Revisiting Instantons in Higgs Phase

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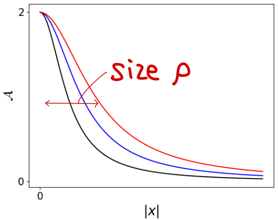
Instanton

Topological soliton in non-abelian (e.g. $\mathrm{SU}(2)$) gauge theories on 4D Euclidean space.

- Classified by **winding number** $w = 0, \pm 1, \ldots$
- Local minima of the action with

$$S_{\text{Euclidean}} = \frac{8\pi^2}{g^2}|w|\,.$$

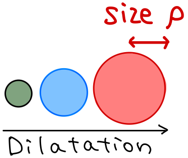


Instanton profile function

Dilatation Zero Mode

The action is classically indepenednt of the **size** ρ.

Dilatation is a **zero mode**, which do not cost the action.

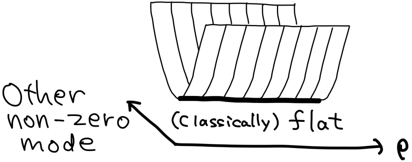


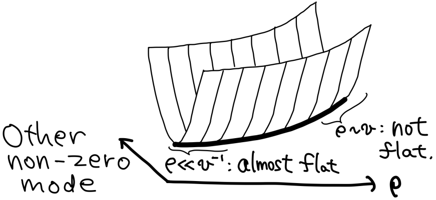
Constrained Instanton

What If Gauge Symmetry is Higgsed? [Affleck (1980)]

$$\mathcal{L}_{\text{YMH}} = \frac{1}{2g^2}\,\mathrm{Tr}\,F_{\mu\nu}F_{\mu\nu} + |D_\mu\phi|^2 + \frac{\lambda}{4}(\phi^\dagger\phi - v^2)^2\,. \quad \text{(SU(2) gauge theory with SU(2) doublet } \phi.\text{)}$$

- **Instanton action increases as the size increases**, since $|A_\mu\phi|^2 \sim |A_\mu|^2 v^2$ at $x \gtrsim v^{-1}$ contributes to the action. Decending the slope leads to $\rho \to 0$.
- There is **no strict minimum** with non-trivial winding number..
- However, small instantons ($\rho \ll v^{-1}$) "do not see" symmetry breaking, effectively. i.e. **dilatation direction is almost flat**, when $\rho \ll v^{-1}$.





Constrained Instanton

- Instantons with $\rho \ll v^{-1}$ have non-negligible effects, since they form an **almost flat "valley"** of the action.
- To compute their effects, those configurations should be extracted.

Minimization with constraint (of size) helps us picking up $\rho \neq 0$ configurations.

Leading order solution at $x \ll v^{-1}$ in singular gauge:

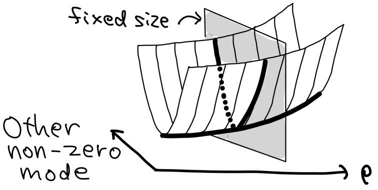
[Affleck (1980), Espinosa (1989)]

$$A_\mu^a = \bar{\eta}_{\mu\nu}^a \frac{x_\nu}{x^2}\left[\frac{2\rho^2}{x^2+\rho^2} + \mathcal{O}\left(\rho^2 v^2\right)\right],$$

$$\phi = \begin{pmatrix} 0 \\ v \end{pmatrix}\left[\sqrt{\frac{x^2}{x^2+\rho^2}} + \mathcal{O}\left(\rho^3 v^3\right)\right].$$

$$S_{\mathrm{YMH}} = \frac{8\pi^2}{g^2} + \boxed{2\pi^2\rho^2 v^2 + \mathcal{O}\left(\rho^4 v^4\right)}\,.$$

Overlapping of instanton profile and $|\phi| \sim v$ at $x \gtrsim v^{-1}$.



Constrained Instanton: Procedure

Minimization with Constraint in Classical Level: **Lagrange Multiplyer Method**

$$
\begin{aligned}
S_{\text{total}}[A,\phi] &= S_{\text{YMH}} + \sigma(S_{\text{constraint}} - f(\rho))\,, \\
S_{\text{constraint}} &= \int \mathrm{d}^4x\,\mathcal{O}_{\text{constraint}}\,.
\end{aligned}
$$

Procedure:

- **Lagrange Multiplier Method**: “For given ρ, minimize S_{total} w.r.t. A, ϕ, σ.”

Constraint term:

- An example: $\mathcal{O}_{\text{constraint}} = \left(\frac{1}{2}\,\mathrm{Tr}\,F\tilde{F}\right)^2$, $\quad \int \mathrm{d}x^4 x\,\mathcal{O}_{\text{constraint}} = \frac{384\pi^2}{7}\rho^{-4}$

Constrained Instanton: Procedure

Procedure of Constraint in **Quantum** Level

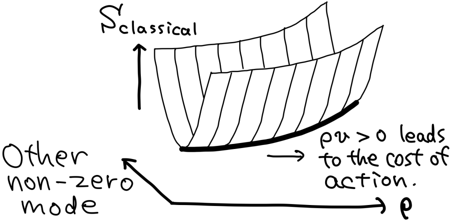
Constrained instaton procedure = "**insertion of 1**" to path integral.

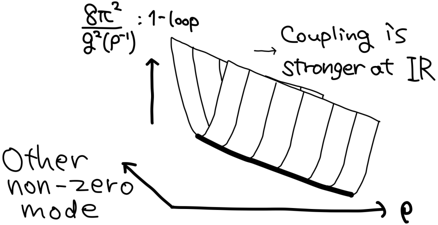
- (Entire configuration space) = $\sum_\rho$(slice of configurations with the fixed sizeρ)

Two size-dependent effects compete, in asymptotically free case.

Classical: Action increases as ρv increases.

Quantum effect: Effective coupling constant depends on the instanton size ρ.





Constrained Instanton: Explicit Construction

Profile Functions $\mathcal{A}$ and $\mathcal{H}$:

$$A^a_\mu = \bar{\eta}^a_{\mu\nu} \frac{x_\nu}{x^2} \mathcal{A}(x^2) , \qquad \phi = \begin{pmatrix} 0 \\ v \end{pmatrix} \mathcal{H}(x^2)$$

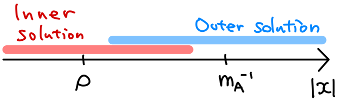
Analytic expansion of $\mathcal{A}$ (and similarly of $\mathcal{H}$) is obtained by

- Inner/Outer solutions at leading order:

$$\mathcal{A}(x) = \begin{cases} \frac{2\rho^2}{x^2+\rho^2} & x \ll m^{-1} \\ (\text{const}) \times K_2(mx) & x \gg \rho \end{cases} \qquad m = gv/\sqrt{2} .$$

$$\left(\text{Modified Bessel function:} \quad (\rho m)^2 K_2(mx) \sim \begin{cases} \frac{2\rho^2}{x^2} & x \ll m^{-1} \\ (\rho m)^2 \sqrt{\frac{\pi}{2mx}} e^{-mx} & x \gg m^{-1} \end{cases} \right)$$

- Solving order by order with respect to ρv.
- **Matching** inner/outer solutions at $\rho \ll x \ll m_A^{-1}$.



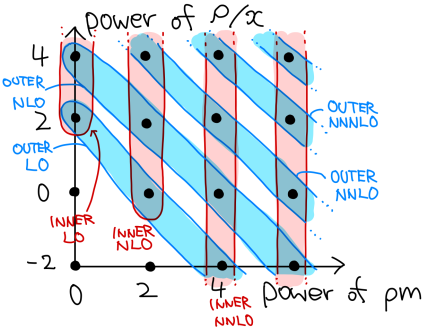
Our Work: Matching

Two expressions of $\mathcal{A}$ at $x \ll m_A^{-1}$ and $x \gg \rho$ should match at $\rho \ll x \ll m_A^{-1}$.

- M. Nielsen and N. K. Nielsn (1999) indicated that **matching fails** at $(\rho m)^2$-order.

Our Work: We explicitly verified that the **matching is possible**.

Matching procedure is the **double expansion** with respect to ρm and ρ / x.



Example: Leading order outer solution

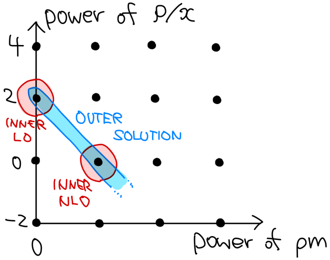
$$\mathcal{A}(x) \sim \underbrace{(\rho m)^2 K_2(mx)}_{\text{Outer LO}}$$

$$= (\rho m)^2 \left[\frac{2}{m^2 x^2} - \frac{1}{2} + \mathcal{O}\left(m^2 x^2\right) \right] \quad (x \ll m^{-1})$$

$$= \underbrace{\frac{2\rho^2}{x^2}}_{\text{Inner LO}} - \underbrace{\tfrac{1}{2}(\rho m)^2}_{\text{Inner NLO}} + \underbrace{\mathcal{O}\left(\rho^4 m^4\right)}_{\text{Inner NNLO}}$$

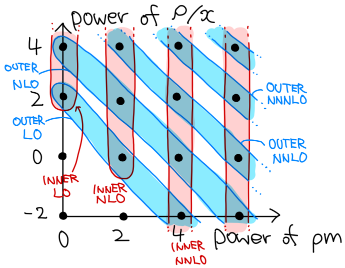
Our Work: Matching

Previous Work



- Corrections to the outer solution at higher-order in ρm are dropped.
- The **appropriate choice of** $\mathcal{O}_{\text{constraint}}$ **is severely restricted** to avoid the mismatch.

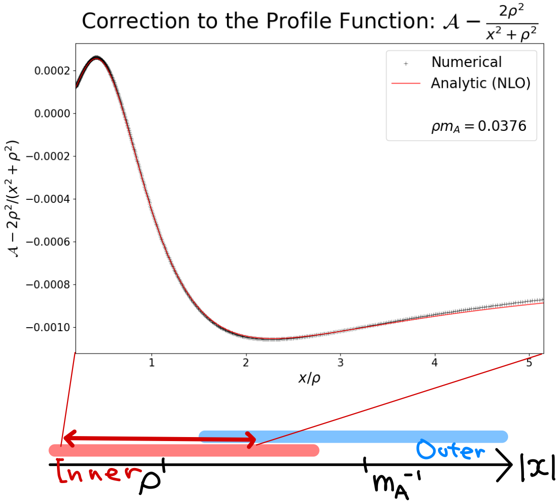
Our Work



- Corrections to the outer solution at higher-order in ρm are taken into account.
- Matching works well **independently to the choice of** $\mathcal{O}_{\text{constraint}}$.

Numerical Check: Fitting the Configuration (Preliminary)

Check: Numerical configuration coincides with analytic NLO ($(\rho m_A)^2$-order) correction.



- Plot: Correction to the instanton profile. $\mathcal{O}\left(10^{-4}\right)$ in profile function.
- The numerical result is consistent with the analytic discussion of the matching.

Summary

- **Constrained instantons** are instanton-like configurations and are minima of action on the constrained surface (of fixed size).
- Nielsen and Nielsen (1999) pointed out that ρv -expanded constrained solutions do not exist for almost every constraint due to **mismatch** between the solution in $x \ll m^{-1}$ and decaying behavior at $x \gg \rho$.
- We clarified that the **matching works well almost independently to the choice of the constraint**, if we take into account corrections to outer solutions coming from $\rho m > 0$, with appropriate ordering using double expansion.
- Numerical check: consistent with the analytic discussion of the matching.

BACKUP

Semiclassical Approximation

$\hbar$-expansion of (part of) $Z = \int \mathcal{D}A \exp(-S[A]/\hbar)$.

- $S[A] = \frac{1}{g^2} \int d^4x \left(\frac{1}{2} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} + \dots \right)$

The procedure of $\hbar$-expansion:

0. $S \simeq$ Classical minima.
1. $S \simeq$ Classical + [Field oscillations around the minima]^2.

[Field oscillations]^2: bare coupling $\rightarrow$ **renormalized coupling** $g(\rho^{-1})$.

Procedure of Constraint in **Quantum** Level [Gervais, Neveu and Virasoro (1977)]

Constrained instaton procedure can be understood as "**insertion of 1**" to Z.

$$\begin{cases} S_{\text{total}}[\Phi, f] := S_{\text{YMH}} + \sigma(S_{\text{constraint}} - f) \\ S^{\text{classical}}_{\text{total}}(f) := S_{\text{total}}[\Phi, f]\big|_{\Phi = (\text{constrained minimum})} \end{cases}$$

$$\begin{aligned}
Z &= \int \mathcal{D}\Phi \, \mathrm{d}f \, \delta(f - S_{\text{constraint}}) \exp(-S_{\text{YMH}}) \\
&= \int \mathcal{D}\Phi \, \mathrm{d}f \, \delta(f - S_{\text{constraint}}) \exp(-S_{\text{YMH}} - \sigma(S_{\text{constraint}} - f(\rho))) \quad \text{(Zero is just added.)} \\
&= \ldots = \int \mathcal{D}\varphi \, \mathrm{d}f \, \frac{\mathrm{d}\mu}{2\pi} \exp\!\left(-S^{\text{classical}}_{\text{total}}(f) - i\mu \int \frac{\delta S_{\text{constraint}}}{\delta \Phi}\varphi - \frac{1}{2}\int \varphi \frac{\delta^2 S_{\text{total}}}{\delta \Phi^2}\varphi + \mathcal{O}\!\left(\varphi^3\right)\right) \\
&= \ldots = \int \mathcal{D}\tilde{\varphi} \, \mathrm{d}f \exp\!\left(-S^{\text{classical}}_{\text{total}}(f) - \frac{1}{2}\int \tilde{\varphi} \frac{\delta^2 S_{\text{total}}}{\delta \Phi^2}\tilde{\varphi} + \mathcal{O}\!\left(\tilde{\varphi}^3\right)\right) \int \frac{\mathrm{d}\mu}{2\pi} \exp\!\left(-\frac{\mu^2}{2}\left(-\frac{\partial f}{\partial \sigma}\right)\right) \\
&\simeq \int \mathrm{d}f \left(-\frac{\partial f}{\partial \sigma}\right)^{-1/2} \exp\!\left(-S^{\text{classical}}_{\text{total}}(f)\right) \int \mathcal{D}\tilde{\varphi} \exp\!\left(-\frac{1}{2}\int \tilde{\varphi} \frac{\delta^2 S_{\text{total}}}{\delta \Phi^2}\tilde{\varphi}\right)
\end{aligned}$$

Explicit Matching

$$
\begin{aligned}
\mathcal{A}^{(\mathrm{LO})}_{\mathrm{inner}}\left(\rho^2/x^2\right) + \mathcal{A}^{(\mathrm{NLO})}_{\mathrm{inner}}\left(\rho^2/x^2\right)
&= \frac{2\rho^2}{x^2} - \frac{2\rho^4}{x^4} + (\rho m)^2\left[-\left(c_1 - \tfrac{1}{12}\right)\left(\frac{\rho^2}{x^2}\right)^{-1} - 6c_1 + 12c_1\frac{\rho^2}{x^2}\ln\frac{\rho^2}{x^2} + c_2\frac{\rho^2}{x^2} + \mathcal{O}\left(\frac{\rho^4}{x^4}\right)\right].\\
\mathcal{A}^{(\mathrm{LO})}_{\mathrm{outer}}(m^2x^2) + \mathcal{A}^{(\mathrm{NLO})}_{\mathrm{outer}}(m^2x^2)
&= \frac{2\rho^2}{x^2} - \frac{2\rho^4}{x^4} + (\rho m)^2\left[-\frac{1}{2} + \frac{\rho^2}{x^2}\ln\frac{\rho^2}{x^2} + (\mathrm{const})\frac{\rho^2}{x^2}\right] + \mathcal{O}\left(\rho^6\right).
\end{aligned}
$$

- Matching is possible by adjusting c_1, c_2. Especially, $c_1 = 1/12$.
- In the previous work, (part of) $\mathcal{A}^{(\mathrm{NLO})}_{\mathrm{outer}}$ was missing, leading to the mismatch.