Cosmological post-Newtonian approximation of a Lorentz-violating vector field

The rise and fall of *a* relativistic MOND theory

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JH-Noh, Cosmological perturbations of a relativistic MOND theory, e-Print: [2410.10205](2410.10205)
JH-Noh, Cosmological post-Newtonian approximation of a Lorentz-violating vector field, e-Print: [2503.20151](2503.20151)

MOND as an alternative to Dark Matter

MOND: **Milgrom (1983)**

$$\boxed{f(|\vec{a}|/a_M)\vec{a} = \vec{g}_N\,, \quad \begin{cases} f(x) = 1 & \text{for } x \gg 1 \text{ (Newtonian)} \\ f(x) = x & \text{for } x \ll 1 \text{ (MOND)} \end{cases}}$$

Low acceleration limit

$$\text{MOND regime}: \ |\vec{a}| = \sqrt{a_M|\vec{g}_N|} > |\vec{g}_N|$$

Gravity becomes stronger

$$\text{with } g_N = GM/r^2, \ |\vec{a}| = v^2/r \ \Rightarrow \ v = (GMa_M)^{1/4} \propto M^{1/4}$$

$\therefore$ **Explaining galactic rotations and Tully-Fisher relation**

Poisson's form: **Bekenstein-Milgrom (1984)**

$$\nabla \cdot [f(|\nabla\Phi|/a_M)\nabla\Phi] = 4\pi G\varrho \ \text{ with } \ \vec{a} = \nabla\Phi$$

Due to its **nonlinear nature**, internal dynamics is affected by the external gravitational field (**external field effect**), **violating the strong equivalence principle**

$$\boxed{\begin{aligned} &\text{Milgrom's universal constant}: \quad a_M \simeq 1.2 \times 10^{-8} cm/sec^2 \\ &\text{Intriguingly}: \quad cH_0/(2\pi) \simeq 1.1 \times 10^{-8} cm/sec^2 \end{aligned}}$$

$$\text{with } H_0 = 70(km/sec)/Mpc$$

Bekenstein-Milgrom, Does the missing mass problem signal the breakdown of Newtonian gravity? (1984)

A Relativistic MOND

Relativistic extension: with MOND limit — **Blanchet-Marsat (2011)** ← Motivated by Einstein-aether theory (Jacobson-Mattingly 2001) and quantum gravity proposal with **local Lorentz invariance violation** (Horava 2009, Blas et al. 2010)

Cosmological addition: for large-scale structure and CMB — **Blanchet-Skordis (2024)**

For MOND **For Cosmology**

$$\mathcal{L} = \sqrt{-g}\Big\{\frac{c^4}{16\pi G}\big[R - 2\mathcal{J}(A) + 2\mathcal{K}(Q)\big] + L_\mathrm{m}\Big\}$$

Time-like Four-vector → $U_a \equiv -\frac{c}{Q}\nabla_a\tau, \quad Q \equiv c\sqrt{-(\nabla^a\tau)\nabla_a\tau},$ ← **Amplitude**

Acceleration Involves gravitational potential → $A_a \equiv c^2 U_{a;b}U^b = -c^2 q^b_a \nabla_b \ln Q, \quad A \equiv \frac{1}{c^4}A^c A_c,$ ← Kinetic part of U_a

$U^c U_c \equiv -1, \quad q_{ab} \equiv g_{ab} + U_a U_b$ ← Projection tensor orthogonal to U_a

Blanchet-Marsat-Skordis (BMS) model of the relativistic MOND

Blanchet-Marsat, Modified gravity approach based on a preferred time foliation, e-Print: 1107.5264;
Blanchet-Skordis, Relativistic khronon theory in agreement with modified Newtonian dynamics and large-scale cosmology, e-Print: 2404.06584

Background constraint

Metric: $ds^2 = -a^2 dx^0 dx^0 + a^2 \delta_{ij} dx^i dx^j \qquad Q = \dot{\tau} = 1 + \dot{\sigma}/c^2$

Model: $\mathcal{K} = \frac{2\nu^2}{n+1}\mathcal{K}_{n+1}(Q-1)^{n+1}$ **No $\mathcal{J}$ contribiutes to BG**

$L_\nu \equiv 1/(\nu\sqrt{\mathcal{K}_{n+1}}) \qquad L_H \equiv c/H_0 = 4.3Gpc$

$H_0 = 70(km/sec)/Mpc$

$$\mu_\tau = \mu_{\tau 0}\frac{1}{a^3}\frac{nQ+1}{n+1}, \qquad Q = 1 + \left[\frac{3}{2a^3}\left(\frac{L_\nu}{L_H}\right)^2 \Omega_{\tau 0}\right]^{1/n}$$

$$w_\tau \equiv \frac{p_\tau}{\mu_\tau} = \frac{Q-1}{nQ+1}, \qquad c_\tau^2 \equiv \frac{\dot{p}_\tau}{\dot{\mu}_\tau} = \frac{Q-1}{nQ}$$

Dust (CDM) for Q ≈ 1

Can be achieved by reducing L_ν

Demand: $w_{\tau *} \leq 0.0164$ at $a_* \sim 10^{-4.5}$ **for a dust-like behavior**

$$L_\nu \leq \left[\frac{2c^2}{3H_0^2\Omega_{\tau 0}}\left(\frac{(n+1)w_{\tau *}}{1-nw_{\tau *}}\right)^n a_*^3\right]^{1/2} \equiv L_*$$

$L_* = 220, 62, 22pc$ for $n = 1, 2, 3$

$\Omega_{\tau 0} = 0.26$

***n* = 1 is excluded**, due to a conflict in 0PN order

Blanchet-Skordis, 2404.06584

Post-Newtonian approximation

Zeroth-order post-Newtonian metric:

$$g_{00} = -\left(1 + \frac{2}{c^2}\Phi\right), \quad g_{0i} = 0, \quad g_{ij} = a^2\delta_{ij}$$

Ansatz: $$\tau(\mathbf{x}, t) = t + \frac{1}{c^2}\bar{\sigma}(t) + \frac{1}{c^2}\sigma(\mathbf{x}, t)$$

$$Q = 1 + \frac{1}{c^2}\dot{\bar{\sigma}} - \frac{1}{c^2}\Xi, \quad \Xi \equiv \Phi - \dot{\sigma} + \frac{1}{2a^2}\sigma^{,i}\sigma_{,i}$$

No 0PN contribution from $\mathcal{K}$ for n $\geq$ 2

$$\boxed{\frac{1}{a^2}\nabla\cdot\left[(1+\mathcal{J}_{,A})\nabla\Phi\right] = 4\pi G\delta\varrho_b} + \underline{\frac{1}{a^2}\nabla\cdot\left\{\mathcal{J}_{,A}\nabla\left[\dot{\sigma} - \frac{1}{2a^2}(\nabla\sigma)\cdot\nabla\sigma\right]\right\}}$$

$\mathcal{J}_{,A} = 0$	: Newtonian
$1 + \mathcal{J}_{,A} = x \equiv \frac{1}{a}\lvert\nabla\Phi\rvert/a_M < 1$	: MOND

Bekenstein-Milgrom's MOND

Can we REMOVE it?
Or, SUPRESS its effect?

Baryon + τ-field system

Conservation equations :

$$\frac{1}{a^3}(a^3\varrho_\tau)^{\cdot} - \frac{1}{a^2}(\varrho_\tau\sigma^{,i})_{,i} = 0.$$

$$\frac{1}{a^3}(a^3\varrho_b)^{\cdot} + \frac{1}{a}\nabla\cdot(\varrho_b\mathbf{v}_b) = 0,$$

Momentum conservation eq. identically valid

$$\frac{1}{a}(a\mathbf{v}_b)^{\cdot} + \frac{1}{a}\mathbf{v}_b\cdot\nabla\mathbf{v}_b + \frac{1}{a}\nabla\Phi = 0.$$

Poisson's equation:

$$\frac{\Delta}{a^2}\Phi = 4\pi G(\delta\varrho_b + \delta\varrho_\tau), \quad \varrho_\tau = \bar{\varrho}_\tau - \frac{1}{4\pi G}\frac{1}{a^2}(\mathcal{J}_{,A}\Xi^{,i})_{,i}.$$

$$\therefore \boxed{\frac{1}{a^2}\nabla\cdot\left[(1+\mathcal{J}_{,A})\nabla\Phi\right] = 4\pi G\delta\varrho_b + \frac{1}{a^2}\nabla\cdot\left\{\mathcal{J}_{,A}\nabla\left[\dot{\sigma} - \frac{1}{2a^2}(\nabla\sigma)\cdot\nabla\sigma\right]\right\}}$$

Dynamic nature of σ, using $\mathbf{v}_\tau \equiv -\frac{1}{a}\nabla\sigma$:

$$\frac{1}{a^3}(a^3\varrho_\tau)^{\cdot} + \frac{1}{a}\nabla\cdot(\varrho_\tau\mathbf{v}_\tau) = 0,$$

$$\frac{1}{a}\nabla\cdot\left\{\mathcal{J}_{,A}\left[\frac{1}{a}(a\mathbf{v}_\tau)^{\cdot} + \frac{1}{a}\mathbf{v}_\tau\cdot\nabla\mathbf{v}_\tau + \frac{1}{a}\nabla\Phi\right]\right\} = -4\pi G\delta\varrho_\tau.$$

σ = 0 as a physical condition

For σ = 0: $\rightarrow$ $\dot{\delta}_\tau = 0$,

$$\boxed{\begin{aligned} &\ddot{\delta}_b + 2H\dot{\delta}_b = \frac{\Delta}{a^2}\Phi, \\ &\frac{1}{a^2}\nabla\cdot\left[(1+\mathcal{J}_{,A})\nabla\Phi\right] = 4\pi G\varrho_b\delta_b. \end{aligned}}$$

Linearized in density and velocity perturbations

Nonlinear gravitational potential in MOND regime

This is what we naïvely expect in Bekenstein-Milgrom's MOND

Solutions: $$\frac{1}{a}|\nabla\Phi| = \frac{3}{10}a_M\Omega_{b0}, \qquad \delta_b = \frac{\frac{3}{10}\Omega_{b0}\frac{1}{a}\Delta\Phi}{4\pi G\varrho_b a^3}a^2.$$

MOND	**vs**	**CDM**
$\delta_b \propto a^2$		$\delta_b \propto a$

We expect fast growth of structures in MOND (Sanders 1998; Nusser 2002)

However, $$\delta_\tau = \frac{\frac{10}{3} - \Omega_b}{1-\Omega_b}\delta_b \propto a^2, \qquad \dot{\delta}_\tau \propto \dot{\delta}_b \neq 0$$

Thus, inconsistent, or possible for stationary system only (Flanagan 2023)

Flanagan, Khronometric theories of modified Newtonian dynamics, e-Print: 2302.14846; Blanchet-Skordis 2404.06584

$\sigma = 0$ using a coordinate transformation

$(\mathbf{x}, t) \to (\mathbf{x}, t + \sigma/c^2)$ with $\bar{\sigma} \equiv 0 \;\to\; \tau = t \;\to\; g_{0i} = \frac{1}{c}\sigma_{,i}$:

New 0PN expansion:

0PN modified!

$$g_{00} = -\left(1 + \frac{2}{c^2}\Phi\right), \quad g_{0i} = \frac{1}{c}\sigma_{,i}, \quad g_{ij} = a^2\delta_{ij}.$$

Baryon conservation equations:

$$\frac{1}{a^3}(a^3\varrho)^{\cdot} + \frac{1}{a}(\varrho v^i)_{,i} = \frac{1}{a^2}(\varrho_b \sigma^{,i})_{,i},$$

Non-Newtonian corrections

$$\frac{1}{a}(a v_i)^{\cdot} + \frac{1}{a}v^j v_{i,j} + \frac{1}{a}\Phi_{,i} = -\frac{2}{a}H\sigma_{,i} + \frac{1}{a}\sigma_{,ij}\left(v^j - \frac{1}{a}\sigma^{,j}\right) + \frac{1}{a}v_{i,j}\sigma^{,j}.$$

Poisson's equation:

$$\frac{1}{a^2}\nabla \cdot \left[(1 + \mathcal{J}_{,A})\nabla\Phi\right] = 4\pi G\delta\varrho_b - \frac{\Delta}{a^2}\dot{\sigma}.$$

No Newtonian limits in both conservation eqs and Poisson's eq

Density perturbations, properly considering dynamic σ

Baryon + τ-field system:

linear density and velocity perturbations, but *nonlinear* in potential

$$\ddot{\delta}_b + 2H\dot{\delta}_b = \frac{\Delta}{a^2}\Phi, \quad \frac{\Delta}{a^2}\Phi = 4\pi G(\varrho_b\delta_b + \varrho_\tau\delta_\tau),$$

$$\dot{\delta}_\tau = \frac{\Delta}{a^2}\sigma, \quad \frac{1}{a^2}\nabla\cdot\left[\mathcal{J}_{,A}\nabla(\Phi - \dot{\sigma})\right] = -4\pi G\varrho_\tau\delta_\tau\,.$$

Newtonian regime, $\mathcal{J}_{,A} = 0$**:**

$$\ddot{\delta}_b + 2H\dot{\delta}_b = 4\pi G\varrho_b\delta_b, \quad \delta_\tau = 0, \quad \sigma = 0.$$

Assuming matter-dominated era without Λ

Solution: $a \propto t^{2/3} \quad \delta_b \propto t^n$

$n = \frac{1}{6}(-1 \pm \sqrt{1 + 24\Omega_{b0}})$

MOND regime, $1 + \mathcal{J}_{,A} = x < 1$**:**

$$\boxed{\begin{aligned}\ddot{\delta}_b + 2H\dot{\delta}_b &= 4\pi G(\varrho_b\delta_b + \varrho_\tau\delta_\tau),\\ \ddot{\delta}_\tau + 2H\dot{\delta}_\tau &= 4\pi G\varrho_b\delta_b - \frac{1}{a^2}\nabla\cdot(x\nabla\Phi).\end{aligned}}$$

Solution: For $x \ll 1$

$$n = \frac{1}{6}\left[-1 \pm \sqrt{1 + 12\Omega_{b0}\left(1 \pm \sqrt{1 + 4\frac{\Omega_{\tau 0}}{\Omega_{b0}}}\right)}\right]$$

$n < \frac{2}{3}$ **∴No faster growth as expected in MOND regime**

∴ non-MONDian!

Perturbation Theory

Metric perturbation:

$x^0 = \eta$

Associated with δ_{ij}

Arbitrary amplitude

$$g_{00} = -a^2(1+2\alpha), \quad g_{0i} = -a\chi_i, \quad g_{ij} = a^2(1+2\varphi)\delta_{ij}$$

Ignored transverse-tracefree perturbation and imposed spatial gauge conditions

Field perturbation:

$$\boxed{\tau \equiv \bar{\tau}(t) + \frac{1}{c^2}\sigma(\mathbf{x}, t)}$$

Linear perturbation

To linear order:

$$Q = (1-\alpha)\bar{Q} + \frac{1}{c^2}\dot{\sigma}, \quad \bar{Q} = \dot{\bar{\tau}}$$

Gauge-invariant notation

$$\Upsilon \equiv \alpha - \frac{1}{c^2 Q}\left(\dot{\sigma} - \frac{\dot{Q}}{Q}\sigma\right) \equiv \alpha_\sigma \qquad A_i = c^2 \Upsilon_{,i}$$

Fluid quantities:

$$\delta\mu_\tau = \frac{c^4}{8\pi G}\left(Q\mathcal{K}_{,QQ}\delta Q - 2\mathcal{J}_{,A}\frac{\Delta}{a^2}\Upsilon\right),$$

$$\delta p_\tau = \frac{c^4}{8\pi G}\mathcal{K}_{,Q}\delta Q, \quad v_\tau = \frac{1}{aQ}\sigma, \qquad v_i \equiv -v_{,i} + v_i^{(v)}$$

No vector and tensor perturbations, No anisotropic stress

EoS:

$$\boxed{\frac{\delta p_\tau}{\mu_\tau} = c_\tau^2\left(1 - \mathcal{J}_{,A}\frac{c_\tau^2 c^2 k^2}{4\pi G \varrho_\tau a^2}\right)^{-1}\delta_\tau}$$

$\therefore$ σ = 0 implies $v = 0$.

JH-Noh, e-Print: [2410.10205](2410.10205)

Gauge issue

Gauge transformation: $\widehat{x}^a = x^a + \xi^a$

$$\frac{\widehat{\sigma}}{c^2} = \frac{\sigma}{c^2} - \tau'\xi^0, \quad \widehat{Q} = Q - c\left(\frac{\tau'}{a}\right)'\xi^0, \quad \widehat{\mathcal{K}} = \mathcal{K} - \mathcal{K}_{,Q}c\left(\frac{\tau'}{a}\right)'\xi^0$$

$$\widehat{\mu} = \mu - \mu'\xi^0, \quad \widehat{p} = p - p'\xi^0, \quad \widehat{v} = v - c\xi^0$$

$\therefore$ σ = 0 can be used as a gauge condition → $v = 0$ ∴ **comoving gauge**

with τ-fluid

Gauge-invariant combination: $\delta\mu_\sigma \equiv \delta\mu - \dfrac{\mu'}{\tau'}\dfrac{\sigma}{c^2}$

Momentum conservation **Energy conservation**

By combining: $\alpha_\sigma \doteq -\dfrac{\delta p_\sigma}{\mu + p}, \quad \kappa_\sigma \doteq \dfrac{(a^3\delta\mu_\sigma)^{\cdot}}{a^3(\mu+p)}$

Einstein eq.
Raychaudhury eq. ⟶ $\dot{\kappa}_\sigma + 2H\kappa_\sigma + \left(c^2\dfrac{\Delta}{a^2} + \cancel{3\dot{H}}\right)\alpha_\sigma = \dfrac{4\pi G}{c^2}(\delta\mu_\sigma + \cancel{3\delta p_\sigma})$

Jeans criterion

Density perturbation equation: valid for a fluid without anisotropic stress and τ-field, with Λ in the background

$$\boxed{\frac{1+w}{a^2 H}\left[\frac{H^2}{a(\mu+p)}\left(\frac{a^3\mu}{H}\delta_\sigma\right)^{\cdot}\right]^{\cdot} = c^2\frac{\Delta}{a^2}\frac{\delta p_\sigma}{\mu}}$$

$\delta \equiv \delta\mu/\mu \qquad \delta\mu_\sigma \equiv \delta\mu - \frac{\mu'}{\tau'}\frac{\sigma}{c^2}$ Gauge-invariant combination

Zero-pressure background:

$$\frac{1}{a^2 H}\left[a^2 H^2\left(\frac{\delta_\sigma}{H}\right)^{\cdot}\right]^{\cdot} = \boxed{\ddot{\delta}_\sigma + 2H\dot{\delta}_\sigma - \underbrace{4\pi G\varrho}_{\text{Gravity}}\delta_\sigma = c^2\underbrace{\frac{\Delta}{a^2}}_{\text{Pressure}}\frac{\delta p_\sigma}{\mu}}$$

Jeans scale:

$$\Delta = -k^2 \longrightarrow \boxed{\frac{k_J}{a} = \sqrt{\frac{4\pi G\delta\varrho_\sigma}{c^2\delta p_\sigma/\mu}} = \left[2c_\tau^2\left(\frac{w_\tau}{\mathcal{K}} + \frac{a^3}{I_0}\frac{1}{Q}\mathcal{J}_{,A}\right)\right]^{-1/2}}$$

Perturbation constraint

Model: $$\mathcal{K} = \frac{2\nu^2}{n+1}\mathcal{K}_{n+1}(Q-1)^{n+1}$$

Jeans scale: $$\frac{k_J}{a} = \left[\frac{2a^3}{I_0}\frac{Q-1}{nQ}\left(\frac{n+1}{nQ+1}+\mathcal{J}_{,A}\right)\right]^{-1/2}$$

n = 1: $$\frac{k_J}{a} \simeq \frac{\nu}{\sqrt{1+\mathcal{J}_{,A}}}$$

Excluded already

n = 2: $$\frac{k_J}{a} \simeq \sqrt{\sqrt{\frac{2I_0}{a^3}}\frac{\nu\sqrt{\mathcal{K}_3}}{1+\mathcal{J}_{,A}}}$$

$1+\mathcal{J}_{,A} = 1$: Newtonian

$1+\mathcal{J}_{,A} = |\nabla\Phi|/a_M < 1$: MOND

$L_H \equiv c/H_0 = 4.3Gpc$

$$\boxed{\frac{\lambda_J}{2\pi} = \frac{a}{k_J} \leq \sqrt{\frac{L_H L_* a^{3/2}}{\sqrt{6\Omega_{\tau 0}}}(1+\mathcal{J}_{,A})} \sim \sqrt{1+\mathcal{J}_{,A}}\,a^{3/4}1.1Mpc}$$

JH-Noh, e-Print: [2410.10205](2410.10205)

Baryon + τ-field system:

In super-Jeans scale: τ-field = CDM

<u>Newtonian regime</u>, $\mathcal{J}_{,A} = 0$:

$$\boxed{\begin{aligned} &\ddot{\delta}_b + 2H\dot{\delta}_b - 4\pi G(\varrho_b\delta_b + \varrho_\tau\delta_\tau) = 0, \\ &\ddot{\delta}_\tau + 2H\dot{\delta}_\tau - 4\pi G(\varrho_b\delta_b + \varrho_\tau\delta_\tau) = c_\tau^2 c^2 \frac{\Delta}{a^2}\delta_\tau \end{aligned}}$$

<u>MOND regime</u>, sub-Jeans scale

$$\boxed{\begin{aligned} &\ddot{\delta}_b + 2H\dot{\delta}_b - 4\pi G(\varrho_b\delta_b + \varrho_\tau\delta_\tau) = 0, \\ &\ddot{\delta}_\tau + 2H\dot{\delta}_\tau - 4\pi G\left(\varrho_b\delta_b - \frac{x}{1-x}\varrho_\tau\delta_\tau\right) = 0 \end{aligned}}$$

$$x \equiv 1 + \mathcal{J}_{,A}$$

The same as PN case ∴ no faster growth as expected in MOND

∴ non-MONDian!

Summary

- **MOND** = modified gravity in regions of **low acceleration**
- **Direct evidence** from wide binary systems supports MOND
- **Early presence of galaxies** is in tension with the CDM paradigm; MOND has stronger gravity, thus provide **faster growth of baryon pert.**
- Numerous versions of MOND and several relativistic extensions
- We studied **Blanchet-Marsat-Skordis' relativistic MOND** with successful cosmology based on a Lorentz-violating vector field
- **In BMS theory:** Cosmology aspects appear sound
- Dynamic nature of vector field is a challenge to achieve MOND
- Difficult to remove the dynamic parts
- BMS theory achieves MOND only in stationary systems
- Dynamic parts lead to **non-MONDian behavior** of baryon density pert.
- The rise and fall of BMS' proposal as a relativistic MOND theory