CP CONSERVATION IN THE STRONG INTERACTIONS

BJÖRN GARBRECHT, TUM

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Outline

- Introduction: QCD θ-parameter, EFT, topology, neutron EDM
- Functional quantization and path integral contour
- Canonical quantization and θ-vacua

INTRODUCTION

CP-odd terms in effective field theories

Topology

CP violation in the strong interactions?

No empirical evidence—neutron electric dipole moment (EDM) strongly constrained:

$$d_n = (0.0 \pm 1.1_{\text{stat}} \pm 0.2_{\text{sys}}) \times 10^{-26}\, e\,\text{cm}$$ [2020 @ PSI]

QCD with massive quarks

$$\mathcal{L} \supset \frac{1}{2g^2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} + \sum_{j=1}^{N_f} \bar{\psi}_j \left(\mathrm{i} \not{D} - m_j \mathrm{e}^{\mathrm{i}\alpha_j \gamma^5} \right) \psi_j + \frac{1}{16\pi^2} \theta \operatorname{tr} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

Believed to cause a neutron electric dipole moment (EDM) $d_n \sim 10^{-15}\, e\,\text{cm} \left(\theta + \sum_j \alpha_j \right)$

[Baluni (1979); Crewther, Di Vecchia, Veneziano, Witten (1979)]

Or does it?

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$\propto \vec{E}^2 + \vec{B}^2$

γ^5, parity-odd

$\propto \vec{E} \cdot \vec{B}$ parity-odd

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Or does it?

Effective interactions with θ

$\mathrm{SU}(N_f)_\mathrm{L} \times \mathrm{SU}(N_f)_\mathrm{R}$ global symmetry in the limit of massless quarks

Chiral $\mathrm{U}(1)_A$ symmetry of the quarks is anomalous however
$\longrightarrow \mathcal{L}$ invariant under [Fujikawa (1979,80)]

chiral trafo		"spurion" trafo
$\psi \to \mathrm{e}^{\mathrm{i}\beta\gamma_5}\psi$ $\bar{\psi} \to \bar{\psi}\mathrm{e}^{\mathrm{i}\beta\gamma_5}$	plus	$m_j \mathrm{e}^{\mathrm{i}\alpha_j\gamma^5} \to m_j \mathrm{e}^{\mathrm{i}(\alpha_j - 2\beta)\gamma^5}$ $\theta \to \theta + 2N_f\beta$

Spurions break the symmetries explicitly. $\longrightarrow$ Approximate symmetries
This pattern should be replicated by any effective theory.

Rephasing invariant: $\bar{\theta} = \theta + \bar{\alpha}$, where $\bar{\alpha} = \sum_{j=1}^{N_f} \alpha_j$, $\longrightarrow$ θ is an angle

Integrating out gauge fields: Effective interactions

Topological effects described by effective 't Hooft vertex (Γ_{N_f} some coefficient) ['t Hooft (1976,86)] [Kaplan, Sen ('24)]

$$\mathcal{L} + \frac{1}{16\pi^2}\theta\,\mathrm{tr}\,F_{\mu\nu}\tilde{F}^{\mu\nu} \to \mathcal{L} - \Gamma_{N_f}\mathrm{e}^{\mathrm{i}\xi}\prod_{j=1}^{N_f}(\bar{\psi}_j P_\mathrm{L}\psi_j) - \Gamma_{N_f}\mathrm{e}^{-\mathrm{i}\xi}\prod_{j=1}^{N_f}(\bar{\psi}_j P_\mathrm{R}\psi_j)$$

As a spurion, $\xi \to \xi + 2N_f\beta \longrightarrow$

Two options:
- $\xi = \theta$ (in general misaligned with masses) $\to$ CP violation
- $\xi = -\bar{\alpha}$ (present claim, aligned with mass terms) $\to$ no CP violation

χPT at low energies

$$U = U_0\mathrm{e}^{\frac{\mathrm{i}}{f_\pi}\Phi} \qquad U_0:\ \textit{chiral condensate} \qquad \Phi = \begin{bmatrix} \pi^0 + \eta' & \sqrt{2}\,\pi^+ \\ \sqrt{2}\,\pi^- & -\pi^0 + \eta' \end{bmatrix}$$

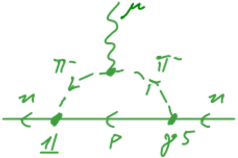
$\xi = \theta$: χral condensate aligned with θ

$\xi = -\bar{\alpha}$: χral condensate aligned with quark mass phases

$$\mathcal{L} = \frac{f_\pi^2}{4}\mathrm{Tr}\,\partial_\mu U\partial^\mu U^\dagger + \frac{f_\pi^2 B_0}{2}\mathrm{Tr}(M U + U^\dagger M^\dagger) + |\lambda|\mathrm{e}^{-\mathrm{i}\xi}f_\pi^4\det U + |\lambda|\mathrm{e}^{\mathrm{i}\xi}f_\pi^4\det U^\dagger \quad M = \mathrm{diag}\{m_u\mathrm{e}^{\mathrm{i}\alpha_u}, m_d\mathrm{e}^{\mathrm{i}\alpha_d}\}$$

$$\mathcal{L}_\text{neutron} \supset -\frac{c_1}{f_\pi}\partial_\mu\pi^a\bar{\mathcal{N}}T^a\gamma^\mu\gamma_5\mathcal{N} \qquad CP \text{ even}$$

$$\mathcal{N} = \begin{pmatrix} p \\ n \end{pmatrix} \qquad +\frac{c_2\bar{m}}{f_\pi}(\xi + \alpha_u + \alpha_d + \alpha_s)\bar{\mathcal{N}}\pi^a T^a\mathcal{N} \qquad CP \text{ odd}$$



Topology in four-dimensional spacetime—winding number Δn

$$U = \begin{pmatrix} a_\mathrm{R} + \mathrm{i} a_\mathrm{I} & -b_\mathrm{R} + \mathrm{i} b_\mathrm{I} \\ b_\mathrm{R} + \mathrm{i} b_\mathrm{I} & a_\mathrm{R} - \mathrm{i} a_\mathrm{I} \end{pmatrix} \in \mathrm{SU}(2) \text{ for } a_\mathrm{R}^2 + a_\mathrm{I}^2 + b_\mathrm{R}^2 + b_\mathrm{I}^2 = 1$$

$$\Rightarrow \text{Homotopy: } \mathrm{SU}(3) \supset \mathrm{SU}(2) \cong S^3 \longrightarrow \pi_3(\mathrm{SU}(2)) = \pi_3(S^3) = \mathbb{Z}$$

Theta-term/topological term is a total divergence:

gauge invariant

$$\frac{1}{4}\mathrm{tr} F_{\mu\nu}\tilde{F}_{\mu\nu} = \partial_\mu K_\mu$$

gauge dependent

$$K_\mu = \epsilon_{\mu\nu\alpha\beta}\mathrm{tr}\left[\frac{1}{2}A_\nu \partial_\alpha A_\beta + \frac{1}{3}A_\nu A_\alpha A_\beta\right]$$

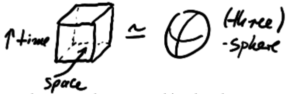
Topological quantization for pure gauge $A_\mu \to -\frac{\mathrm{i}}{g}(\partial_\mu U)U^{-1}$ at $\partial\Omega \cong S^3$

$$\Delta n = \frac{1}{16\pi^2}\int_\Omega \mathrm{d}^4 x F_{\mu\nu}\tilde{F}_{\mu\nu} = \frac{1}{4\pi^2}\oint_{\partial\Omega} \mathrm{d}^3\sigma K_\perp \in \mathbb{Z}$$

Haar measure for pure gauge
$K_\mu = \frac{1}{6}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}[(U^{-1}\partial_\nu U)(U^{-1}\partial_\lambda U)(U^{-1}\partial_\rho U)]$

[see e.g. Coleman, "Aspects of symmetry" (1985)]

E.g. take boundary of $\Omega = \mathbb{R}^4$ as a sphere S^3:



Or $\Omega = T^4$(lattice), $\Omega = S^4$(Euclidean dS): $\Delta n \in \mathbb{Z}$ based on slightly more involved argument

Topology on spatial hypersurfaces—point compactification, large gauge transformations

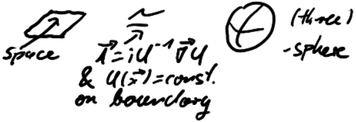
Consider temporal gauge $A^0 = 0$ (in view of canonical quantization)

Chern–Simons functional:

$$W[\vec{A}] = \frac{1}{4\pi^2}\varepsilon_{ijk}\int_V \mathrm{d}^3x\,\mathrm{tr}\left[\frac{1}{2}A_i\partial_j A_k - \frac{\mathrm{i}}{3}A_i A_j A_k\right] \equiv \frac{1}{4\pi^2}\int_V \mathrm{d}^3x\, K_0$$

Define $\vec{A}_U = U\vec{A}U^{-1} + \mathrm{i}\,U^{-1}\vec{\nabla}U$ (residual gauge freedom in temporal gauge)

With extra constraint $U(\vec{x}) \to$ const. on ∂V (periodic on T^3)
$\to$ Point compactification, homotopy $V \cong S^3$ ($V \cong T^3$)



$U^{(\nu)}$: equivalence classes of "large" ($\nu \neq 0$) gauge transformation on spacelike ($\tau =$ const.) hypersurface $V \simeq S^3$ ($V \simeq T^3$) with $W[\vec{A}_{U^{(\nu)}}] - W[\vec{A}] = \nu \in \mathbb{Z}$

However: Extra constaint not enforced by $A^0 = 0$ —would require extra provisions

FUNCTIONAL QUANTIZATION

Take time to infinity before summing over topological sectors

Euclidean path integral & topology

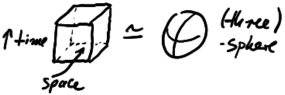
Topological term $F\tilde{F}$ total derivative—how can it contribute?
Does interference of sectors have a material effect?

Recall: Euclidean path integral projects on ground state

$$\lim_{T\to\infty} \frac{\mathrm{e}^{-HT}}{\mathrm{e}^{-E_0 T}} \quad \text{or} \quad \lim_{T\to\infty} \frac{\mathrm{e}^{-\mathrm{i}HT(1-\mathrm{i}\varepsilon)}}{\mathrm{e}^{-\mathrm{i}E_0 T(1-\mathrm{i}\varepsilon)}}$$

H: Hamiltonian
E_0: ground state energy

$\longrightarrow$ Consider $\Omega = \mathbb{R}^4$ (or different *spatial* topologies)



Finite action $\longrightarrow$ pure gauge at infinity
$\longrightarrow$ Topological quantization $\longrightarrow$ Phases $\mathrm{e}^{\mathrm{i}\Delta n\theta}$

No reason for topological quantization in finite $\Omega \subset \mathbb{R}^4$ (finite temperature, see below)

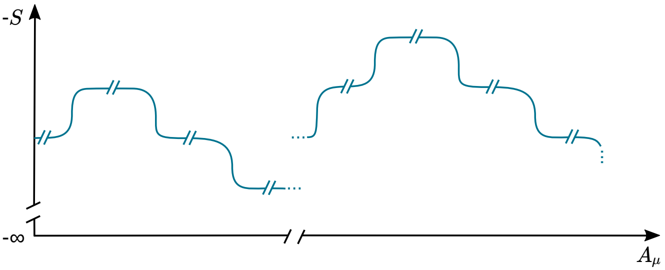
Must take $T \to \infty$ before summing over sectors:

$$Z = \lim_{N\to\infty} \sum_{\Delta n=-N}^{N} \lim_{VT\to\infty} \int_{\Delta n} \mathcal{D}\phi\, \mathrm{e}^{-S_{\mathrm{E}}[\phi]}$$

More technically: Integration contour from Lefschetz thimbles

Parametrization of the path integral through steepest descent contours about classical saddle points $\longrightarrow$ Contour integration on Lefschetz thimbles

$$\frac{\partial\phi(x;u)}{\partial u} = \overline{\frac{\delta S_{\mathrm{E}}[\phi(x;u)]}{\delta\phi(x;u)}} \Longrightarrow -\frac{\partial \mathrm{Re} S_{\mathrm{E}}[\phi(x;u)]}{\partial u} \leq 0 \ \text{ and } \ \frac{\partial \mathrm{Im} S_{\mathrm{E}}[\phi(x;u)]}{\partial u} = 0$$



Each thimble emerges from a critical point and corresponds to one $\Delta n \in \mathbb{Z}$

Keeping VT finite while summing over different Δn ($\Leftrightarrow$ different boundary conditions, infinite distance in field space) does *not* correspond to a nonsingular deformation of the Cauchy contour

Integration contour sweeps over full thimbles first, i.e. $VT \to \infty$ before sum over Δn

So is it $\xi = -\bar{\alpha}$ or $\xi = \theta$?

- Take $\langle F(x)\tilde{F}(x)\rangle$ as measure for CP violation
- Each element in the sequence over N vanishes (not so when limits ordered the other way around):

$$\langle F(x)\tilde{F}(x)\rangle = \lim_{\substack{N\to\infty \\ N\in\mathbb{N}}} \lim_{VT\to\infty} \frac{\sum_{\Delta n=-N}^{N} \frac{\Delta n}{VT} Z_{\Delta n}}{\sum_{\Delta n=-N}^{N} Z_{\Delta n}} = 0$$

- Index theorem: Δn is the difference of the numbers of right and left chiral zero modes
- Left/right chiral quasi-zero modes in spectral representation of fermion correlation regulated by $1/(m\,\mathrm{e}^{\mp \mathrm{i}\alpha})$

$$S(x,x') = \frac{\hat{\psi}_{0\mathrm{L}}(x)\hat{\psi}^{\dagger}_{0\mathrm{L}}(x')}{m\mathrm{e}^{-\mathrm{i}\alpha}} + \sum\!\!\!\!\!\!\int_{\lambda^{\mathrm{E}}\neq 0} \frac{\hat{\psi}_{\lambda}(x)\hat{\psi}^{\dagger}_{\lambda}(x')}{\lambda}$$

$\uparrow$ $\Delta n = -1$

- Contributions from discrete modes to correlation function vanish for $VT \to \infty \to$ Quark correlations remain aligned with quark mass after interference of Δn-sectors

$\rightarrow$ $\xi = -\bar{\alpha}$

- Order of limits matters because series is not positive definite due to phases $\mathrm{e}^{\mathrm{i}\Delta n\theta}$, not absolutely summable

CANONICAL QUANTIZATION

Done consistently without extra gauge constraint, point compactification

Properly normalizable physical states

Theta vacuum, standard story

Take, $A^0 = 0$, *assume* in addition:

For $|\vec{x}| \to \infty$: $\vec{A}(\vec{x}) = \mathrm{i}\, U^{-1}(\vec{x})\vec{\nabla}\, U(\vec{x})$ and $U(\vec{x}) \to$ const.

But why? [cf. Jackiw (1980)]

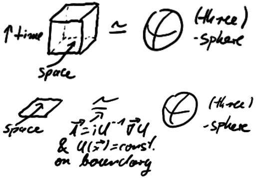
Consider initial and final states, taking $x_4 \to \pm\infty$

$\to$ *Ansatz*: Construct from pure gauge configurations on these surfaces, with

$$n_{\pm\infty} = \frac{1}{4\pi^2} \int\limits_{x^4 = \pm\infty} \mathrm{d}^3\sigma K_\perp \in \mathbb{Z}$$

Chern–Simons number
not gauge invariant

point compactification



Prevacua: $n_{-\infty} \to |n\rangle$, $n_{\infty} \to \langle n|$ (field eigenstates)

Gauge invariant (up to phase) state $|\theta\rangle = \sum_n \mathrm{e}^{-\mathrm{i}n\theta}|n\rangle$ [Callan, Dashen, Gross (1976); Jackiw, Rebbi (1976); Jackiw (1980)]

⚠ States not normalizable in the proper sense: $\langle\theta^{(i)}|\theta^{(j)}\rangle = \sum_n \delta(\theta^{(i)} - \theta^{(j)} + 2\pi n)$

[cf. e.g. Callan, Dashen, Gross (1976); issue taken by Okubo, Marshak (1992)]

Without ado, this contradicts 1st Dirac–von Neumann axiom of quantum mechanics.

[Dirac 1932, von Neumann 1932]

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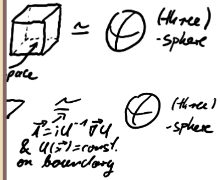
Consider initial an…

$\to$ *Ansatz*: Constr… with



$$n_{\pm\infty} = \frac{1}{4\pi^2} \int_{x^4=\pm\infty}$$

Prevacua: $n_{-\infty}$ — n_{∞} —

B. STATEMENT OF THE POSTULATES

1. Description of the state of a system

In chapter I, we introduced the concept of the quantum state of a particle. We first characterized this state at a given time by a square-integrable wave function. Then, in chapter II, we associated a ket of the state space $\mathscr{E}_{\mathbf{r}}$ with each wave function: choosing $|\psi\rangle$ belonging to $\mathscr{E}_{\mathbf{r}}$ is equivalent to choosing the corresponding function $\psi(\mathbf{r}) = \langle \mathbf{r} | \psi \rangle$. Therefore, the quantum state of a particle at a fixed time is characterized by a ket of the space $\mathscr{E}_{\mathbf{r}}$. In this form, the concept of a state can be generalized to any physical system.

> *First Postulate*: At a fixed time t_0, the state of a physical system is defined by specifying a ket $|\psi(t_0)\rangle$ belonging to the state space $\mathscr{E}$.

[Cohen-Tanoudji, Diu, Laloë]

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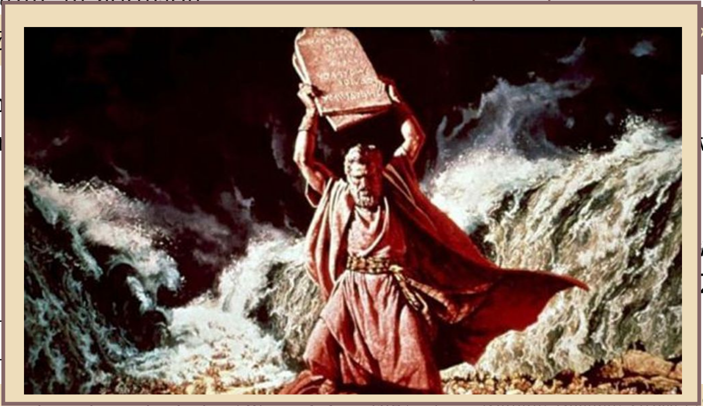

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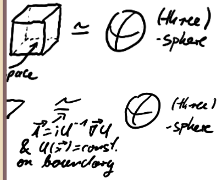

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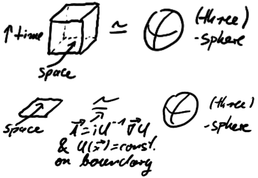

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Canonical quantization of the gauge field

Minkowski spacetime, temporal gauge $A^0 = 0$, no sources $\longrightarrow$

$$g\vec{E}^a = -\partial/\partial t\,\vec{A}^a$$
$$g\vec{B}^a = \vec{\nabla}\times\vec{A}^a - 1/2\,f^{abc}\vec{A}^a\times\vec{A}^b$$

Canonical momentum conjugate to $\vec{A}^a$:

$$g\vec{\Pi}^a = -\vec{E}^a + \frac{g^2}{8\pi^2}\theta\vec{B}^a$$

- **No constraints on ∂V accounted for $\longrightarrow$ $\Psi[\vec{A}]$ must be defined for $U(\vec{x}) \neq \text{const.}$ on ∂V**
- **Residual gauge dofs.: Throw out unphysical states "First quantize, then constrain"**

The corresponding operator must observe the commutation relations:

$$[A^{a,i}(\vec{x}),\Pi^{b,j}(\vec{x}\,')] = \mathrm{i}\delta^{ij}\delta^{ab}\delta^3(\vec{x}-\vec{x}\,'),\quad [\Pi^{a,i}(\vec{x}),\Pi^{b,j}(\vec{x}\,')] = 0,\quad \text{take e.g. } \vec{\Pi}^a = \frac{\delta}{\mathrm{i}\delta\vec{A}^a}$$

Hamiltonian density:

$$\mathcal{H} = \frac{1}{2}\left((\vec{E}^a)^2 + (\vec{B}^a)^2\right) = \frac{1}{2}\left(\left(g\frac{\delta}{\mathrm{i}\delta\vec{A}^a} - \frac{g^2}{8\pi^2}\theta\vec{B}^a\right)^2 + (\vec{B}^a)^2\right)$$

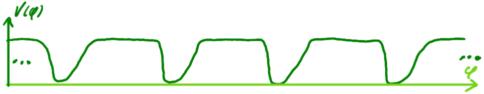
[e.g. Jackiw (1980)]

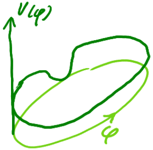
Since $[U^{(\nu)}, H] = 0$, can find states $\Psi_{\theta^{(i)}}[\vec{A}_{(U^{(1)})^\nu}] = \mathrm{e}^{\mathrm{i}\nu\theta^{(i)}}\Psi_{\theta^{(i)}}[\vec{A}]$

Crystal or circle?

Functionals $\Psi_\theta(\vec{A})$ with above periodicity properties can be compared with Bloch states

Bloch states live on a crystal:
$\vec{A}_{U_4^{(1)}}$ is a *different* site than $\vec{A}$





In contrast: In gauge theory
$\vec{A}_{U_4^{(1)}}$ is a *redundant* description of the configuration $\vec{A}$—corresponding to $\varphi \to \varphi + 2\pi n$ on a circle

On a crystal: Bloch states do not correpsond to normalized wave functions, these are rather wave packets made up of Bloch states. Packets, however, not translation (gauge) invariant

On a circle: Truncation of the inner product according to a single period leads to properly normalizable states, corresponding here to *gauge fixing* $\vec{A} \in \mathcal{A}$ so that each physical configuration appears one time and one time only:

$$\int_{\mathcal{A}} \underbrace{\mathcal{D}\vec{A}\, f_{\mathcal{A}}[\vec{A}]}_{\substack{\text{gauge invariant} \\ \text{under change of } \mathcal{A}}} \quad \Psi^{(a)*}_{\theta^{(i)}}[\vec{A}]\, \Psi^{(b)}_{\theta^{(j)}}[\vec{A}]$$

Note: Under gauge fixed inner product, $\Psi^{(a)}_{\theta^{(i)}}$, $\Psi^{(b)}_{\theta^{(j)}}$ no longer orthogonal for $\theta^{(i)} \neq \theta^{(j)}$

Form of the wave functional & Constraining the Hilbert space

Require: Gauge invariance & $\frac{\delta}{\mathrm{i}\delta\vec{A}(\vec{x})}$ should remain Hermitian under restricted inner product

$\Longrightarrow \Psi^{(a)}[\vec{A}] \overset{(*)}{=} \Psi^{(a)}[\vec{A}]_{\mathrm{g.i.}} \exp(\mathrm{i}\varphi[\vec{A}])$ valid for *all* $U(\vec{x})$ (also nonconstant on boundary)

gauge invariant

independent of state (a)

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gauge invariant ↗   ↖ independent of state (a)

Now $\int \mathrm{d}^3x\, \mathrm{tr}\vec{B}\cdot\frac{\delta}{\mathrm{i}\delta\vec{A}}$ leads to mixing of pure gauge and other directions → Separation?

Form of the wave functional & Constraining the Hilbert space

Require: Gauge invariance & $\frac{\delta}{\mathrm{i}\delta\vec{A}(\vec{x})}$ should remain Hermitian under restricted inner product

$\Longrightarrow \Psi^{(a)}[\vec{A}] \overset{(*)}{=} \Psi^{(a)}[\vec{A}]_{\mathrm{g.i.}} \exp(\mathrm{i}\varphi[\vec{A}])$ valid for *all* $U(\vec{x})$ (also nonconstant on boundary)

gauge invariant ↗ independent of state (a)

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⟶ "Diagonalize" H:

$$\begin{aligned} \Psi'[\vec{A}] &= \mathrm{e}^{-\mathrm{i}\theta W[\vec{A}]}\Psi[\vec{A}]\,, \\ H' &= \mathrm{e}^{-\mathrm{i}\theta W[\vec{A}]} H \mathrm{e}^{\mathrm{i}\theta W[\vec{A}]} = \frac{1}{2}\int \mathrm{d}^3x\, \mathrm{tr}\left[-g^2\frac{\delta^2}{\delta\vec{A}^2} + \vec{B}^2\right] \\ &= -\frac{g^2}{2}\sum\!\!\!\!\!\!\int_\sigma \frac{\delta^2}{\delta A^2(\sigma)} + \frac{1}{2}\int \mathrm{d}^3x\, \mathrm{tr}\,\vec{B}^2\,,\ \sigma \in \{\sigma_{\mathrm{gauge}}, \sigma_{\|}\} \end{aligned}$$

$$\frac{\delta}{\delta\vec{A}(\vec{x})} W[\vec{A}] = \frac{g}{8\pi^2}\vec{B}(\vec{x})$$

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Only trivial one-dimensional representations of SU(2)

$$\Psi[\vec{A}_U] = \mathrm{e}^{\mathrm{i}\varphi[\vec{A}_U]}\Psi[\vec{A}]\ \ \text{(eigenstate of } U\text{)}\,, \qquad U_3 = U_2\,U_1$$

$$\mathrm{e}^{\mathrm{i}\varphi[\vec{A}_{U_3}]} = \mathrm{e}^{\mathrm{i}\varphi[\vec{A}_{U_2}]}\mathrm{e}^{\mathrm{i}\varphi[\vec{A}_{U_1}]} \Rightarrow \mathrm{e}^{\mathrm{i}\varphi[\vec{A}_{U_2U_1}]} - \mathrm{e}^{\mathrm{i}\varphi[\vec{A}_{U_1U_2}]} = 0$$

$\Rightarrow \Psi'[\vec{A}]$ is gauge invariant (**)

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Throw states not satisfying $(*,**)$ out of the Hilbert space

$\to CP$ conserved

No "dark backgrounds"

Nondiagonal basis

Redefining derivatives w.r.t. $\vec{A}$ as

$$\vec{D}_{\vec{A}} \Psi[\vec{A}] = \mathrm{i}\left(\frac{\delta}{\mathrm{i}\delta\vec{A}} - \theta\frac{g}{8\pi^2}\vec{B}\right)\Psi[\vec{A}]$$

corresponds to a canonical transformation of the momentum operator.

Induces translation as

$$T[\Delta\vec{A}]\,\Psi[\vec{A}] = \mathrm{e}^{-\mathrm{i}\theta\left(W[\vec{A}+\Delta\vec{A}]-W[\vec{A}]\right)}\Psi[\vec{A}+\Delta\vec{A}]$$

For a shift $\Delta\vec{A}_{\mathrm{gauge}}$ corresponding to a *general* gauge transformation: gauge invariant

$$T[\Delta\vec{A}_{\mathrm{gauge}}]\,\Psi[\vec{A}] = \Psi[\vec{A}] \quad \text{if} \quad \Psi[\vec{A}] = \mathrm{e}^{\mathrm{i}\theta\, W[\vec{A}]}\Psi_{\mathrm{g.i.}}[\vec{A}]$$

Agrees with reasoning & result in the diagonal basis

θ in Ψ_θ is pinned to θ in H so that CP is conserved

Conclusion

There is no CP violation in QCD.

Challenges to the standard calculation and resolutions:

- **Taking $T \to \infty$ after summing over sectors corresponds to an inequivalent deformation of the integration contour**

  Maintain Cauchy contour and order of limits
- **No point compactification/topology in temporal gauge (w/o extra constraint)**

  Drop the constraint, define Ψ for all temporal gauges
- **θ-vacua are not properly normalizable $\to$ not physical states according to postulates of QM**

  Giving up point compactification can integrate over each physical configuration one time and one time only

  $\to$No need to give up Dirac–von Neumann axioms

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THANK YOU!

For hearing me out and to the organizers

Effective chiral Lagrangian (χPT)

$$U = U_0 \mathrm{e}^{\frac{\mathrm{i}}{f_\pi}\Phi} \qquad U_0: \textit{ chiral condensate}$$

$$\Phi = \begin{bmatrix} \pi^0 + \eta' & \sqrt{2}\,\pi^+ \\ \sqrt{2}\,\pi^- & -\pi^0 + \eta' \end{bmatrix}$$

Chiral Lagrangian (lowest order terms) inherits “spurious” symmetries:

$$\mathcal{L} = \frac{f_\pi^2}{4}\mathrm{Tr}\,\partial_\mu U \partial^\mu U^\dagger + \frac{f_\pi^2 B_0}{2}\mathrm{Tr}(M\,U + U^\dagger M^\dagger) + |\lambda|\mathrm{e}^{-\mathrm{i}\xi} f_\pi^4 \det U + |\lambda|\mathrm{e}^{\mathrm{i}\xi} f_\pi^4 \det U^\dagger$$
$$+\mathrm{i}\bar{N}\not{\partial}N - \left(m_N \bar{N}\,\tilde{U}\,P_\mathrm{L}\,N + \mathrm{i}c\bar{N}\,\tilde{U}^\dagger\not{\partial}P_\mathrm{L}\,\tilde{U}\,N + d\,\bar{N}\,\tilde{M}^\dagger P_\mathrm{L}\,N + e\,\bar{N}\,\tilde{U}\,\tilde{M}\,\tilde{U}\,P_\mathrm{L}\,N + \mathrm{h.c.}\right)$$

$$M = \mathrm{diag}\{m_u \mathrm{e}^{\mathrm{i}\alpha_u}, m_d \mathrm{e}^{\mathrm{i}\alpha_d}, m_s \mathrm{e}^{\mathrm{i}\alpha_s}\}$$

$\tilde{M}$, $\tilde{U}$ reduced to subspace (u, d)

nucleon doublet $N = \begin{pmatrix} p \\ n \end{pmatrix}$

Effective interaction $\propto \det U$ cannot be quantitatively reliably handled in χPT but yet represents pattern of broken axial symmetry.

Neutron electric dipole moment

$$\mathrm{i}\mathcal{M} = -2\mathrm{i}D(q^2)\varepsilon^*_\mu(\vec{q})\bar{u}_{s'}(\vec{p}\,')\frac{\mathrm{i}}{4}[\gamma^\mu,\gamma^\nu]\,q_\nu \mathrm{i}\gamma_5\, u_s(\vec{p})$$

$$\rightarrow \mathcal{L}_{\mathrm{eff}} = D(0)\bar{n}\,\underbrace{F_{\mu\nu}\frac{\mathrm{i}}{4}[\gamma^\mu,\gamma^\nu]\mathrm{i}\gamma_5}_{\subset \vec{S}\cdot\vec{E}}\,n$$

- χPT value: $d_n = 3.2\times 10^{-16}(\xi+\bar{\alpha})\,e\,\mathrm{cm}$
- Experimental bound: $|d_n| < 1.8\times 10^{-26}\,e\,\mathrm{cm}$ (90% c.l.) [nEDM/PSI (2020)]
- Calculations e.g. of neutron EDM implicitly *assume* $\xi=\theta$
  [e.g. Baluni (1979); Crewther, Di Vecchia, Veneziano, Witten (1979)]
- However $\xi = -\bar{\alpha}$ also perfectly valid by arguments used to this end
- Another signature—weaker bounds: $\eta' \to \pi\pi$

Why $T \to \infty$

(Implying $\Omega = VT \to \infty$ as opposed to a finite spacetime volume)

To evaluate amplitudes at finite T, project path integral on the state in terms of a wave function(al) $\Psi[\phi(\vec{x})] = \langle\phi(\vec{x})|\Psi\rangle$ (more on this later):

$$\langle\Psi_f, t_f|\Psi_i, t_i\rangle = \int \mathcal{D}\phi_f \mathcal{D}\phi_i \langle\Psi_f, t_f|\phi_f\rangle \int\limits_{\substack{\phi(t_f,\vec{x})=\phi_f(\vec{x})\\ \phi(t_i,\vec{x})=\phi_i(\vec{x})}} \mathcal{D}\phi\, \mathrm{e}^{\mathrm{i}S[\phi]}\langle\phi_i|\Psi_i, t_i\rangle$$

$|\phi_{i,f}\rangle$ field eigenstates, not energy eigenstates

Problem: Neither know $\Psi[\phi(\vec{x})]$ nor kernels of Schrödinger equation

Way out: Euclidean path integral/project on ground state

$$\lim_{T\to\infty} \frac{\mathrm{e}^{-HT}}{\mathrm{e}^{-E_0 T}} \quad \text{or} \quad \lim_{T\to\infty} \frac{\mathrm{e}^{-\mathrm{i}HT(1-\mathrm{i}\varepsilon)}}{\mathrm{e}^{-\mathrm{i}E_0 T(1-\mathrm{i}\varepsilon)}}$$

H: Hamiltonian
E_0: ground state energy

$\rightarrow$ Obtain vacuum correlations without bothering about $\Psi[\phi(\vec{x})]$

Boundary configurations & topological quantization

The parameter θ can be viewed as an angular variable (forced by the anomalous chiral current). $\longrightarrow$

Requires $\Delta n \in \mathbb{Z}$ ("topological quantization") $\rightarrow \exp(\mathrm{i}S)|_\theta = \exp(\mathrm{i}S)|_{\theta+2\pi}$

Readily built into the path integral for $VT \rightarrow \infty$ without constraining boundary conditions by hand:

(Relatively) nonvanishing contributions in *infinite* spacetime only from classical local minima of the Euclidean action & fluctuations about these—these configurations must go to pure gauges at ∞

Indeed, for pure gauge configurations at $\infty \rightarrow \Delta n \in \mathbb{Z}$ (as discussed above)

There is no such restriction/principle to fixed physical bcs. for finite VT.

Consequence: $\Delta n \in \mathbb{Z}$ requires $T \rightarrow \infty$ first $\rightarrow$ In the path integral, take $T \rightarrow \infty$, then sum over all topological sectors Δn weighted $\exp(\mathrm{i}\Delta n\theta)$

Fermion correlations

The effective vertex generates the following correlation functions at tree level:

$$\langle \prod_{j=1}^{N_f} \psi_j(x_j)\bar{\psi}_j(x_j')\rangle_{\text{inst}} = \left(\mathrm{e}^{-\mathrm{i}\xi} \prod_{j=1}^{N_f} P_{\mathrm{L}j} + \mathrm{e}^{\mathrm{i}\xi} \prod_{j=1}^{N_f} P_{\mathrm{R}j} \right) \bar{H}(x_1, \ldots, x_1', \ldots)$$

Goal: Compute correlation function and compare with EFT answer above to fix ξ

Cf. leading contribution to two-point function

$$\begin{aligned}\langle \psi_i(x)\psi_j(x')\rangle =& \mathrm{i}S_{0\text{inst}\,ij}(x,x') \\ \mathrm{i}S_{0\text{inst}\,ij}(x,x') =& (-\gamma^\mu\partial_\mu + \mathrm{i}m_i \mathrm{e}^{-\mathrm{i}\alpha_i\gamma^5}) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}p(x-x')} \frac{\delta_{ij}}{p^2 - m_i^2 + \mathrm{i}\epsilon}\end{aligned}$$

So $\xi = \theta / \xi = -\bar{\alpha}$ implies CP-violation/no CP-violation

Only one explicit calculation based on dilute instanton gas (DIGA) finding $\xi = \theta$
['t Hooft (1986)]

Fermion correlations

- Obtain correlation functions from Green's functions in fixed background of instantons and anti-instantons
- Interfere all instanton configurations
  - First, within one topological sector
  - Then over the different sectors

DIGA to dermine CP phase of 't Hooft vertex—not quantitatively accurate for actual QCD

Green's function in n-instanton, $\bar{n}$-anti-instanton background (DIGA)

$$\mathrm{i}S_{n,\bar{n}}(x,x') \approx \mathrm{i}S_{0\mathrm{inst}}(x,x') + \sum_{\bar{\nu}=1}^{\bar{n}} \frac{\hat{\psi}_{0\mathrm{L}}(x-x_{0,\bar{\nu}})\hat{\psi}^{\dagger}_{0\mathrm{L}}(x'-x_{0,\bar{\nu}})}{m\mathrm{e}^{-\mathrm{i}\alpha}} + \sum_{\nu=1}^{n} \frac{\hat{\psi}_{0\mathrm{R}}(x-x_{0,\nu})\hat{\psi}^{\dagger}_{0\mathrm{R}}(x'-x_{0,\nu})}{m\mathrm{e}^{\mathrm{i}\alpha}}$$

$\hat{\psi}_{0\mathrm{L,R}}$: 't Hooft zero modes

Comments:

- For small masses, zero modes dominate close to the cores of instantons, far away from instantons the solution goes to the zero-instanton configuration [Diakonov, Petrov (1986)]
- Alignment of phase α between Lagrangian mass and instanton-induced χSB $\longrightarrow$ No indication of CP violation here

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cf.

$$\mathrm{i}S_{0\mathrm{inst}}(x,x') = (-\gamma^{\mu}\partial_{\mu} + \mathrm{i}m\mathrm{e}^{-\mathrm{i}\alpha\gamma^5}) \int \frac{\mathrm{d}^4p}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}p(x-x')} \frac{1}{p^2 - m^2 + \mathrm{i}\epsilon}$$

C…

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Interferences within the topological sectors

Within a topological sector, interfere/sum/integrate over

- all instanton/anti-instanton numbers $n + \bar{n}$ with $\Delta n = n - \bar{n}$ fixed
- locations of all instantons/anti-instantons
- remaining collective coordinates

$\longrightarrow$ Dilute instanton gas approximation (skip technicalities)

Can also obtain coincident fermion correlations using the index theorem and anomalous current only

Correlation function for fixed Δn

$$\langle\psi(x)\bar\psi(x')\rangle_{\Delta n} = \sum_{\substack{\bar n,n\geq 0\\ n-\bar n=\Delta n}} \frac{1}{\bar n! n!}\left[\bar h(x,x')\left(\frac{\bar n}{m e^{-i\alpha}}P_{\rm L} + \frac{n}{m e^{i\alpha}}P_{\rm R}\right)(VT)^{\bar n+n-1} + iS_{0\rm inst}(x,x')(VT)^{\bar n+n}\right] \times (i\kappa)^{\bar n+n}(-1)^{n+\bar n}e^{i\Delta n(\alpha+\theta)}$$

$$= \left[\left(e^{i\alpha}I_{\Delta n+1}(2i\kappa VT)P_{\rm L} + e^{-i\alpha}I_{\Delta n-1}(2i\kappa VT)P_{\rm R}\right)\frac{i\kappa}{m}\bar h(x,x') + I_{\Delta n}(2i\kappa VT)iS_{0\rm inst}(x,x')\right] \times (-1)^{\Delta n}e^{i\Delta n(\alpha+\theta)}$$

Instantons per spacetime volume: $i\kappa \propto e^{-S_{\rm E}}$

χSB rank-two spinor-tensor from integrating quark zero-modes over the locations of the instantons: $\bar h(x,x')$

Modified Bessel function: $I_\nu(x)$

Sum is dominated by particular value of $n \approx \bar n$: [Diakonov, Petrov (1986)]

$$\langle n\rangle = \frac{\sum_{n=0}^{\infty} n\frac{(\kappa VT)^n}{n!}}{\sum_{n=0}^{\infty}\frac{(\kappa VT)^n}{n!}} = \kappa VT\,,\quad \frac{\sqrt{\langle(n-\langle n\rangle)^2\rangle}}{\langle n\rangle} = \frac{1}{\sqrt{\kappa VT}}\,,\quad \text{cf. } \lim_{x\to\infty}\frac{I_{\Delta n}(ix\,e^{-i0^+})}{I_{\Delta n'}(ix\,e^{-i0^+})} = 1$$

$\longrightarrow$ No relative CP phase between mass and instanton induced breaking of χral symmetry—alignment in infinite-volume limit

Correspondingly, partition function for fixed Δn: [cf. Leutwyler, Smilga (1992)]

$$Z_{\Delta n} = I_{\Delta n}(2\mathrm{i}\kappa VT)\,(-1)^{\Delta n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$$

Note: The topological phase $\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$ multiplies $\langle\psi(x)\bar{\psi}(x')\rangle_{\Delta n}$ and $Z_{\Delta n}$ entirely—not just the contributions induced by instantons.

Other correlation functions (n point, stress-energy, for some observer,...) are calculated from the Feynman diagram with the Green's function in the n instanton, $\bar{n}$ anti-instanton background.
Then it remains to average over n, $\bar{n}$, locations and remaining collective coordinates.

There is no CP violation/misalignment of phases to this end. It remains to consider the interference between the topological sectors.

Interferences among topological sectors (are immaterial)

Topological quantization $\leftrightarrow$ Interference between sectors for $VT \to \infty$

Fermion correlator

$$\langle\psi(x)\bar\psi(x')\rangle = \lim_{\substack{N\to\infty \\ N\in\mathbb{N}}} \lim_{VT\to\infty} \frac{\sum_{\Delta n=-N}^{N}\langle\psi(x)\bar\psi(x')\rangle_{\Delta n}}{\sum_{\Delta n=-N}^{N} Z_{\Delta n}}$$

$$= \mathrm{i}S_{0\mathrm{inst}}(x,x') + \mathrm{i}\kappa\bar h(x,x') m^{-1} \mathrm{e}^{-\mathrm{i}\alpha\gamma^5} \quad \text{(same as for fixed } \Delta n)$$

Recall: $\mathrm{i}S_{0\mathrm{inst}}(x,x') = (-\gamma^\mu\partial_\mu + \mathrm{i}m\mathrm{e}^{-\mathrm{i}\alpha\gamma^5})\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}p(x-x')}\frac{1}{p^2-m^2+\mathrm{i}\epsilon}$

$\longrightarrow$ No relative CP-phase between mass and instanton term

$\longrightarrow \xi = -\alpha$

$\longrightarrow$ CP is conserved

Limits ordered the other way around

First sum over all Δn as well:

$$\sum_{\bar{n},n\geq 0} \frac{1}{\bar{n}!n!}\left[\bar{h}(x,x')(\bar{n}\,m^{-1}\mathrm{e}^{\mathrm{i}\alpha}P_{\mathrm{L}}+n\,m^{-1}\mathrm{e}^{-\mathrm{i}\alpha}P_{\mathrm{R}})\,(VT)^{\bar{n}+n-1}+\mathrm{i}S_{0\mathrm{inst}}(x,x')\,(VT)^{\bar{n}+n}\right] \times(-mi\kappa)^{\bar{n}+n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$$

$$=\left[-\left(\mathrm{e}^{-\mathrm{i}\theta}P_{\mathrm{L}}+\mathrm{e}^{\mathrm{i}\theta}P_{\mathrm{R}}\right)\frac{\mathrm{i}\kappa}{m}\bar{h}(x,x')+\mathrm{i}S_{0\mathrm{inst}}(x,x')\right]\mathrm{e}^{-2\mathrm{i}\kappa VT\cos(\alpha+\theta)}$$

$$Z\to\sum_{n,\bar{n}}\frac{1}{n!\bar{n}!}(-\mathrm{i}\kappa\,VT)^{\bar{n}+n}\mathrm{e}^{-\mathrm{i}(\bar{n}-n)(\alpha+\theta)}=\mathrm{e}^{-2\mathrm{i}\kappa VT\cos(\alpha+\theta)}$$

Then, $VT\to\infty$ trivial as VT-dependence cancels

$\longrightarrow$ Relative CP phase leading to CP-violating observables

However: Changing the order does not correspond to a nonsingular integration contour.

Gauß' law in the constrained Hilbert space

For $\Omega(\vec{x})$ an infinitesimal generator of gauge transformations
$\longrightarrow$ Noether charge:

$$
\begin{aligned}
Q(\Omega) =& \frac{1}{g}\int \mathrm{d}^3x\,\mathrm{tr}\left[\Pi^i(D^i\Omega)\right] = \int_V \mathrm{d}^3x\,\mathrm{tr}\left[\left(-E^i + \frac{g^2}{8\pi^2}\theta B^i\right) D^i\Omega\right] \\
&= \int \mathrm{d}^3x\,\mathrm{tr}\left[\Omega D^i\left(E^i - \frac{g^2}{8\pi^2}\theta B^i\right)\right] + \int_{\partial V} \mathrm{d}a^i\,\mathrm{tr}\left[\Omega\left(-E^i + \frac{g^2}{8\pi^2}\theta B^i\right)\right]
\end{aligned}
$$

For $\Omega(\vec{x}) = 0$ when $\vec{x} \in \partial V$ and since Ψ' is gauge invariant
$\rightarrow$ Gauß' law: $\vec{D}\cdot\vec{E}\,\Psi'[\vec{A}] = 0$

Usually, the argument is made the other way around: Impose Gauß' law to throw states out of the Hilbert space

Since $[Q(\Omega), W[\vec{A}]] = 0$ for $\Omega(\vec{x}) = 0$ when $\vec{x} \in \partial V$ this also holds when $\Psi'[\vec{A}] \rightarrow \mathrm{e}^{\mathrm{i}\tilde{\theta}W[\vec{A}]}\Psi'[\vec{A}]$, so imposing Gauß' law does not fix $\tilde{\theta}$, does not tell us about large gauge transformations