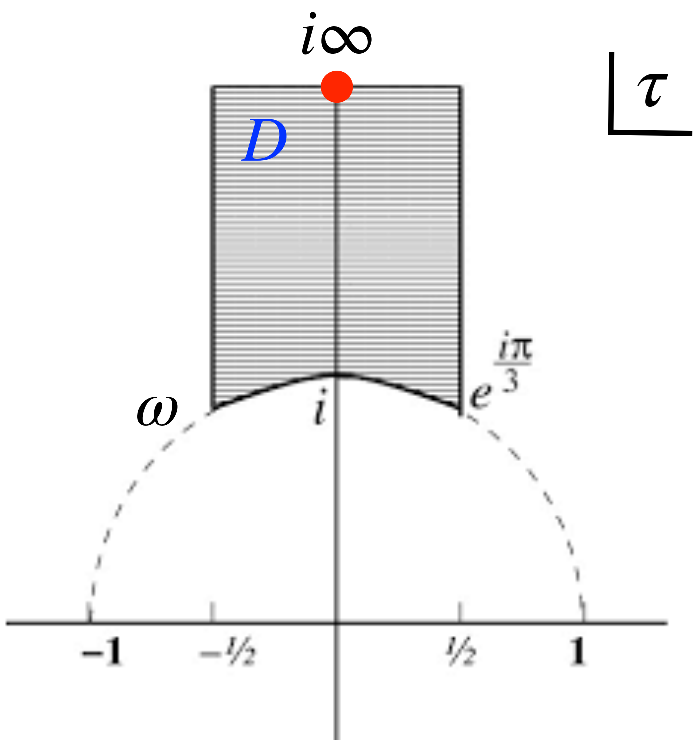
- $k_{f_3} < 0$ and $k_{\det} > 0$ (or vice versa)
  - $e^{-8\pi^2 f_3(S,\tau)}$ is singular at some point(s) in the moduli space
  - $\det\left[Y^u(\tau)Y^d(\tau)\right]$ is holomorphic everywhere

For $A(S,\tau)$ to be τ-independent, singularities of $e^{-8\pi^2 f_3(S,\tau)}$ must be cancelled by zeroes of $\det\left[Y^u(\tau)Y^d(\tau)\right]$ occurring at the same points in the moduli space

[Solution proposed in Feruglio, Marrone, Strumia, **AT**, 2505.20395]

Singularities in the moduli space

$e^{-8\pi^2 f_3(S,\tau)}$ is expected to be singular
at the cusp $\tau = i\infty$ (decompactification limit)
Gonzalo, Ibanez, Uranga, 1812.06520



Distance Conjecture Ooguri, Vafa, hep-th/0605264
$\tau = i\infty$ is at infinite distance from any point in D

Explicit computations in string compactifications

$$f_3(S,\tau) = \kappa_3 S - \frac{k_{f_3}}{8\pi^2} \ln \eta^2(\tau)$$ Kaplunovsky, Louis, hep-th/9502077

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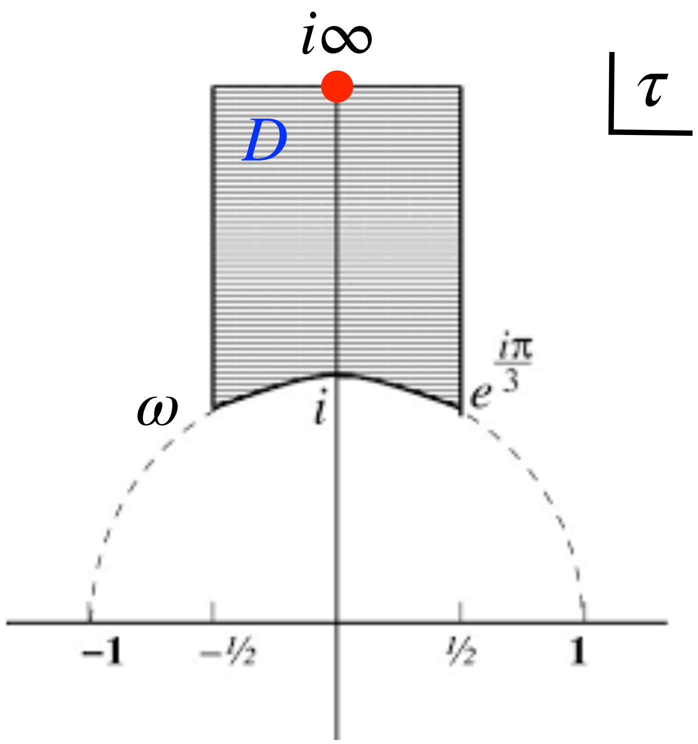

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We assume that $\det\left[Y^u(\tau)Y^d(\tau)\right]$ has a unique zero at $\tau = i\infty$ (*quark masses never vanish except in this limit*), i.e., it is a cusp form of weight $k_{\det} = 12\,m$:

$$\det\left[Y^u(\tau)Y^d(\tau)\right] \propto \Delta(\tau)^m$$

$$\Delta(\tau) = \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728} = \eta(\tau)^{24} = q\prod_{n=1}^{\infty}\left(1-q^n\right)^{24} \quad \text{with} \quad q = e^{2\pi i\tau} \quad \left(k_\Delta = 12\right)$$

Modular invariance and CP

Fields

$$\tau \xrightarrow{\text{CP}} -\tau^\dagger \qquad \text{and} \qquad \Phi \xrightarrow{\text{CP}} \Phi^\dagger$$

Modular forms

$$Y(\tau) \xrightarrow{\text{CP}} Y(-\tau^*) = Y(\tau)^*$$

Novichkov, Penedo, Petcov, **AT**, 1905.11970; Baur, Nilles, Trautner, Vaudrevange, 1901.03251

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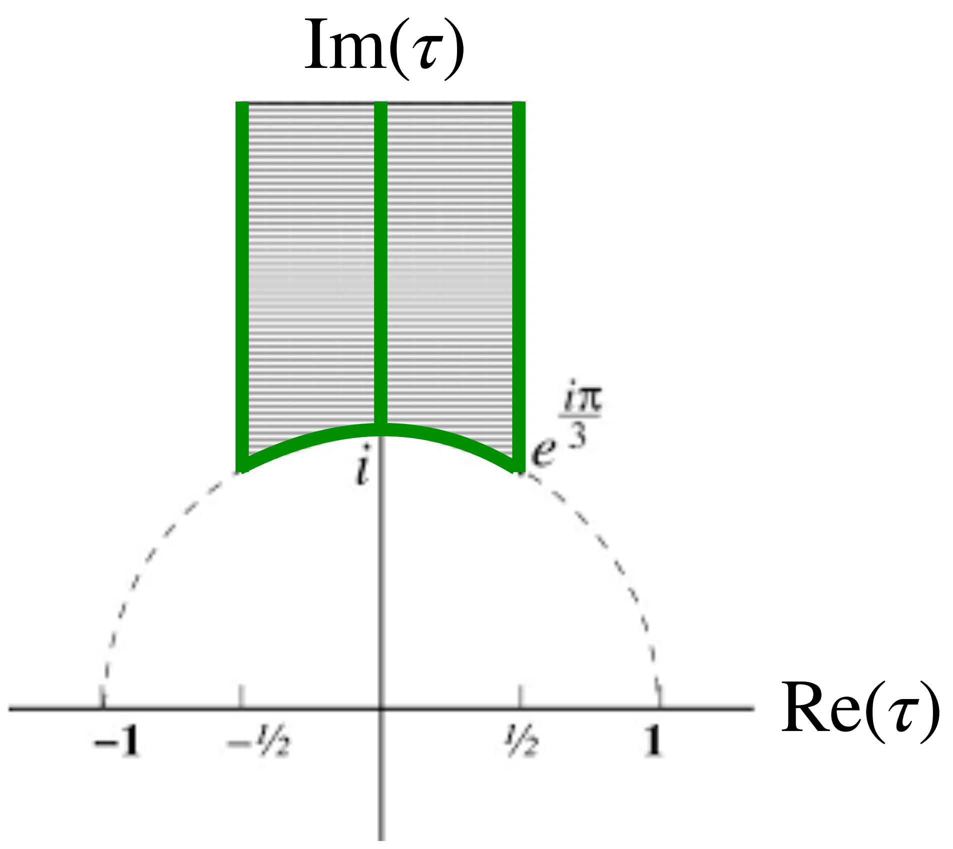
CP-conserving values of τ

$\tau \xrightarrow{\text{CP}} -\tau^* = \gamma\tau$ (goes to itself up to γ)

1. $\tau = iy \xrightarrow{\text{CP}} iy$

2. $\tau = -\dfrac{1}{2} + i\,y \xrightarrow{\text{CP}} \dfrac{1}{2} + i\,y = T\tau$

3. $\tau = e^{i\varphi} \xrightarrow{\text{CP}} -e^{-i\varphi} = S\tau$



Novichkov, Penedo, Petcov, **AT**, 1905.11970; Baur, Nilles, Trautner, Vaudrevange, 1901.03251

Example model

$$k_{H_u} = k_{H_d} = 0 \qquad k_{Q_i} = k_{U_i^c} = k_{D_i^c} = (2,\ 4,\ 6) \quad \Rightarrow \quad k_{\det} = 48 \quad (m = 4)$$

$$Y^{u,d}(\tau) = \begin{pmatrix} E_4 & E_6 & E_8 \\ E_6 & E_8 & E_{10} \\ E_8 & E_{10} & E_{12} \end{pmatrix} \quad \Rightarrow \quad \det Y^{u,d} \propto \Delta(\tau)^2$$

$$E_8 = E_4^2\,, \quad E_{10} = E_4 E_6\,, \quad E_{12} = a\,E_4^3 + (1-a)\,E_6^2$$

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Slightly non-minimal Kähler potential

$$K = \sum_{i=1}^{3} \left[c_{Q_i}^{-2}\, y^{-k_{Q_i}} |Q_i|^2 + c_{U_i^c}^{-2}\, y^{-k_{U_i^c}} |U_i^c|^2 + c_{D_i^c}^{-2}\, y^{-k_{D_i^c}} |D_i^c|^2 \right] \quad \text{with} \quad y \equiv 2\,\mathrm{Im}\,\tau$$

In the basis where kinetic terms of quarks are canonical

$$Y^q_{\text{can}} = \begin{pmatrix} c_{Q_1^c} c_{Q_1} y^2 E_4 & c_{Q_1^c} c_{Q_2} y^3 E_6 & c_{Q_1^c} c_{Q_3} y^4 E_8 \\ c_{Q_2^c} c_{Q_1} y^3 E_6 & c_{Q_2^c} c_{Q_2} y^4 E_8 & c_{Q_2^c} c_{Q_3} y^5 E_{10} \\ c_{Q_3^c} c_{Q_1} y^4 E_8 & c_{Q_3^c} c_{Q_2} y^5 E_{10} & c_{Q_3^c} c_{Q_3} y^6 E_{12} \end{pmatrix} \quad \text{with} \quad Q^c = \begin{cases} U^c & \text{for } q = u \\ D^c & \text{for } q = d \end{cases}$$

Example model: quarks

Observable	Central value $\pm 1\sigma$
m_u/m_c	$(1.93 \pm 0.60)\, 10^{-3}$
m_c/m_t	$(2.82 \pm 0.12)\, 10^{-3}$
m_d/m_s	$(5.05 \pm 0.62)\, 10^{-2}$
m_s/m_b	$(1.82 \pm 0.10)\, 10^{-2}$
m_t/GeV	87.5 ± 2.1
m_b/GeV	0.97 ± 0.01

Observable	Central value $\pm 1\sigma$
$\sin^2\theta^q_{12}$	$(5.08 \pm 0.03)\, 10^{-2}$
$\sin^2\theta^q_{13}$	$(1.22 \pm 0.09)\, 10^{-5}$
$\sin^2\theta^q_{23}$	$(1.61 \pm 0.05)\, 10^{-3}$
$\delta_{\mathrm{CKM}}/\pi$	0.385 ± 0.017

Quark masses and mixing parameters renormalised at 2×10^{16} GeV assuming $M_{\mathrm{SUSY}} = 10$ TeV and $\tan\beta = 10$

Antusch, Maurer, 1306.6879

Best-fit point ($\chi^2 \approx 0$):

$$\tau = -0.286 + 1.096i$$

$$q_{13} = 0.037\,,\ q_{23} = 0.075\,,\ u_{13} = 0.035\,,\ u_{23} = 19.98\,,\ d_{13} = 3.44\,,\ d_{23} = 0.203$$

$$c_{U_3^c} c_{Q_3} = 4.15 \times 10^{-4} \quad \text{and} \quad c_{D_3^c} c_{Q_3} = 4.76 \times 10^{-4} \quad \text{(overall scales)}$$

$$(q_{i3} \equiv c_{Q_i}/c_{Q_3}\,, \quad u_{i3} \equiv c_{U_i^c}/c_{U_3^c}\,, \quad d_{i3} \equiv c_{D_i^c}/c_{D_3^c}\,, \quad i = 1, 2)$$

8 real parameters + τ vs. 10 observables

Example model: leptons

Observable	Central value $\pm 1\sigma$
m_e/m_μ	$(4.74 \pm 0.04)\, 10^{-3}$
m_μ/m_τ	$(5.88 \pm 0.05)\, 10^{-2}$
$\delta m^2/\lvert\Delta m^2\rvert$	$(2.95 \pm 0.06)\, 10^{-2}$
	$(2.99 \pm 0.07)\, 10^{-2}$
m_τ/GeV	1.293 ± 0.007
$\delta m^2/\mathrm{eV}^2$	$(7.37 \pm 0.17)\, 10^{-5}$
$\lvert\Delta m^2\rvert/\mathrm{eV}^2$	$(2.495 \pm 0.020)\, 10^{-3}$
	$(2.465 \pm 0.020)\, 10^{-3}$

Observable	Central value $\pm 1\sigma$
$\sin^2\theta^\ell_{12}$	$(3.03 \pm 0.14)\, 10^{-1}$
$\sin^2\theta^\ell_{13}$	$(2.23 \pm 0.05)\, 10^{-2}$
$\sin^2\theta^\ell_{23}$	$(4.73 \pm 0.24)\, 10^{-1}$
	$(5.45 \pm 0.23)\, 10^{-1}$
$\delta_{\mathrm{PMNS}}/\pi$	1.20 ± 0.22
	1.48 ± 0.12

Lepton masses and mixing parameters
Antusch, Maurer, 1306.6879
Capozzi et al., 2503.07752

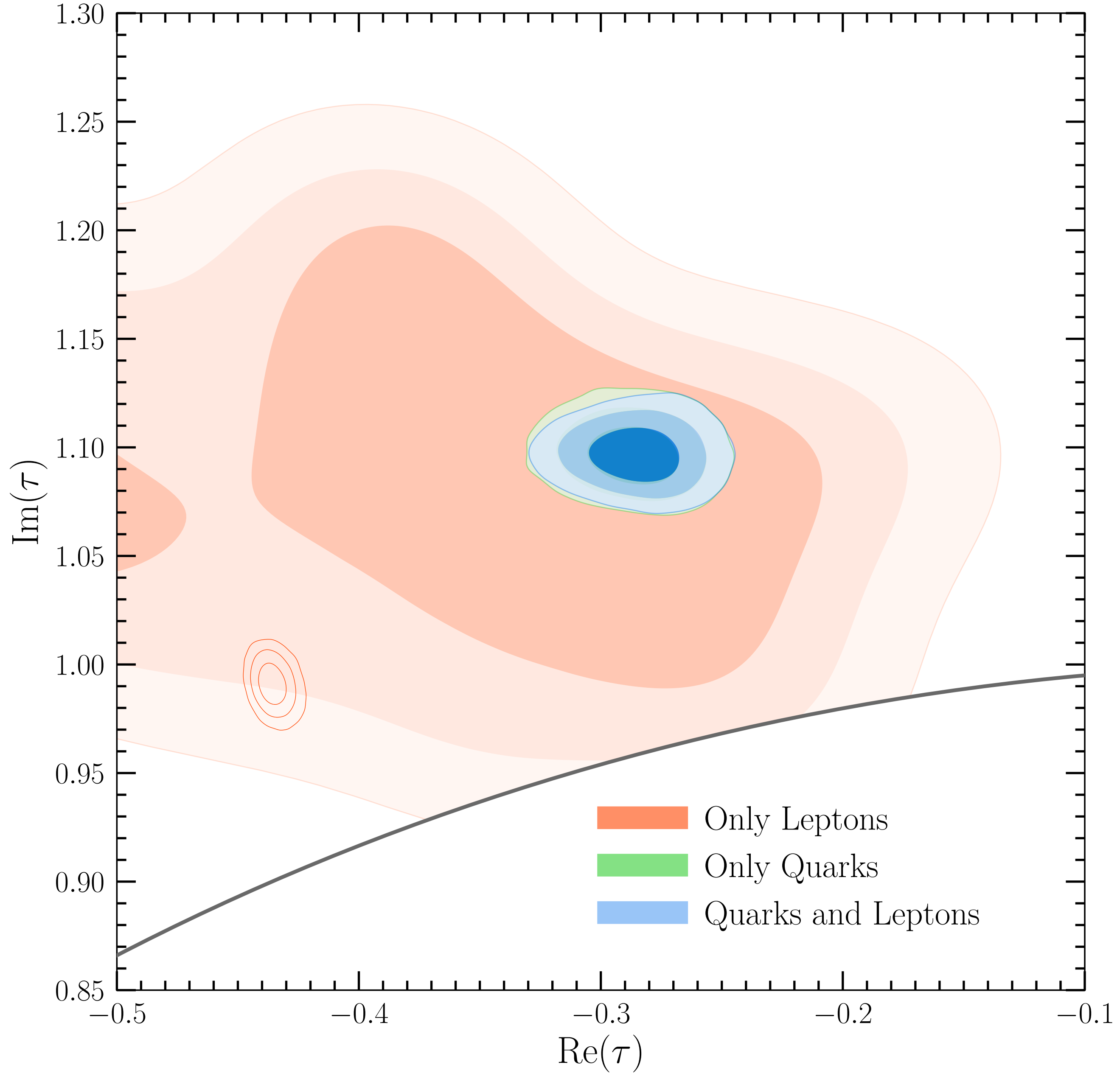
Best-fit point ($\chi^2 \approx 0$) for the same value of τ:

$$l_{13} = 2.51\,,\; l_{23} = 1.94\,,\; e_{13} = 2.21\,,\; e_{23} = 0.0079\,,\; c^e_{33} = 5.61\,,\; c^\nu_{33} = 0.076$$

$$c_{E^c_3} c_{L_3} = 7.66\;10^{-5} \quad \text{and} \quad \frac{c^2_{L_3}}{2\Lambda_L} = \frac{6.60\;10^{-2}}{10^{16}\ \text{GeV}} \quad \text{(overall scales)}$$

8 parameters vs. 8 observables + 4 predictions

Example model: combined fit



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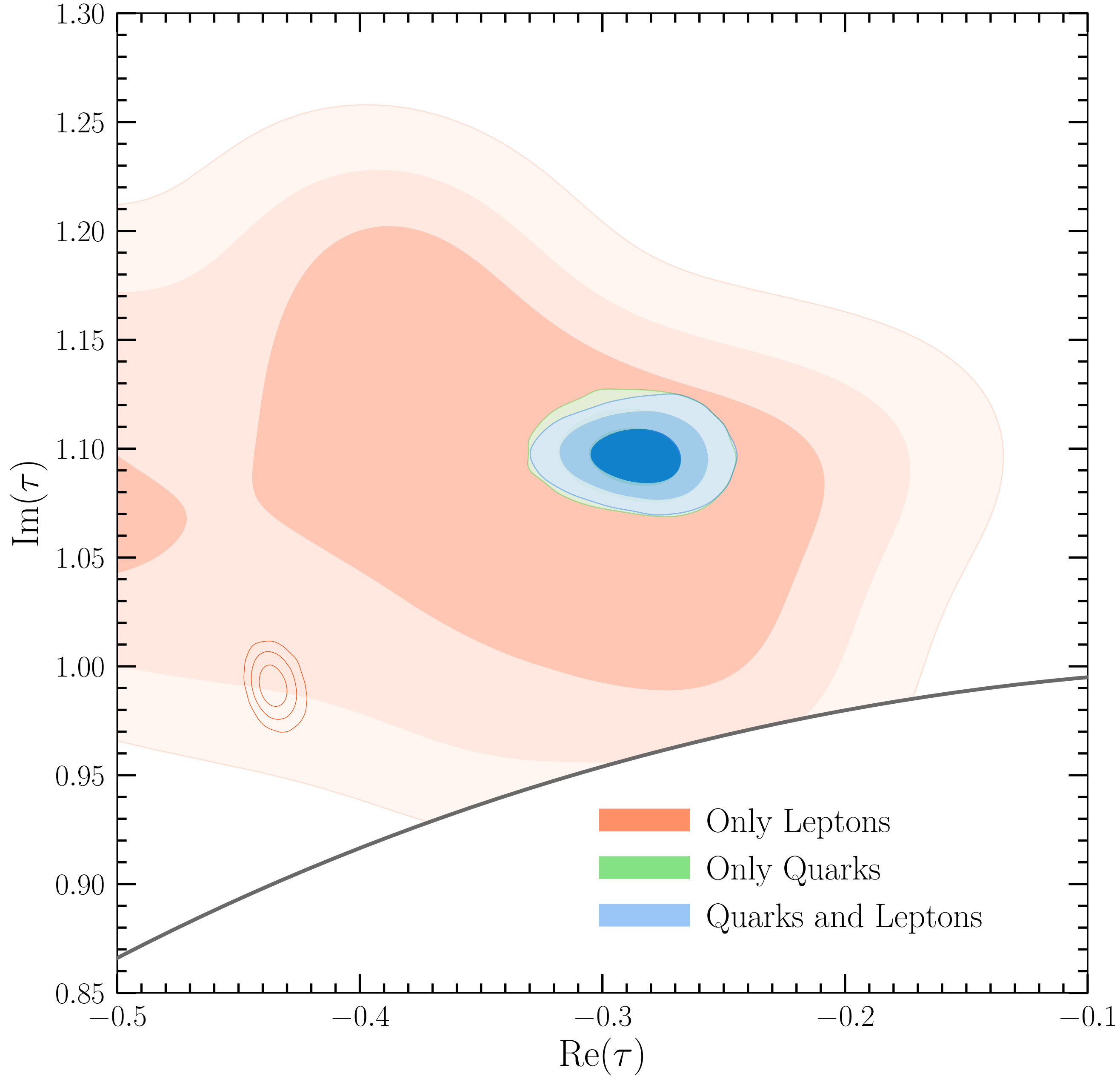


Predictions in the neutrino sector

Masses

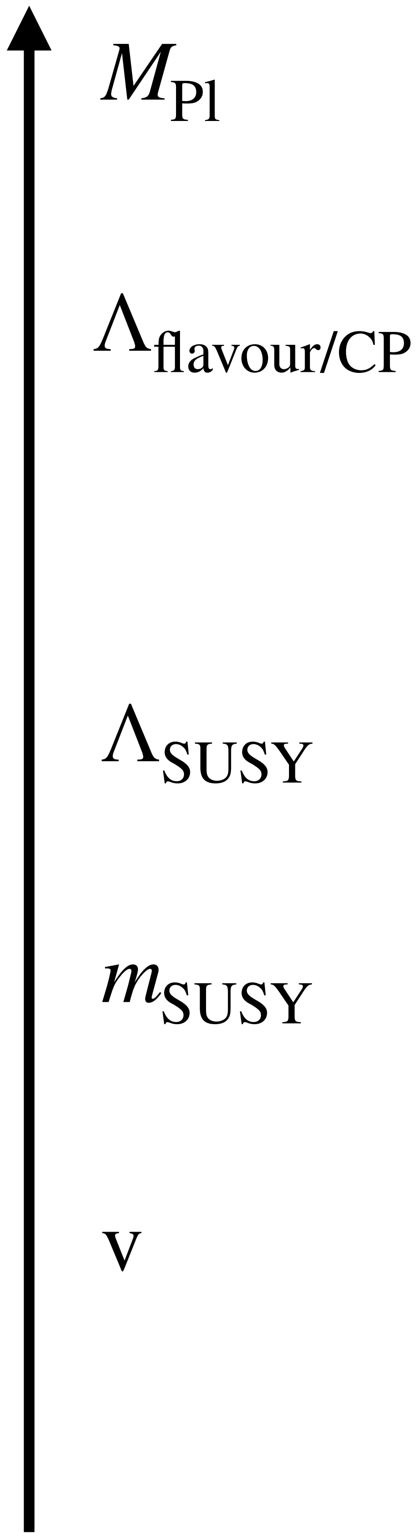
- Normal ordering with $m_1 \approx 10$ meV

$$m_{\beta\beta} \approx 6 \text{ meV} \qquad \sum_{i=1}^{3} m_i \approx 74 \text{ meV}$$

CPV phases

- $\delta_{\text{PMNS}} \approx 0.94\,\pi$
- $\alpha_{21} \approx 1.29\,\pi \quad \alpha_{31} \approx 0.14\,\pi$

Corrections to $\bar{\theta} = 0$



Corrections to $\bar{\theta} = 0$

M_{Pl}

$\Lambda_{\text{flavour/CP}}$

Λ_{SUSY}

m_{SUSY}

v

SUSY unbroken

- Modular invariance determines completely (up to real couplings) the functional dependence $W(\tau)$
- It is not the case for K, but $\bar{\theta}$ is insensitive to K
- No-renormalisation theorems Ellis, Ferrara, Nanopoulos, PLB **114** (1982) 231

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$\Lambda_{\rm SUSY}$

SUSY breaking corrections

- In general, can be large

$m_{\rm SUSY}$

- Small if $\Lambda_{\rm flavour/CP} \gg \Lambda_{\rm SUSY}$ (as e.g. in gauge mediation) and soft breaking terms respect the flavour structure of the SM

$\rm v$

$$\bar{\theta} \lesssim \frac{m_t^4 m_b^4 m_c^2 m_s^2}{\rm v^{12}} J_{\rm CP} \tan^6\beta \sim 10^{-28} \tan^6\beta$$

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SM corrections are negligible

- $\bar{\theta} \sim 10^{-18}$ at four loops Khriplovich, PLB **173** (1986) 193
Ellis, Gaillard, NPB **150** (1979) 141

Conclusions

- Modular invariance is inherent to toroidal compactifications in string theory and can be consistently implemented in a supersymmetric QFT

- Quarks with positive weights and non-trivial gauge kinetic functions

- The VEV of the modulus τ is the only source of spontaneous CP violation, while the dilaton S has a CP-conserving VEV

Conclusions

- Modular invariance is inherent to toroidal compactifications in string theory and can be consistently implemented in a supersymmetric QFT

- Quarks with positive weights and non-trivial gauge kinetic functions

- The VEV of the modulus τ is the only source of spontaneous CP violation, while the dilaton S has a CP-conserving VEV

$$\bar{\theta} = \arg A(S,\tau) \quad \text{with} \quad A(S,\tau) \equiv e^{-8\pi^2 f_3(S,\tau)} \det\left[Y^u(\tau) Y^d(\tau)\right]$$

- f_3 is singular at $\tau = i\infty$, while $\det\left(Y^u Y^d\right)$ has the only zero at this point, such that $A(S,\tau)$ is a real constant and hence $\bar{\theta} = 0$

- Quark and lepton masses and mixings are accommodated for the same $\langle\tau\rangle$

- Corrections to $\bar{\theta} = 0$ are small under certain assumptions on SUSY breaking

Back-up slides

What can solve the strong CP problem?

Kaplan, Melia, Rajendran, 2505.08358

- θ_{QCD} cannot be set to zero by CP/P, since it is not a parameter (like other couplings), but a label of a vacuum state
- Viable solutions should be dynamical, like the axion solution

What can solve the strong CP problem?

Kaplan, Melia, Rajendran, 2505.08358

- $\theta_{\rm QCD}$ cannot be set to zero by CP/P, since it is not a parameter (like other couplings), but a label of a vacuum state
- Viable solutions should be dynamical, like the axion solution

In our framework, CP is conserved, but $\bar{\theta}$ is a dynamical field-dependent quantity, i.e., we do not set $\theta_{\rm QCD} = 0$ by CP

$$\bar{\theta} = \arg A(S,\tau) \quad \text{with} \quad A(S,\tau) \equiv e^{-8\pi^2 f_3(S,\tau)} \det\left[Y^u(\tau) Y^d(\tau)\right]$$

- CP invariance $\Rightarrow$ $f_3(S^*, -\tau^*) = f_3(S,\tau)^*$
- $\theta_{\rm QCD}$ is inside S $\Rightarrow$ $f_3(S,\tau) = S + \ldots$

Phenomenology and cosmology

- Couplings to matter are suppressed by $1/h$ ($1/M_{\mathrm{Pl}}$ in SUGRA)
- No couplings to gauge bosons in the exact SUSY limit
- $m_\tau \gtrsim 10$ TeV not to spoil BBN
- Fermionic component of τ could be LSP and maybe DM
- Scalar potential $V(\tau) = V(-\tau^*)$ => CP-conjugated minima
  (domain walls are inflated away if CP breaking occurs before inflation)

Matter fields and canonical normalisation

Gauge quantum numbers

	Q	U^c	D^c	L	E^c	H_u	H_d
$\mathrm{SU}(3)_c$	$\mathbf{3}$	$\overline{\mathbf{3}}$	$\overline{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathrm{SU}(2)_L$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$
$\mathrm{U}(1)_Y$	$\frac{1}{6}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$

Canonical normalisation

$$K \supset \frac{\Phi^\dagger \Phi}{(-i\tau + i\tau^\dagger)^{k_\Phi}} = \Phi^\dagger_{\text{can}} \Phi_{\text{can}} \qquad \Phi_{\text{can}} = \{\phi_{\text{can}}, \psi_{\text{can}}\}$$

$$\psi_{\text{can}} \to \left(\frac{c\tau + d}{c\tau^\dagger + d}\right)^{-\frac{k_\Phi}{2}} \psi_{\text{can}} = e^{-ik_\Phi \alpha(\tau)} \psi_{\text{can}} \qquad \alpha(\tau) = \arg(c\tau + d)$$

Modular symmetry acts on canonically normalised fields as a τ-dependent phase rotation (with $\tau = \tau(x)$)

Modular-SM anomalies

Conditions for modular-gauge anomaly cancellation

$$\mathrm{SU}(3)_c: \quad A \equiv \sum_{i=1}^{3}\left(2k_{Q_i} + k_{U_i^c} + k_{D_i^c}\right) = 0$$

$$\mathrm{SU}(2)_L: \quad \sum_{i=1}^{3}\left(3k_{Q_i} + k_{L_i}\right) + k_{H_u} + k_{H_d} = 0$$

$$\mathrm{U}(1)_Y: \quad \sum_{i=1}^{3}\left(k_{Q_i} + 8k_{U_i^c} + 2k_{D_i^c} + 3k_{L_i} + 6k_{E_i^c}\right) + 3\left(k_{H_u} + k_{H_d}\right) = 0$$

Simplest solution

$$k_Q = k_{U^c} = k_{D^c} = k_L = k_{E^c} = (-k, 0, k) \quad \text{and} \quad k_{H_u} + k_{H_d} = 0$$

Cancellation of modular-QCD anomaly along with $k_{H_u} + k_{H_d} = 0$ implies

$$k_{\det} = \sum_{i=1}^{3}\left(2k_{Q_i} + k_{U_i^c} + k_{D_i^c}\right) + 3\left(k_{H_u} + k_{H_d}\right) = 0$$

More on modular-gauge anomalies

$$\Phi \to \Phi' = (c\tau + d)^{-k_\Phi}\, \Phi$$

Jacobian J: $\mathscr{D}\Phi' = J\, \mathscr{D}\Phi$ Arkani-Hamed, Murayama, hep-th/9707133

$$\log J = -\frac{i}{64\pi^2} \int d^4x\, d^2\theta \left[\sum_\Phi T(\Phi)\, k_\Phi \right] W^a W^a \log(c\tau + d)$$

$T(\Phi)$ is the Dynkin index of the rep of Φ: $\mathrm{tr}\left(t_a t_b\right) = T(\Phi)\, \delta_{ab}$

$$\boxed{\sum_\Phi T(\Phi)\, k_\Phi = 0}$$

$$\mathrm{SU}(3)_c: \quad \sum_i \left(2k_{Q_i} + k_{U_i^c} + k_{D_i^c}\right) = 0$$

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Simplest example: quarks

Simplest non-trivial example giving $k_{\text{det}} = 0$ and $A = 0$

$$k_Q = k_{U^c} = k_{D^c} = (-6, 0, 6) \quad \text{and} \quad k_{H_u} = k_{H_d} = 0$$

Yukawa matrices

$$Y^q = \begin{pmatrix} 0 & 0 & c^q_{13} \\ 0 & c^q_{22} & c^q_{23} E_6 \\ c^q_{31} & c^q_{32} E_6 & c^q_{33} E_4^3 + c'^q_{33} E_6^2 \end{pmatrix} \quad \Rightarrow \quad Y^q|_{\text{can}} = \begin{pmatrix} 0 & 0 & c^q_{13} \\ 0 & c^q_{22} & c^q_{23} (2\text{Im}\tau)^3 E_6 \\ c^q_{31} & c^q_{32} (2\text{Im}\tau)^3 E_6 & (2\text{Im}\tau)^6 \left[c^q_{33} E_4^3 + c'^q_{33} E_6^2 \right] \end{pmatrix}$$

$$\det Y^q|_{\text{can}} = -\, c^q_{13} c^q_{22} c^q_{31} \in \mathbb{R}$$

Fixing $\tau = 1/8 + i$ and $\tan\beta = 10$

$$c^u_{ij} \approx 10^{-3} \begin{pmatrix} 0 & 0 & 1.56 \\ 0 & -1.86 & 0.87 \\ 1.29 & 4.14 & 3.51,\ 1.40 \end{pmatrix} \qquad c^d_{ij} \approx 10^{-3} \begin{pmatrix} 0 & 0 & 1.55 \\ 0 & -2.59 & 4.59 \\ 0.378 & 0.710 & 0.734,\ 1.76 \end{pmatrix}$$

reproduce the quark masses, mixing angles and δ_{CKM} at the GUT scale of 2×10^{16} GeV

Simplest example: leptons

$k_L = k_{E^c} = (-6, 0, 6)$

Weinberg operator $\mathscr{C}^\nu_{ij}(L_i H_u)(L_j H_u)$ for neutrino masses

Charged lepton Yukawa matrix and coefficient of the Weinberg operator

$$Y^e = \begin{pmatrix} 0 & 0 & c^e_{13} \\ 0 & c^e_{22} & c^e_{23}E_6 \\ c^e_{31} & c^e_{32}E_6 & c^e_{33}E_4^3 + c'^e_{33}E_6^2 \end{pmatrix} \qquad \mathscr{C}^\nu = \begin{pmatrix} 0 & 0 & c^\nu_{13} \\ 0 & c^\nu_{22} & c^\nu_{23}E_6 \\ c^\nu_{13} & c^\nu_{23}E_6 & c^\nu_{33}E_4^3 + c'^\nu_{33}E_6^2 \end{pmatrix}$$

Fixing $\tau = 1/8 + i$ and $\tan\beta = 10$

$$c^e_{ij} = 10^{-3}\begin{pmatrix} 0 & 0 & 1.29 \\ 0 & 5.95 & 0.35 \\ -2.56 & 1.47 & 1.01,\ 1.32 \end{pmatrix} \qquad c^\nu_{ij} = \frac{1}{10^{16}\ \text{GeV}}\begin{pmatrix} 0 & 0 & 3.4 \\ 0 & 7.1 & 1.2 \\ 3.4 & 1.2 & 0.19,\ 0.95 \end{pmatrix}$$

reproduce the lepton masses and mixings, including δ_{PMNS}

Models with larger modular weights

Yukawa matrices	Modular weights			Alternative bigger weights		
$Y_{u,d}$	$(u_L, d_L)_{1,2,3}$	$u_{R1,2,3}$	$d_{R1,2,3}$	$(u_L, d_L)_{1,2,3}$	$u_{R1,2,3}$	$d_{R1,2,3}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_6 \\ 1 & E_6 & E_4^3 + E_6^2 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -4 \\ 2 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ -2 \\ 4 \end{pmatrix}$	$\begin{pmatrix} -8 \\ -2 \\ 4 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_4^2 \\ 1 & E_4 & E_4^3 + E_6^2 \end{pmatrix}$	$\begin{pmatrix} -6 \\ -2 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 2 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 2 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -4 \\ -4 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 4 \\ 4 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 4 \\ 4 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_4^2 \\ 1 & E_4^2 & E_4(E_4^3 + E_6^2) \end{pmatrix}$				$\begin{pmatrix} -8 \\ 0 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 0 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 0 \\ 8 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_4 E_6 \\ 1 & E_6 & E_4(E_4^3 + E_6^2) \end{pmatrix}$				$\begin{pmatrix} -8 \\ -2 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 2 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 2 \\ 8 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_4^3 + E_6^2 \\ 1 & E_4 & E_4(E_4^3 + E_6^2) \end{pmatrix}$				$\begin{pmatrix} -8 \\ -4 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 4 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 4 \\ 8 \end{pmatrix}$

Heavy quarks and singularities

- Heavy quarks are not needed for the mechanism to work, but assume they exist

$$k_q = (-6,\ -2, 0,\ +2,\ +6) \quad \text{and} \quad k_{H_u} = k_{H_d} = 0$$

Light chiral quarks Heavy vector-like quarks

- In the full theory $f_{\mathrm{UV}} \in \mathbb{R}$ and $\det M_{\mathrm{all}} \in \mathbb{R}$ => $\bar{\theta}_{\mathrm{UV}} = 0$

$$M_{\mathrm{all}} = \begin{pmatrix} M_{LL} & M_{LH} \\ M_{HL} & M_{HH} \end{pmatrix}$$

$$M_{\mathrm{light}} \approx M_{LL} - M_{LH} M_{HH}^{-1} M_{HL} \qquad M_{\mathrm{heavy}} \approx M_{HH}$$

Singularities where $\det M_{\mathrm{heavy}}(\tau) = 0$
(breakdown of EFT)

$$\det M_{\mathrm{all}} = \det M_{\mathrm{light}} \det M_{\mathrm{heavy}}$$

$$\det M_{\mathrm{light}} \to (c\tau + d)^{k_{\mathrm{light}}} \det M_{\mathrm{light}} \qquad \det M_{\mathrm{heavy}} \to (c\tau + d)^{k_{\mathrm{heavy}}} \det M_{\mathrm{heavy}}$$

$$k_{\mathrm{all}} = k_{\mathrm{light}} + k_{\mathrm{heavy}} = 0$$

EFT of light quarks

In the EFT of light quarks

$$\bar{\theta}_{\rm IR} = \theta_{\rm QCD} + \arg\det M_{\rm light} = -8\pi^2\,{\rm Im}\,f_{\rm IR} + \arg\det M_{\rm light}$$

The EFT has anomalous field content with $k_q = (-6, -2, 0)$

Anomaly is cancelled by a new contribution to the gauge kinetic function arising from the integration over the heavy quarks

$$f_{\rm IR} = f_{\rm UV} - \frac{1}{8\pi^2}\log\det M_{\rm heavy}$$

Thus

$$\bar{\theta}_{\rm IR} = \arg\det M_{\rm heavy} + \arg\det M_{\rm light} = \arg\det M_{\rm all} = 0$$

Model with VLQs based on $\Gamma_2 \cong S_3$

Feruglio, Parriciatu, Strumia, **AT**, 2406.01689

	SM quarks			Extra vector-like quarks			
	Q	D^c	U^c	D'^c	D'	U'^c	U'
$\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$	$2_{1/6}$	$1_{1/3}$	$1_{-2/3}$	$1_{1/3}$	$1_{-1/3}$	$1_{-2/3}$	$1_{2/3}$
Flavor symmetry Γ_2	$\mathbf{2} \oplus \mathbf{1}_0$	$\mathbf{2} \oplus \mathbf{1}_1$	$\mathbf{2} \oplus \mathbf{1}_0$	$\mathbf{2} \oplus \mathbf{1}_0$	$\mathbf{2} \oplus \mathbf{1}_1$	$\mathbf{2} \oplus \mathbf{1}_0$	$\mathbf{2} \oplus \mathbf{1}_0$
Modular weights k_Φ	-2	-2	-2	$+2$	$+2$	$+2$	$+2$

$$W_{\mathrm{UV}} \supset Q^T m^d D^c + Q^T n^d D'^c + D'^T N^d D^c + D'^T M^d D'^c$$

$$\mathcal{M}_d = \begin{pmatrix} m^d & n^d \\ N^d & M^d \end{pmatrix} \qquad m^d = v_d Y^d \qquad n^d = v_d Y'^d$$

$$m^d = 0_{3\times 3} \qquad n^d = n_d \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha_d \end{pmatrix} \qquad N^d = N_d \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta_d \end{pmatrix} \qquad M^d = M_d \begin{pmatrix} -Z_1^{(4)} + \gamma_{d1} Z_3^{(4)} & Z_2^{(4)} & \gamma_{d2} Z_1^{(4)} \\ Z_2^{(4)} & Z_1^{(4)} + \gamma_{d1} Z_3^{(4)} & \gamma_{d2} Z_2^{(4)} \\ \gamma_{d3} Z_2^{(4)} & -\gamma_{d3} Z_1^{(4)} & 0 \end{pmatrix}$$

$(Z_1^{(4)}(\tau), Z_2^{(4)}(\tau))^T \sim \mathbf{2}$ $\qquad Z_3^{(4)}(\tau) \sim \mathbf{1}_0$ $\qquad$ level $N = 2$ modular forms of weight $k = 4$

$$\det \mathcal{M}_d = \det \left[m^d - n^d \left(M^d \right)^{-1} N_d \right] \det M^d = -\det \left[n_d N_d \right] = -\, n_d^3 N_d^3 \alpha_d \beta_d \in \mathbb{R}$$

Modular group

Homogeneous modular group

$$\Gamma = \langle S,\ T \mid S^4 = (ST)^3 = I \rangle \cong \mathrm{SL}(2,\mathbb{Z})$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\tau \xrightarrow{S} -\frac{1}{\tau} \qquad\qquad \tau \xrightarrow{T} \tau + 1$$

duality — discrete shift symmetry

Inhomogeneous modular group

$$\overline{\Gamma} = \langle S,\ T \mid S^2 = (ST)^3 = I \rangle \cong \mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{I, -I\}$$

In other words, $\mathrm{SL}(2,\mathbb{Z})$ matrices γ and $-\gamma$ are identified

$$\tau \xrightarrow{\gamma} \gamma\tau = \frac{a\tau + b}{c\tau + d} \qquad\qquad \tau \xrightarrow{-\gamma} (-\gamma)\,\tau = \frac{-a\tau - b}{-c\tau - d} = \gamma\tau$$

Modular forms

Holomorphic functions on $\mathscr{H} = \{\tau \in \mathbb{C} : \ \mathrm{Im}(\tau) > 0\}$ transforming under Γ as

$$f\left(\gamma\tau\right) = \left(c\tau + d\right)^k f(\tau)\,, \qquad \gamma \in \Gamma$$

k is weight, a non-negative even integer

$$\gamma = -I \quad \Rightarrow \quad f(\tau) = (-1)^k f(\tau) \quad \Rightarrow \quad k \text{ is even}$$

Modular forms are periodic and admit q-expansions

$$\gamma = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad f(\tau + 1) = f(\tau) \quad \Rightarrow \quad f(\tau) = \sum_{n=0}^{\infty} a_n q^n\,, \quad q = e^{2\pi i \tau}$$

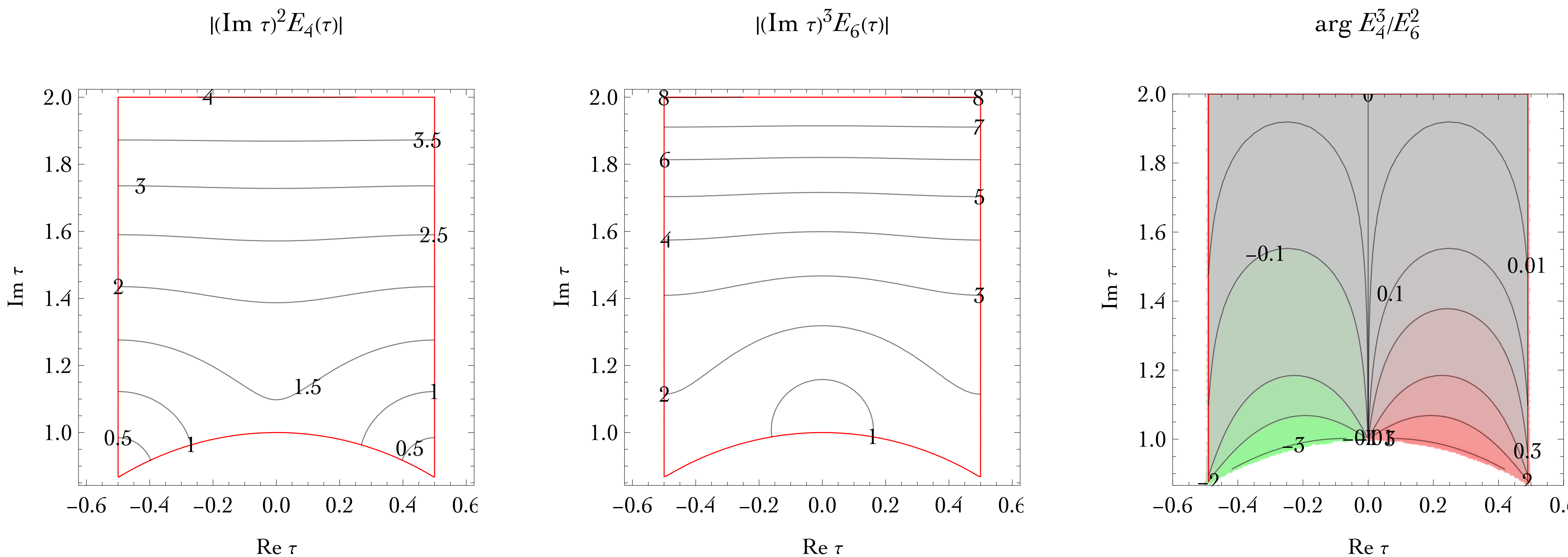
Modular forms of weight k form a linear space $\mathscr{M}_k$ of finite dimension

$$\dim \mathscr{M}_k = \begin{cases} 0 & \text{if } k \text{ is negative or odd} \\ \lfloor k/12 \rfloor & \text{if } k = 2 \pmod{12} \\ \lfloor k/12 \rfloor + 1 & \text{if } k \neq 2 \pmod{12} \end{cases}$$

E4 and E6

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \mathcal{O}(q^5)$$

$$E_6(\tau) = 1 - 540\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n} = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + \mathcal{O}(q^5)$$



Modular invariance and SUGRA

$\mathcal{N} = 1$ SUGRA action depends on

$$G = \frac{K}{M_{\rm Pl}^2} + \log \left| \frac{W}{M_{\rm Pl}^3} \right|^2$$

For G to be invariant, both K and W have to transform

$$K \to K + M_{\rm Pl}^2 \left(F + F^\dagger \right) \qquad \text{and} \qquad W \to e^{-F} W$$

In the case of modular transformations

$$F = \frac{h^2}{M_{\rm Pl}^2} \log(c\tau + d)$$

$$W \to \left(c\tau + d \right)^{-k_W} W \qquad \text{with} \qquad k_W = \frac{h^2}{M_{\rm Pl}^2} > 0$$

The superpotential is a modular function, having singularities at some values of τ

$k_W \to 0$ rigid SUSY limit

Modular invariance and SUGRA

$$W = Y^u_{ij}(\tau)\, U^c_i\, Q_j\, H_u + Y^d_{ij}(\tau)\, D^c_i\, Q_j\, H_d$$

$$Y^q_{ij}(\tau) \to (c\tau + d)^{k^q_{ij}}\, Y^q_{ij}(\tau) \qquad \text{with} \qquad k^{u(d)}_{ij} = k_{U^c_i(D^c_i)} + k_{Q_j} + k_{H_{u(d)}} - k_W$$

Furthermore, the Kähler transformation must be accompanied by a U(1) rotation

$$\psi \to e^{\frac{F - F^\dagger}{4}}\, \psi \qquad\qquad \lambda \to e^{-\frac{F - F^\dagger}{4}}\, \lambda$$

how gaugino enters the game

$$\psi_{\text{can}} \to \left(\frac{c\tau + d}{c\tau^\dagger + d}\right)^{\frac{k_W}{4} - \frac{k_\Phi}{2}} \psi_{\text{can}} \qquad\qquad \lambda \to \left(\frac{c\tau + d}{c\tau^\dagger + d}\right)^{-\frac{k_W}{4}} \lambda$$

Modular-QCD anomaly modifies as

$$A = \sum_{i=1}^{3} \left(2k_{Q_i} + k_{U^c_i} + k_{D^c_i} - 2k_W\right) + Ck_W$$

$C = 3$ is quadratic Casimir of $\mathbf{8}$ of $\mathrm{SU}(3)_C$

Gluino mass

$$\bar{\theta} = \theta_{\rm QCD} + \arg\det M_q + C\arg M_3$$

Assume $k_{\rm det} = 0$ and the quark contribution to A vanishes. Then

$$\bar{\theta} = \theta_{\rm QCD} + C\arg M_3$$

Gluino mass requires SUSY breaking

$$M_3 = \frac{g_3^2}{2} e^{K/2M_{\rm Pl}^2} K^{i\bar{j}} D_{\bar{j}} W^\dagger f_i$$

Assuming $D_\tau W = 0$ and no additional phases from SUSY breaking

$$\arg M_3 = -\arg W$$

$$W = \ldots + \frac{c_0 M_{\rm Pl}^3}{\eta(\tau)^{2k_W}} \qquad \text{and} \qquad f = \ldots + \frac{Ck_W}{4\pi^2}\log\eta(\tau)$$

$$\bar{\theta} = -8\pi^2 {\rm Im} f - C\arg W = 0$$

More on modular invariance in SUGRA

$$\det M_q \to \left(\frac{c\tau + d}{c\tau^\dagger + d}\right)^{\frac{k_{\det}}{2}} \det M_q \qquad k_{\det} = \sum_{i=1}^{3}\left(2k_{Q_i} + k_{U_i^c} + k_{D_i^c} - 2k_W\right) + 3\left(k_{H_u} + k_{H_d}\right)$$

$$M_3 \to \left(\frac{c\tau + d}{c\tau^\dagger + d}\right)^{\frac{k_W}{2}} M_3$$

(gluino mass arises only if SUSY is broken)

Finite modular groups

Infinite normal subgroups of $SL(2,\mathbb{Z})$, $N = 2\,,\,3\,,\,4\,,\ldots$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

Principal congruence subgroups of the modular group

$$\overline{\Gamma}(2) \equiv \Gamma(2)/\{I, -I\} \qquad \overline{\Gamma}(N) \equiv \Gamma(N)\,, \quad N > 2$$

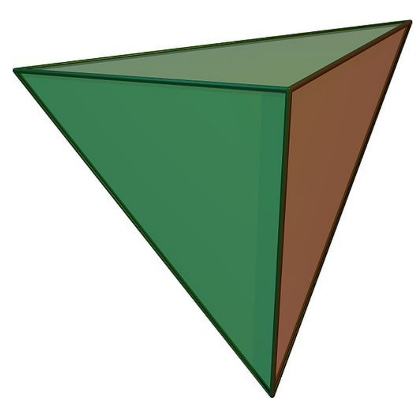
Finite modular groups

$$\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$$

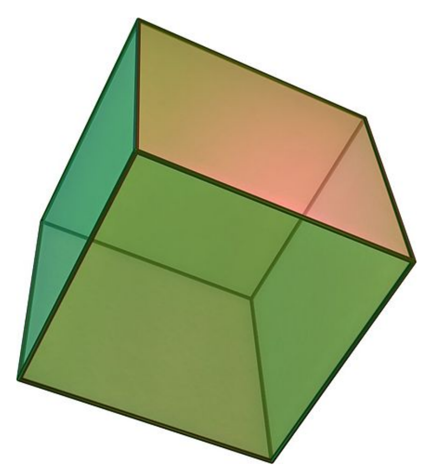
$$\Gamma_N = \left\langle S,\, T \mid S^2 = (ST)^3 = T^N = I \right\rangle\,, \quad N = 2\,,\,3\,,\,4\,,\,5$$

$\Gamma_2 \cong S_3$

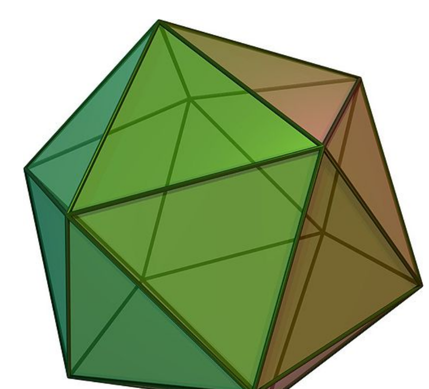
$\Gamma_3 \cong A_4$



$\Gamma_4 \cong S_4$



$\Gamma_5 \cong A_5$



Theories based on finite modular groups

$\mathcal{N} = 1$ rigid SUSY matter action

$$\mathcal{S} = \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\; K(\tau, \bar{\tau}, \psi, \bar{\psi}) + \int \mathrm{d}^4x\, \mathrm{d}^2\theta\; W(\tau, \psi) + \int \mathrm{d}^4x\, \mathrm{d}^2\bar{\theta}\; \overline{W}(\bar{\tau}, \bar{\psi})$$

Ferrara, Lust, Shapere, Theisen, PLB **225** (1989) 363

Ferrara, Lust, Theisen, PLB **233** (1989) 147

$$\begin{cases} \tau \to \dfrac{a\tau + b}{c\tau + d} \\ \psi_i \to \left(c\tau + d\right)^{-k_i} \rho_i(\tilde{\gamma})\, \psi_i \end{cases} \Rightarrow \begin{cases} W(\tau, \psi) \to W(\tau, \psi) \\ K(\tau, \bar{\tau}, \psi, \bar{\psi}) \to K(\tau, \bar{\tau}, \psi, \bar{\psi}) + f_K(\tau, \psi) + \overline{f_K}(\bar{\tau}, \bar{\psi}) \end{cases}$$

unitary representation of Γ_N

Feruglio, 1706.08749

$$W(\tau, \psi) = \sum_n \sum_{\{i_1, \ldots, i_n\}} \sum_s g_{i_1 \ldots i_n, s} \left(Y_{i_1 \ldots i_n, s}(\tau)\, \psi_{i_1} \ldots \psi_{i_n} \right)_{\mathbf{1}, s}$$

$$Y(\tau) \xrightarrow{\gamma} (c\tau + d)^{k_Y} \rho_Y(\tilde{\gamma})\, Y(\tau)$$

$$k_Y = k_{i_1} + \ldots + k_{i_n}$$

$$\rho_Y \otimes \rho_{i_1} \otimes \ldots \otimes \rho_{i_n} \supset \mathbf{1}$$

Yukawa couplings are modular forms!

Models with finite modular symmetries

$$\begin{cases} \Phi \to (c\tau + d)^{-k_\Phi} \rho_\Phi(\gamma)\, \Phi \\ Y \to (c\tau + d)^{k_Y} \rho_Y(\gamma)\, Y \end{cases}$$

Feruglio, 1706.08749

ρ is a unitary representation of the finite modular group $\Gamma_N = \Gamma/\Gamma(N)$

$$\Gamma_2 \cong S_3 \qquad \Gamma_3 \cong A_4' = T' \qquad \Gamma_4 \cong S_4' \qquad \Gamma_5 \cong A_5'$$

Invariance of the superpotential $W \sim Y(\tau)\, \Phi_I \Phi_J \Phi_K$ under Γ and Γ_N requires

$$\begin{cases} k_Y = k_I + k_J + k_K \\ \rho_Y \otimes \rho_I \otimes \rho_J \otimes \rho_K \supset \mathbf{1} \end{cases}$$

Models with finite modular symmetries

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Feruglio, 1706.08749

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$$\begin{cases} k_Y = k_I + k_J + k_K \\ \rho_Y \otimes \rho_I \otimes \rho_J \otimes \rho_K \supset \mathbf{1} \end{cases}$$

Penedo, Petcov, 2404.08032

Multi-dim irreps $\rho \sim \mathbf{r}\,, \quad \mathbf{r} > 1$ for SM quarks do not allow to simultaneously

- realise the proposed mechanism for $\bar{\theta} = 0$
- successfully describe quark masses and mixings

For the mechanism to work, SM quarks must furnish 1D irreps $\rho \sim \mathbf{1},\ \mathbf{1}',\ \mathbf{1}'', \dots$

Models with finite modular symmetries

Minimal models (6 Lagrangian parameters per sector) — Penedo, Petcov, 2404.08032

$$Y^q = \begin{pmatrix} c^q_{11} & 0 & c^q_{13} F^{(k')}_{\mathbf{1}^*} + c'^q_{13} F'^{(k')}_{\mathbf{1}^*} \\ 0 & c^q_{22} & c^q_{23} F^{(k)}_{\mathbf{1}^*} \\ 0 & 0 & c^q_{33} \end{pmatrix}$$

(up to weak basis transformations)

Minimal models (I and II)	
All Γ'_N	$(k, k') = (10, 12),\ (12, 14),\ (14, 16)$
S_3 only	$(10', 12),\ (10, 18'),\ (10', 18),\ (12, 12'),\ (12, 14'),\ (12, 16'),\ (12', 16),\ (12, 20'),\ (12', 20),\ (14', 16),\ (14, 18'),\ (14', 18),\ (14, 22'),\ (14', 22),\ (16, 16'),\ (16', 18),\ (16', 18'),\ (16, 20'),\ (16', 20),\ (18, 20'),\ (18', 20'),\ (20, 20'),\ (20', 22),\ (20', 22')$
A'_4 only	$(8', 12),\ (8', 18),\ (10', 12),\ (10, 16'),\ (10', 16),\ (10, 20''),\ (10', 20),\ (12, 12'),\ (12, 12''),\ (12, 14'),\ (12, 14''),\ (12, 16''),\ (12', 16),\ (12'', 16'),\ (12, 18'),\ (12, 18''),\ (12', 18),\ (12'', 18),\ (12, 22''),\ (12', 22),\ (12'', 22'),\ (14, 16'),\ (14', 16),\ (14', 16'),\ (14'', 16),\ (14'', 16'),\ (14', 18),\ (14, 20'),\ (14, 20''),\ (14', 20),\ (14', 20''),\ (14'', 20),\ (14'', 20'),\ (14, 24''),\ (14'', 24'),\ (16, 16''),\ (16', 16''),\ (16, 18'),\ (16, 18''),\ (16', 18'),\ (16', 18''),\ (16'', 18),\ (16'', 20'),\ (16, 22''),\ (16', 22''),\ (16'', 22),\ (16'', 22'),\ (16'', 26'),\ (18, 18'),\ (18, 18''),\ (18', 20),\ (18', 20'),\ (18', 20''),\ (18'', 20),\ (18'', 20'),\ (18'', 20''),\ (18, 22''),\ (18', 22),\ (18'', 22'),\ (18', 24''),\ (18'', 24'),\ (20, 22''),\ (20', 22''),\ (20'', 22''),\ (22, 22''),\ (22', 22''),\ (22'', 24'),\ (22'', 24''),\ (22'', 26')$

Modular weights are large

String compactifications
lead to smaller weights

(For models based on modular A_4
see also Petcov, Tanimoto, 2404.00858)