Towards a non-Hermitian cosmology

Many thanks to: Jean Alexandre, Richard Battye, Carl M. Bender, Maxim N. Chernodub, Edmund J. Copeland, Steven Cotterill, Madeleine Dale, John Ellis, Robert Mason, Esra Sablevice, and Dries Seynaeve

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Monday 21 July 2025

The state of fundamental physics at 17:40 on Monday 21 July 2025 …

Yes, let’s add more stuff!

No, let’s get rid of stuff!

The state of fundamental physics at 17:40 on Monday 21 July 2025 …

Actually, it’s time for the reception.

Agreed.

But first this talk:

What if Hermiticity (of the Hamiltonian) turns out to be a really good thing to get rid of?

Forget Hermiticity, get *pseudo*-Hermiticity:
antilinear symmetry, not Hermiticity.

Some pseudo-Hermitian truths and untruths ...

Truth 1: All Hermitian matrices have real eigenvalues,
but not all matrices with real eigenvalues are Hermitian.

Truth 2: Hermiticity is a mathematical statement,
anti-linear symmetry is a physical statement.

An untruth: Only real eigenvalues are interesting. (This talk.)

What is a pseudo-Hermitian theory?

A Hamiltonian is **pseudo-Hermitian**, if there exists a **Hermitian operator** $\hat{\eta}$ s.t.

$$\hat{H}^\dagger = \hat{\eta}\hat{H}\hat{\eta}^{-1}$$

Expectation values of $\hat{H}$ are **real** with respect to the inner product

$$\langle \cdot | \cdot \rangle_\eta := \langle \cdot | \hat{\eta} \cdot \rangle$$

and the **time evolution** is η**-pseudo-unitary**.

PT-(parity-time-reversal-)symmetric theories are a subset of pseudo-Hermitian theories.

Bender and Boettcher, Phys. Rev. Lett. 80 (1998) 5243, physics/9712001; Mostafazadeh, J. Math. Phys. 43 (2002) 205, math-ph/0107001;
Bender, Brody and Jones, Phys. Rev. Lett. 89 (2002) 270401, quant-ph/0208076

Example: non-Hermitian mixing of 2 complex scalar fields

$$\mathscr{L} = -\,\partial\tilde{\phi}_a^* \cdot \partial\phi_a - m_a^2\tilde{\phi}_a^*\phi_a - \mu^2\left(\tilde{\phi}_1^*\phi_2 - \tilde{\phi}_2^*\phi_1\right), \qquad a = 1,2$$

PT-symmetric under: $\phi_a \to P_{ab}\tilde{\phi}_b^*, \qquad P = \pm\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = P^{-1}$

Pseudo-Hermitian mass matrix: $M^2 = \begin{pmatrix}m_1^2 & \mu^2\\ -\mu^2 & m_2^2\end{pmatrix} \neq \left(M^2\right)^\dagger, \qquad PM^2P = \left(M^2\right)^\dagger$

Eigenspectrum: $m_\pm^2 = \dfrac{m_1^2 + m_2^2}{2} \pm \left[\left(\dfrac{m_1^2 - m_2^2}{2}\right)^2 - \mu^4\right]^{1/2} \in \mathbb{R}$ if $\zeta = \dfrac{2\,|\,\mu^2\,|}{|\,m_1^2 - m_2^2\,|} \leq 1$

Exceptional point at $\zeta = 1$, and **complex-conjugate pairs** of eigenvalues for $\zeta > 1$ (**PT-broken regime**).

Alexandre, PM and Seynaeve, Phys. Rev. D 96 (2017) 065027, 1707.01057; Alexandre, Ellis and PM, Phys. Rev. D 102 (2020) 12, 125030, 2006.06656; Sablevice and PM, Phys. Rev. D 109 (2024) 6, 2307.16805

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Alexandre, PM and Seynaeve, Phys. Rev. D 96 (2017) 065027, 1707.01057; Alexandre, Ellis and PM 0) 12, 125030, 2006.06656; Sable . D 109 (2024) 6, 2307.16805

What is the tilde notation all about?

Back to representations of the Poincaré group

Consider the **Heisenberg equation** for a scalar field operator $\hat{\phi}(t,\vec{x})$:

$$\left[\hat{\phi}(t,\vec{x}),\hat{H}\right] = i\partial_t\hat{\phi}(t,\vec{x}) \;\Rightarrow\; \left[\hat{\phi}^\dagger(t,\vec{x}),\hat{H}^\dagger\left(\neq \hat{H}\right)\right] = i\partial_t\hat{\phi}^\dagger(t,\vec{x})$$

$\hat{\phi}(t,\vec{x})$ evolves with the Hamiltonian $\hat{H} \;\Rightarrow\; \hat{\phi}^\dagger(t,\vec{x})$ evolves with the Hamiltonian $\hat{H}^\dagger$

Thus, if $\hat{P}^0 = \hat{H}$ is not Hermitian, so too are other **generators of the Poincaré group**, and $\hat{\phi}^\dagger(t,\vec{x})\hat{\phi}(t,\vec{x})$ does **not** transform properly under the **Poincaré group**.

Which is the operator $\hat{\tilde{\phi}}^\dagger(x)$ transforming in the representation "dual" to $\hat{\phi}(x)$?

Chernodub and PM, Phys. Rev. D 105 (2022) 7, 076020, 2110.05289; Sablevice and PM, Phys. Rev. D 109 (2024) 6, 2307.16805

The **dual field operator** is

defines the pseudo-unitarity
of the finite dim. rep.

$$\hat{\tilde{\phi}}^{\dagger}(x) = \hat{\eta}^{-1}\hat{\phi}^{\dagger}(x_{\eta})\hat{\eta}\pi$$

the same map η that determines the pseudo-Hermiticity of the Hamiltonian

since the map η can involve spacetime transformations

Sablevice and PM, Phys. Rev. D 109 (2024) 6, 2307.16805

An observation about field theory on dynamical spacetimes:

Eigenspectra are, in general, not real.

So this talk:

Can we exploit the PT-broken regime for cosmology?

Example: a non-Hermitian theory with symmetry breaking

$$-\mathscr{L} = -g^{\rho\sigma}\partial_\rho\tilde{\phi}_a^\dagger\partial_\sigma\phi_a + m_1^2\tilde{\phi}_1^\dagger\phi_1 - m_2^2\tilde{\phi}_2^\dagger\phi_2 - \mu^2\left(\tilde{\phi}_1^\dagger\phi_2 - \epsilon\tilde{\phi}_2^\dagger\phi_1\right) - \frac{\lambda}{4}\left(\tilde{\phi}_1^\dagger\phi_1\right)^2$$

Non-Hermitian for $\epsilon = +1$; **Hermitian** for $\epsilon = -1$. (Note the overall sign of the Lagrangian.)

The $U(1)$-breaking minima of the potential lie at

$$\phi_1 \to v_1 = \sqrt{(2/\lambda)(m_1^2 - \epsilon\mu^4/m_2^2)} \qquad \tilde{\phi}_1^\dagger \to v_1 \qquad \phi_2 \to v_2 = \epsilon\mu^2 v_1/m_2^2 \qquad \tilde{\phi}_2^\dagger \to -\epsilon v_2$$

Expanding around the minima in Minkowski spacetime, the eigenmasses are

$$M_G^2 = 0 \qquad M_0^2 = m_2^2 - \frac{\epsilon\mu^2}{m_2^2} \qquad M_\pm^2 = \frac{m_2^2}{2}\left[1 + 2\frac{m_1^2}{m_2^2} + 3\frac{\epsilon\mu^4}{m_2^4} \pm \sqrt{\left(1 - 2\frac{m_1^2}{m_2^2}\right)^2 + 2\left(1 - 6\frac{m_1^2}{m_2^2}\right)\frac{\epsilon\mu^4}{m_2^4} + 9\frac{\epsilon^2\mu^8}{m_2^8}}\,\right]$$

There is an **exceptional point** at $\mu^2/m^2 = 1$, where $M_0^2 \to M_G^2$.

Alexandre, Ellis, PM and Seynaeve, Phys. Rev. D 98 (2018) 045001, 1805.06380; Mannheim, Phys. Rev. D 99 (2019) 4, 045006, 1808.00437;
see also related works by Fring and Taira

Now on FLRW ...

In conformal time τ, the de-Sitter phase FLRW line element $\mathrm{d}s^2 = 1/(H\tau)^2(-\mathrm{d}\tau^2 + \mathrm{d}\vec{x}^2)$ is consistent with anti-linear symmetry under $\tau \longmapsto -\tau$ and $\vec{x} \longmapsto -\vec{x}$.

Defining $\varphi_i = \phi_i/v_i$ and $\varphi_i^\dagger = \tilde{\phi}_i^\dagger/\tilde{v}_i$ (with $\tilde{v}_1 = v_1$ and $\tilde{v}_2 = -\epsilon v_2$), and $\bar{H} = H/m_2$, $\bar{m} = m_1/m_2$, $\bar{\mu} = \mu^2/m_2^2$ and $\bar{t} = m_2 t$, the **equations of motion** and **Friedmann constraint** take simpler forms:

$$-\ddot{\varphi}_1 - 3\bar{H}\dot{\varphi}_1 + \bar{m}^2\varphi_1 - \epsilon\bar{\mu}^2\varphi_2 - \left(\bar{m}^2 - \epsilon\bar{\mu}^2\right)\varphi_1^\dagger\varphi_1^2 = 0$$

$$-\ddot{\varphi}_2 - 3\bar{H}\dot{\varphi}_2 - \varphi_2 + \varphi_1 = 0$$

$$\bar{H}^2 = \frac{8\pi G v_1^2}{3}\left[\epsilon\bar{\mu}^2\dot{\varphi}_2^\dagger\dot{\varphi}_2 - \dot{\varphi}_1^\dagger\dot{\varphi}_1 + \bar{m}^2|\varphi_1|^2 + \epsilon\bar{\mu}^2|\varphi_2|^2 - \epsilon\bar{\mu}^2\left(\varphi_1^\dagger\varphi_2 + \varphi_2^\dagger\varphi_1\right) - \frac{1}{2}\left(\bar{m}^2 - \epsilon\bar{\mu}^2\right)|\varphi_1|^4\right]$$

Copeland, Lazanu, PM and Sablevice to appear this week

At late times ...

In terms of **radial** and **azimuthal field components** ($\varphi_a = R_a e^{i\theta_a}$), and assuming: R_1, R_2 and $\bar{H}$ constant, and $\ddot{\theta}_a \to 0$ at late times, the asymptotic behaviour is governed by

$$R_1\dot{\theta}_1^2 + \bar{m}^2 R_1 - \epsilon\bar{\mu}^2 R_2 \cos\delta\theta - \left(\bar{m}^2 - \epsilon\bar{\mu}^2\right) R_1^3 = 0 \qquad R_2\dot{\theta}_2^2 - R_2 + R_1\cos\delta\theta = 0$$

$$-3\bar{H}R_1\dot{\theta}_1 + \epsilon\bar{\mu}^2 R_2 \sin\delta\theta = 0 \qquad -3\bar{H}R_2\dot{\theta}_2 + R_1\sin\delta\theta = 0$$

and the solution for the **relative phase** $\delta\theta = \theta_1 - \theta_2$ is

$$\delta\theta = 2\mathrm{arccot}\left\{A\exp\left[\left(\frac{R_1}{R_2} - \epsilon\bar{\mu}^2\frac{R_2}{R_1}\right)\frac{\bar{t}}{3\bar{H}}\right]\right\} = \begin{cases} 0, & \epsilon\mu^4/m_2^4\frac{R_2^2}{R_1^2} < 1 \text{ (PT unbroken at vevs)} \\ \text{const.}, & \epsilon\mu^4/m_2^4\frac{R_2^2}{R_1^2} \geq 1 \text{ (PT broken at vevs)} \end{cases}$$

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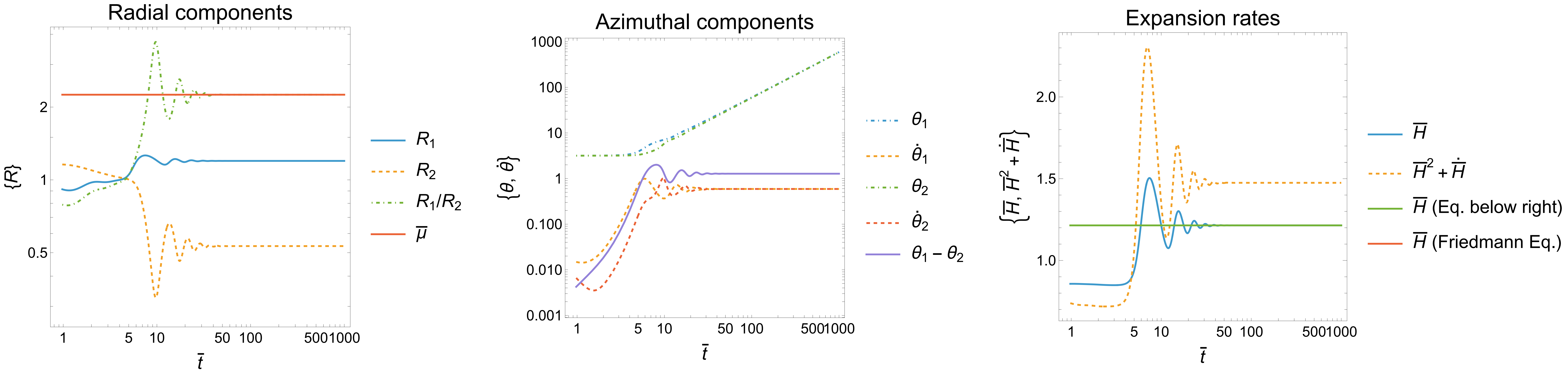
Self-sustaining accelerated expansion

This late-time attractive behaviour is realised numerically:



Equations of motion $\Rightarrow \bar{H}^2 = \frac{\bar{\mu}^2}{9} \frac{\sin^2 \delta\theta}{1 - \bar{\mu}\cos\delta\theta}$

Copeland, Lazanu, PM and Sablevice to appear this week

Growth instability vs. Hubble damping

In the **PT-broken regime** (with complex eigenspectrum), the eigenfrequency of one of the fluctuations about the minimum of the potential has a **positive imaginary part**:

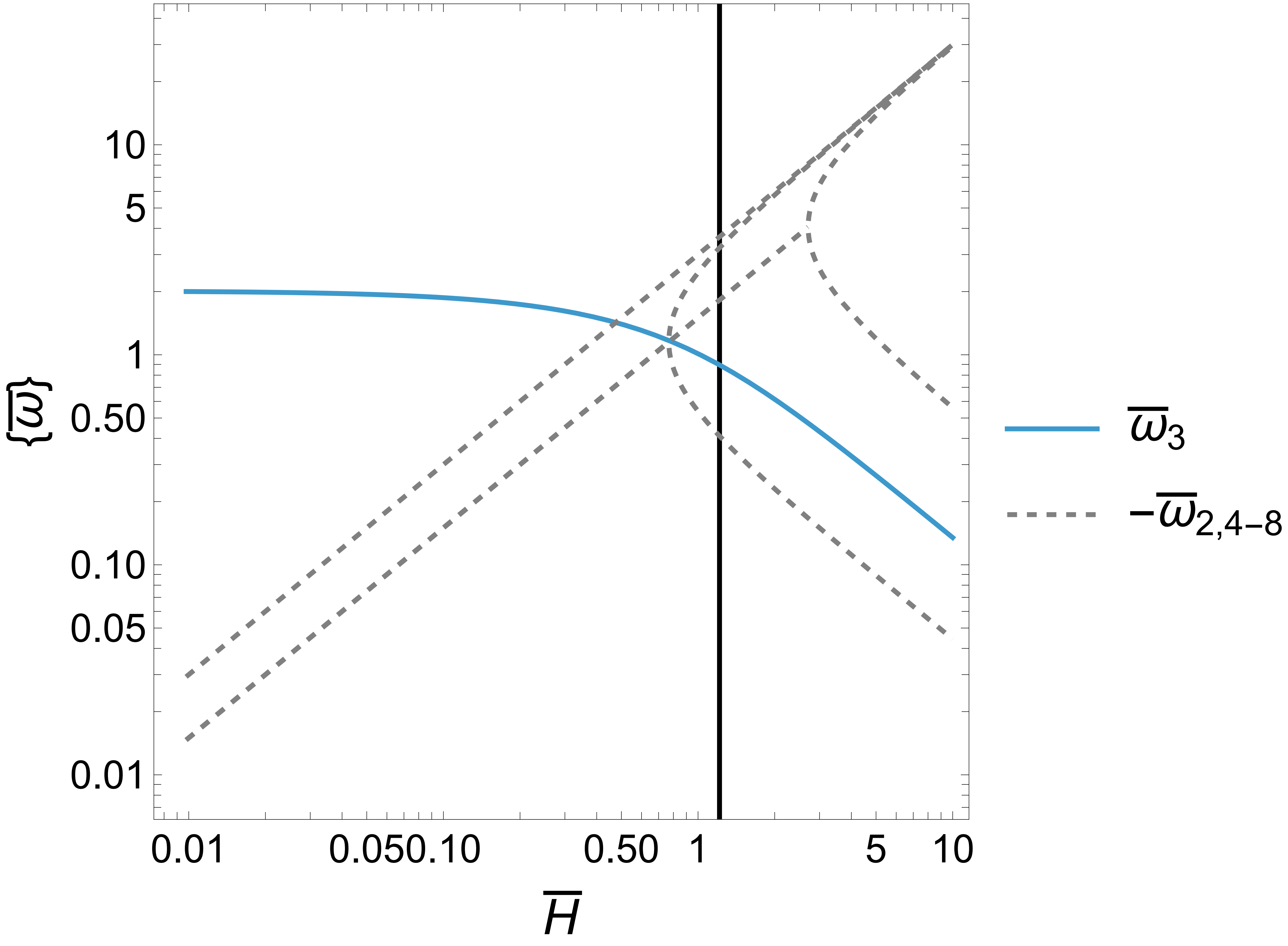
$$\bar{\omega}_{3(4)} = -\frac{3i\bar{H}}{2} + (-)\frac{i}{2}\sqrt{9\bar{H}^2 + 4\bar{\mu}^2 - 4}$$

Late-time equation of motion for the phases is

$$\dot{\theta} = \text{const}\,.$$

⇒ effectively subject to a linear potential.

cf. M. C. Escher's "Ascending and Descending"



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This model (and similar ones) also support defects.

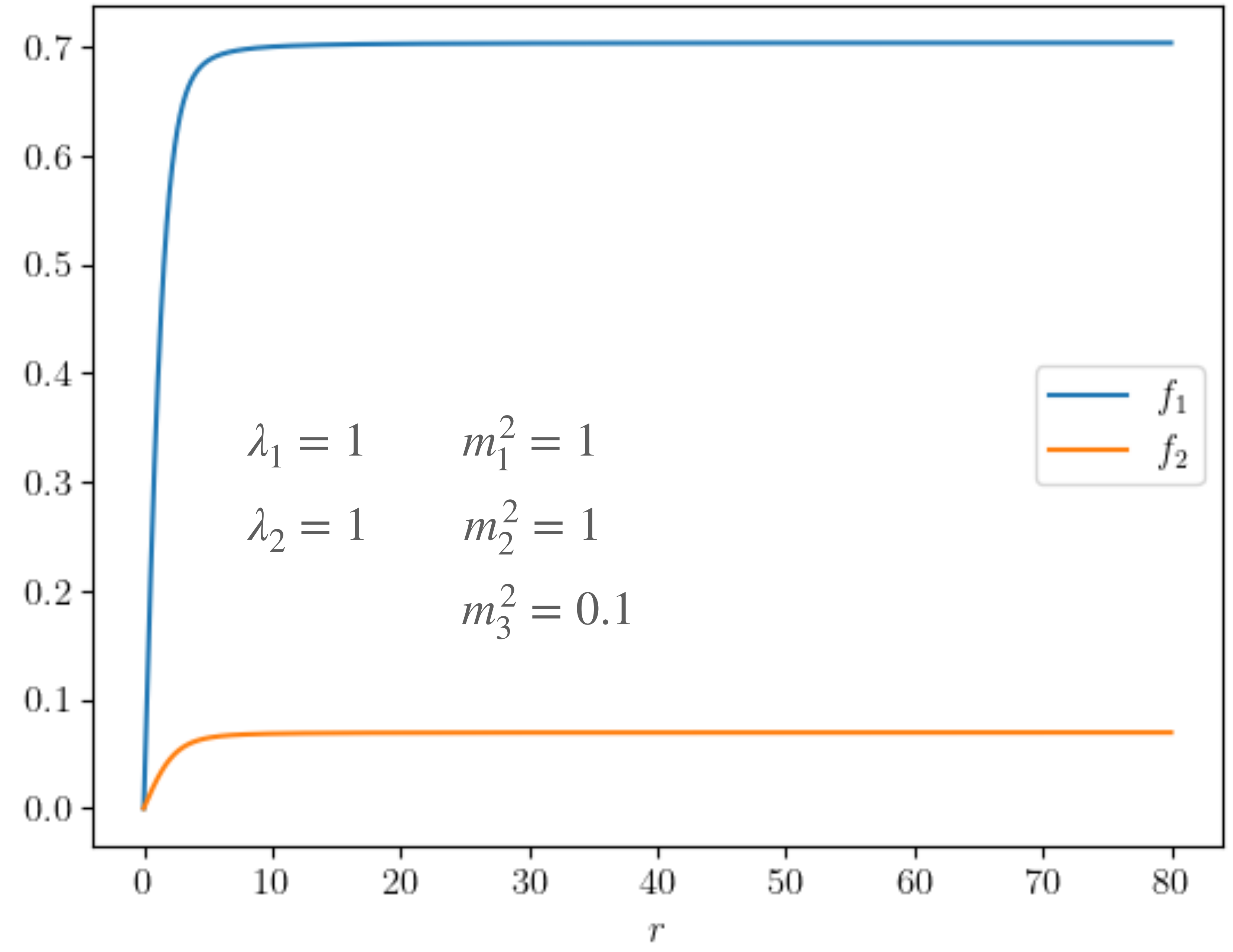
Vortex solutions found by Begun, Chernodub and Molochkov (Phys. Rev. D 104 (2021) 5, 056024, 2105.07453).

See also Fring and Taira, Phys. Lett. B 807 (2020) 135583, 2006.02718; J. Phys. A 53 (2020) 45, 455701, 2007.15425; Correa, Fring and Taira, Nucl. Phys. B 971 (2021) 115516, 2102.05781; Nucl. Phys. B 979 (2022) 115783, 2110.06825; JHEP 10 (2022) 109, 2208.03199

Revisiting a similar model, with quartic self-couplings for both fields, the usual **vortex Ansatz** $\phi_a = f_a(r)e^{in\theta}$ leads to the equations of motion

$$\frac{\mathrm{d}^2 f_1}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f_1}{\mathrm{d}r} - \frac{n^2}{r^2}f_1 + m_1^2 f_1 - \mu^2 f_2 - 2\lambda_1 f_1^3 = 0$$

$$\frac{\mathrm{d}^2 f_2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f_2}{\mathrm{d}r} - \frac{n^2}{r^2}f_2 - m_2^2 f_2 + \mu^2 f_1 - 2\lambda_2 f_2^3 = 0$$



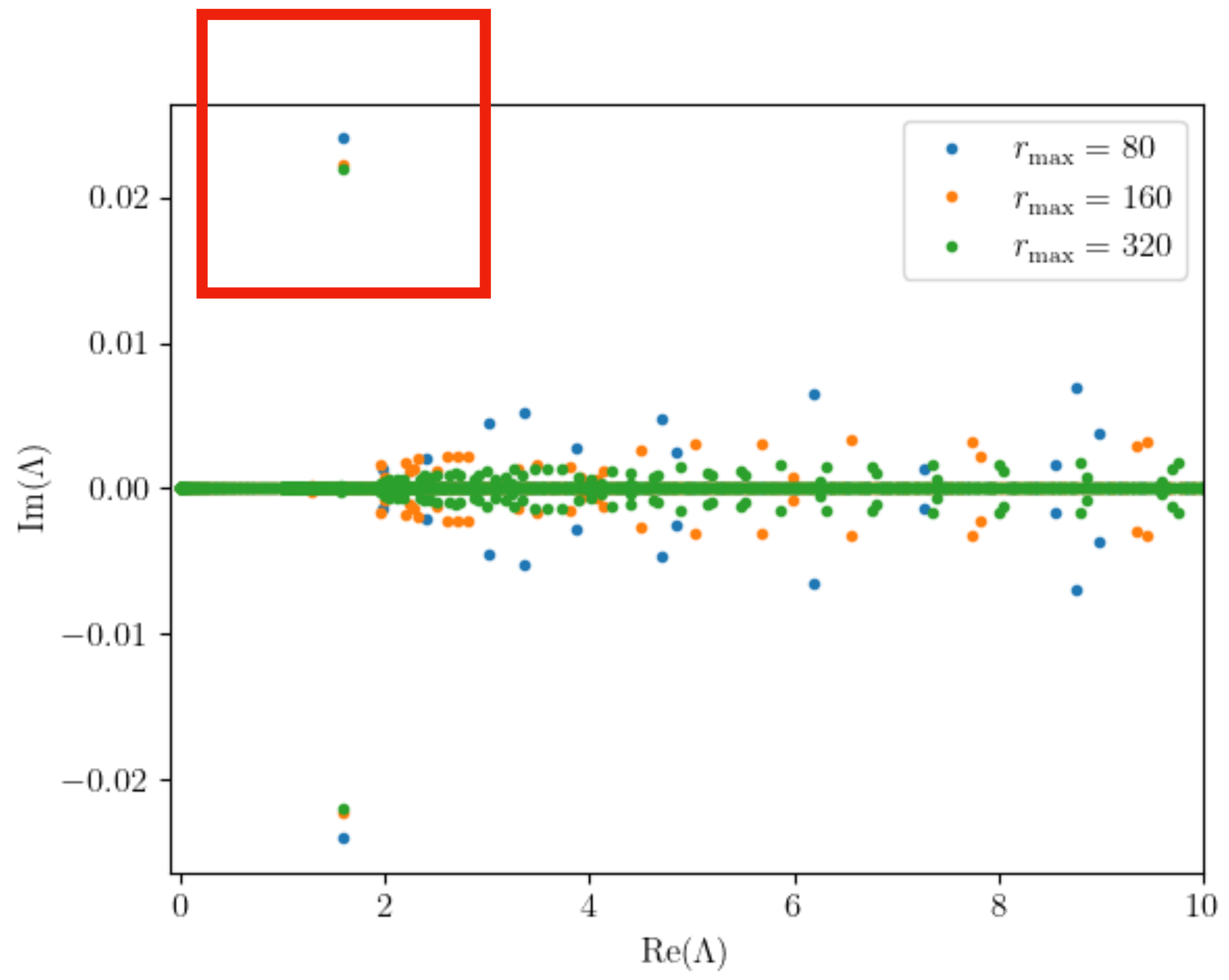
Battye, Cotterill and PM in prep 2025

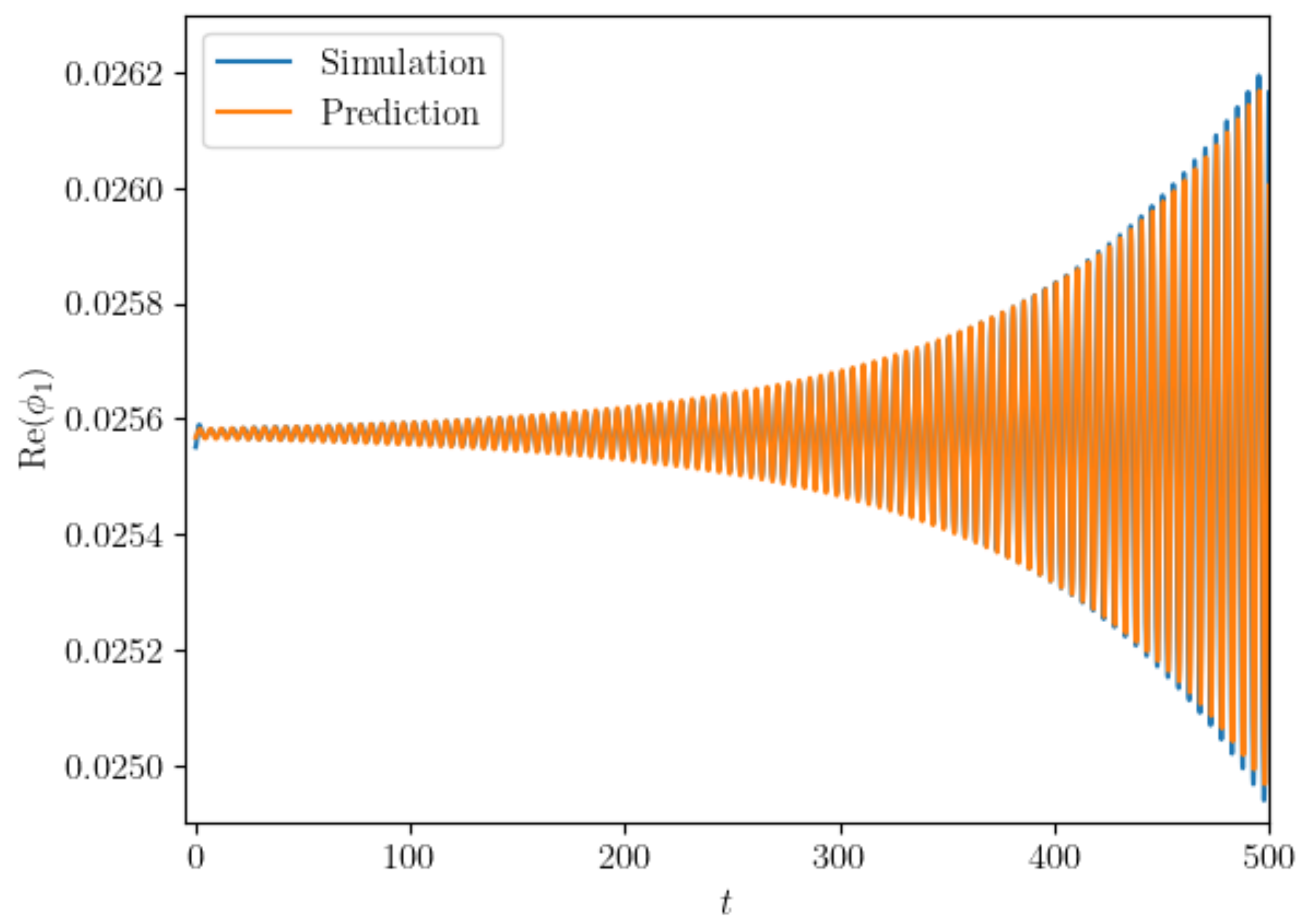
We find that these solutions are metastable. (Preliminary.)

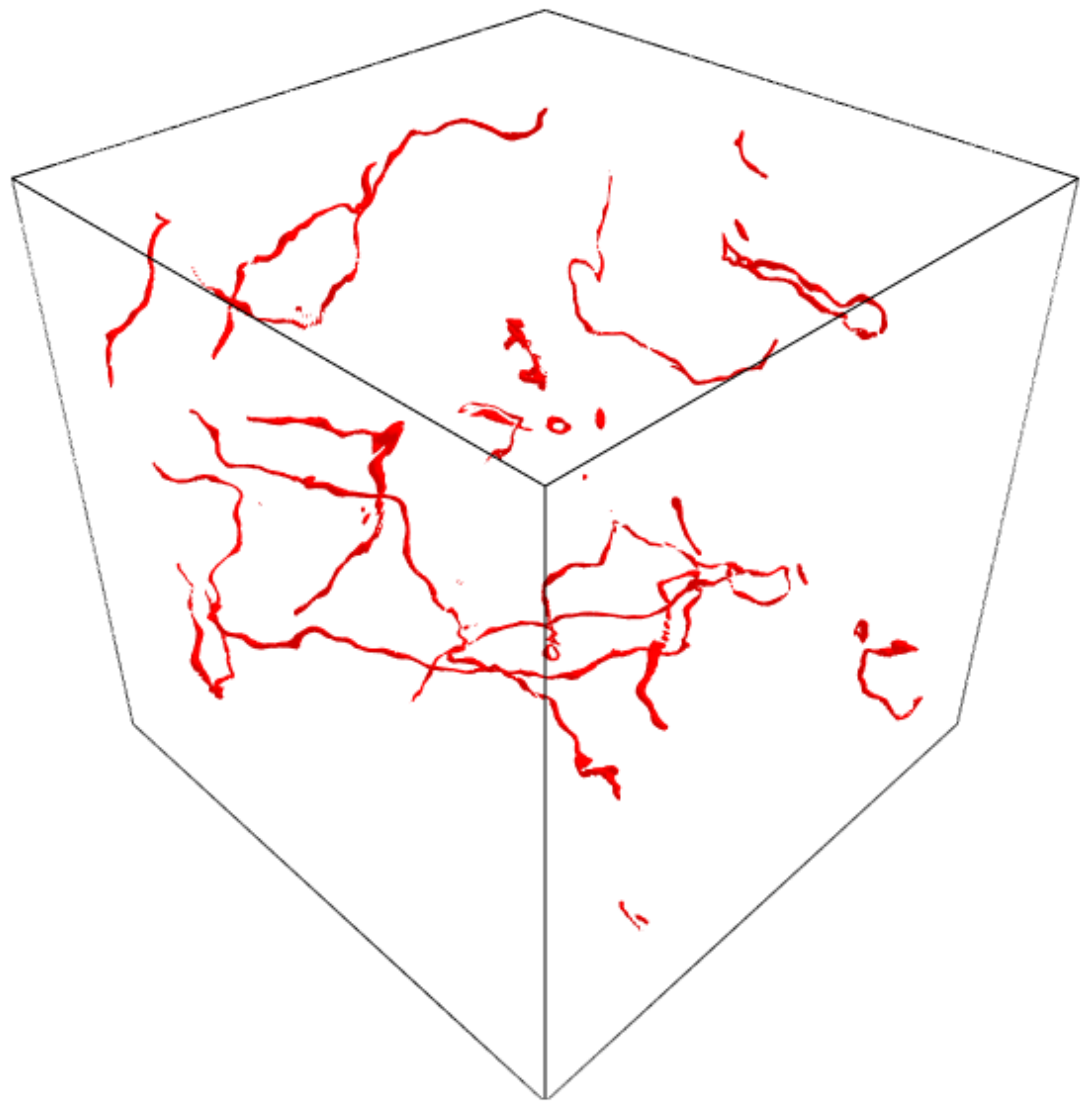
Despite being in the **PT-symmetric regime**, we find evidence of an **unstable fluctuation**, which may be due to the loss of spatial homogeneity via an effect identified by Chernodub and PM.

See Chernodub and PM, Phys. Rev. D 105 (2022) 7, 076020, 2110.05289; Phys. Rev. D 109 (2024) 10, 105006, 2401.06097.

This means that the **string networks** (right) may be **metastable**, with intriguing possibilities for cosmology.







Battye, Cotterill and PM in prep 2025

Closing remarks

- We can **relax Hermiticity** and still realise **physically viable theories**.
- The **PT-broken regime** can be viable in **dynamical spacetimes**.
- Example: **self-sustaining cosmological expansion** from PT-breaking.
- Example: **metastable networks** of **topological defects**.
- Phenomenologically relevant pseudo-Hermitian extensions of the Standard Models of particle physics and cosmology …

Backup slides

Some intuition from thinking about QFTs as $\bigotimes$ QMs

Proximity to the exceptional points is determined by the parameters of the theory.

In QM: We are in (1) the regime of exact antilinear symmetry (real eigenvalues), (2) at an exceptional point, or (3) in the broken regime (complex conjugate eigenvalues).

In QFT: Modes do not need to be in the same regime of the antilinear symmetry!

To see this, we just need to make the parameters of the theory spacetime dependent …

Local similarity transformations

$$\mathscr{L} = \bar{\psi}\left(i\gamma\cdot\partial - m - m_5\gamma^5\right)\psi \qquad \text{with eigenspectrum} \qquad M^2 = m^2 - m_5^2$$

Bender, Jones and Rivers, Phys. Lett. B 625 (2005) 333, hep-th/0508105

The theory belongs to a one-parameter family of similar theories $m + m_5\gamma^5 = Me^{2\gamma^5\theta}$ related by the **similarity transformation**

$$\psi \to e^{\omega_5\gamma^5}\psi, \qquad \bar{\psi} \to \bar{\psi}e^{\omega_5\gamma^5}.$$

If we make $\omega_5 \equiv \omega_5(x)$ local, we must promote the global similarity transformation to a local one and introduce a **“similarity gauge field”** C_μ with $C_\mu \to C_\mu - \partial_\mu\omega_5$:

$$\mathscr{L} = \bar{\psi}(i\gamma\cdot\partial + i\gamma\cdot C - m - m_5\gamma^5)\psi$$

Chernodub and PM, Phys. Rev. D 105 (2022) 076020, 2110.05289; Phys. Rev. D 109 (2024) 105006, 2401.06097;
similar vector fields found in non-Hermitian holographic theories, Morales-Tejera and Landsteiner, SciPost Phys. 14 (2023) 030, 2203.02524

Two-flavour fermion mixing

$$\mathscr{L} = \bar{\psi}_a \left(i\gamma \cdot \partial - m_a\right) \psi_a - m_5 \left(\bar{\psi}_1 \gamma^5 \psi_2 + \bar{\psi}_2 \gamma^5 \psi_1\right) \quad \equiv \quad \mathscr{L} = \bar{\Psi} \left(i\gamma \cdot \partial - M\right) \Psi$$

$$M = \begin{pmatrix} m_1 & m_5 \\ -m_5 & m_2 \end{pmatrix} \quad \text{with eigenspectrum} \quad M_\pm = \frac{m_1 + m_2}{2} \pm \left[\left(\frac{m_1 - m_2}{2}\right)^2 - m_5^2\right]^{1/2}$$

Now make the mass parameters local s.t. $M_\pm = M_0 \pm m_0$ remain constant:

$$\hat{M}(x) = M_0 \mathbb{I} + m_0[\sigma_3 \cosh(2\theta(x)) + i\sigma_2 \sinh(2\theta(x))]$$

Taking a time-local $\theta(x) = \theta_0 + C_0 x^0$ (time-like similarity gauge field), the **dispersion relation** is

$$\omega_{\pm,\mathbf{p}} = \mathbf{p}^2 + M_0^2 + m_0^2 - C_0^2 \pm 2\sqrt{M_0^2(m_0^2 - C_0^2) - C_0^2 \mathbf{p}^2}$$

Chernodub and PM, Phys. Rev. D 109 (2024) 105006, 2401.06097

Inhomogeneous two-flavour fermion mixing

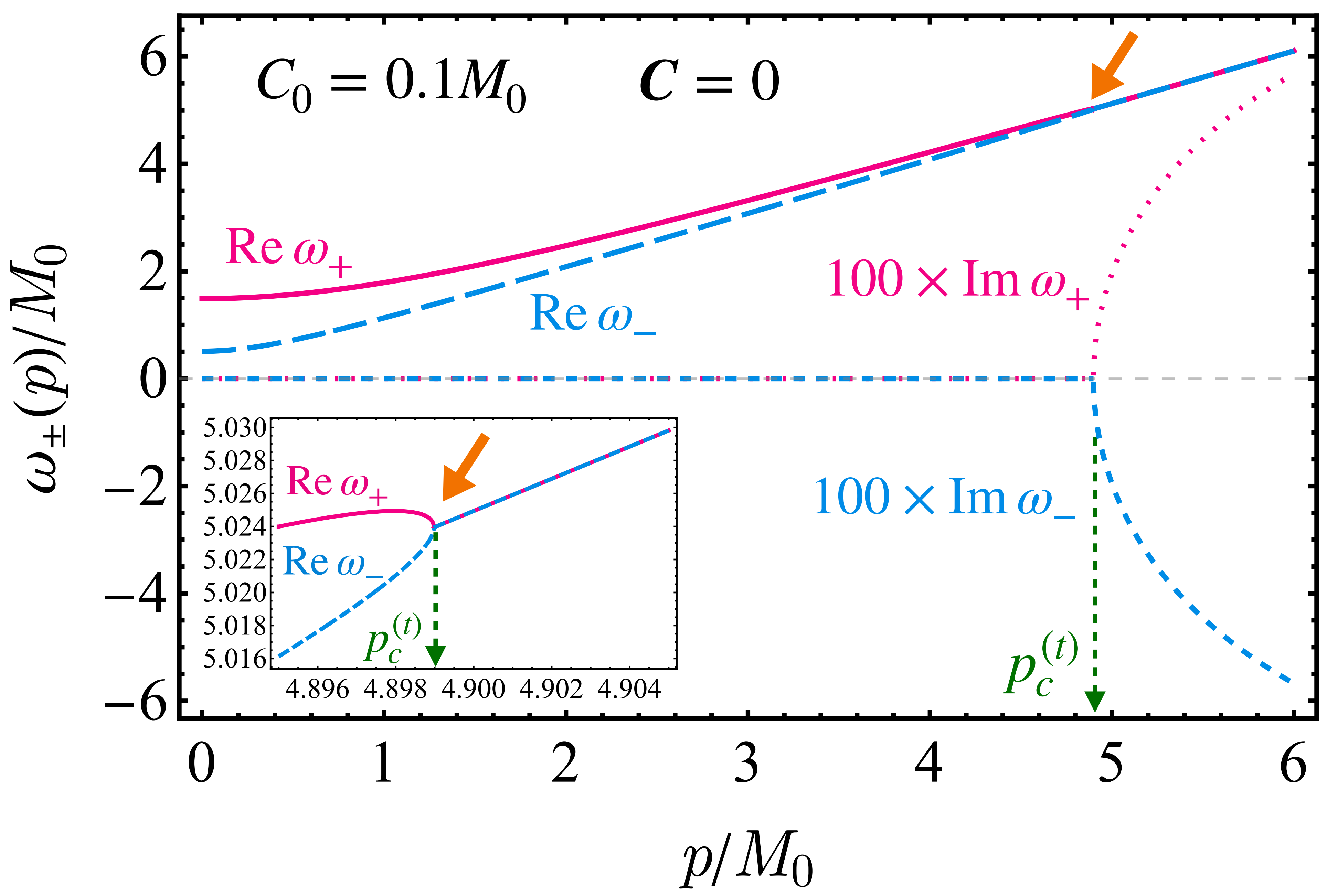
Momentum-dependent exceptional point: $p = p_c = M_0\sqrt{\frac{m_0^2}{C_0^2} - 1}$

Superluminal propagation: $p > p_{\rm SL}, \quad v_p = \frac{\partial \omega_{\pm,p}}{\partial p} > 1$

Stopped propagation: $p = p_{\rm stop} = \frac{1}{|C_0|}\sqrt{M_0^2(m_0^2 - C_0^2) - C_0^4}$

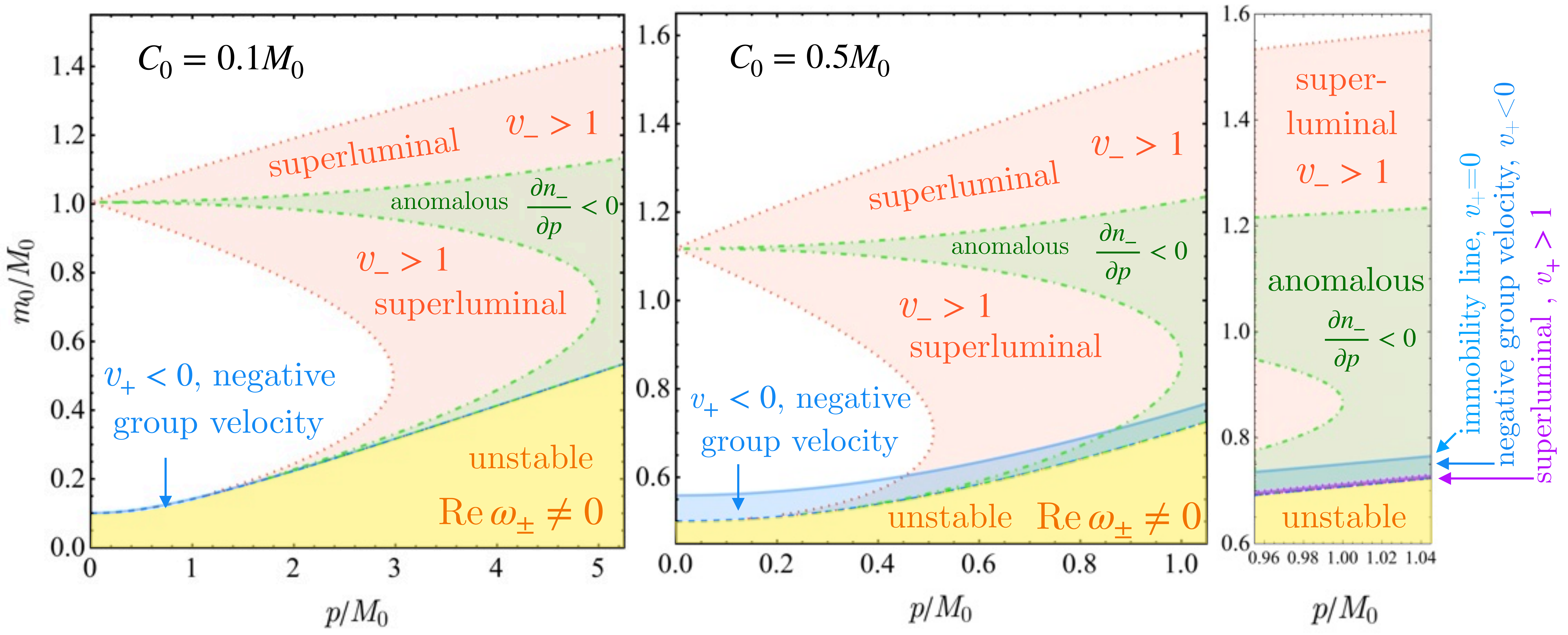
Negative group velocities: $p_{\rm stop} < p < p_c, \quad v_{+,p} < 0$

Anomalous dispersion: $\frac{\partial n_-}{\partial p} < 0$



Chernodub and PM, Phys. Rev. D 109 (2024) 105006, 2401.06097

Phase diagram



Chernodub and PM, Phys. Rev. D 109 (2024) 105006, 2401.06097

Pseudo-reality

We can (in fact have to) generalise what we mean by a **real field**:

$$\hat{\tilde{\phi}}^\dagger(x) = \hat{\eta}^{-1}\hat{\phi}^\dagger(x_\eta)\hat{\eta}\pi = \hat{\phi}(x) \in \mathbb{R}_\eta$$

- **Two degenerate constraints** (real and imaginary parts) define a hypersurface in the complex field space, such that we still have **one degree of freedom**.
- **Similarity transformations** of pseudo-Hermitian theories **map this hypersurface**.
- This is the self-consistent resolution of the so-called **Hermiticity puzzle**.

See Mannheim, Phys. Rev. D 99 (2019) 045006, 1808.00437

- **Import:** We can treat **continuous symmetries** in pseudo-Hermitian QFTs consistently.

Chernodub, PM and Sablevice, to appear in Phys. Rev. D (2025), 2501.09111