Black Holes of Palatini Nonlinear Electrodynamics

PASCOS 2025
21-25 July,
Durham

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PRD 111, 084061 (2025)

Thanks to:

Carlos Herdeiro
Jutta Kunz
Syksy Rasanen

Simple Question:

Does there exist a formalism for electrodynamics and its non-linear generalizations, which is analogous to the Palatini formalism of General Relativity and its extensions and modifications?

Answer for Maxwell is easy and simple: Yes

Answer for NLED: Yes, but not as simple

Maxwell Lagrangian and Field Equations – First Order

$$\mathcal{L}_{Max-Pal} = +\frac{1}{4}P^{\mu\nu}P_{\mu\nu} - \frac{1}{2}P^{\mu\nu}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - J^\mu A_\mu \ .$$

Notice change of notation: $F_{\mu\nu} \to P_{\mu\nu}$. $F_{\mu\nu}$ kept for: $F \equiv dA$.

Dynamical variables (for performing variation): A_ν and $P_{\mu\nu}$.

- Variation with respect to $A_\nu \Rightarrow$ inhomogeneous FEqs: $\nabla_\mu P^{\mu\nu} = J^\nu$
- variation with respect to $P_{\mu\nu} \Rightarrow P_{\mu\nu}(A_\sigma)$ or $P_{\mu\nu}(F_{\rho\sigma})$ relations:

$$P_{\mu\nu} = F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \Rightarrow \quad \nabla_\mu {}^*P^{\mu\nu} = 0 \ , \quad {}^*P^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} P_{\rho\sigma}/2\sqrt{-g}$$

$\therefore$ The field equations are the same, but the homogeneous ones result from the variational principle.

Non-Linear Electrodynamics

second order formalism: $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \to f(X,\Xi)$.

$$\mathcal{L}^{(2)}_{NLED} = f(X,\Xi) - J^{\mu}A_{\mu} \ , \quad X = F_{\mu\nu}F^{\mu\nu} \ , \ \Xi = F_{\mu\nu}{}^{*}F^{\mu\nu} \ .$$

$$-4\nabla_{\mu}\left(f_X F^{\mu\nu} + f_{\Xi}\,{}^{*}F^{\mu\nu}\right) = J^{\nu} \ , \quad f_X = \partial f/\partial X \ , \ f_{\Xi} = \partial f/\partial \Xi \ .$$

Energy-momentum: $T_{\mu\nu} = 4 f_X F_{\mu\alpha}F_{\nu}{}^{\alpha} + (\Xi f_{\Xi} - f(X,\Xi))g_{\mu\nu}$

Two NLED Examples

- Born-Infeld (1934): $f_{BI}(X,\Xi) = b^2\left(1 - \sqrt{1 + \frac{1}{2b^2}X - \frac{1}{16b^4}\Xi^2}\right)$
- Euler-Heisenberg (1936): $f_{HE}(X,\Xi) = -\frac{1}{4}X + \frac{\alpha^2}{90m_e^4}(X^2 + \frac{7}{4}\Xi^2) + \ldots$

(+B. Kockel, 1935)

Why NLED?

- Answer 2: QED radiative corrections; Euler-Kockel-Heisenberg (1935-6)

- Answer 1: less singularities; Born-Infeld (1934)

- Answer 3: regular black holes; Ayon-Beato & Garcia (2000)…

Some simple NLED solutions – Second order formalism

Generalized Euler-Heisenberg Lagrangian

$$f_{GHE}(X, \Xi) = -\frac{1}{4}X + \frac{\gamma}{2n}X^n + \frac{\beta}{2n}\Xi^n$$

Field Equations:

$$\nabla_\mu \left[\left(1 - 2\gamma X^{n-1} \right) F^{\mu\nu} - 2\beta \Xi^{n-1} \, {}^*F^{\mu\nu} \right] = J^\nu \ .$$

Electric Field of a Point Charge: $F_{tr} + (-2)^n \gamma F_{tr}^{2n-1} = \frac{Q}{r^2}$.

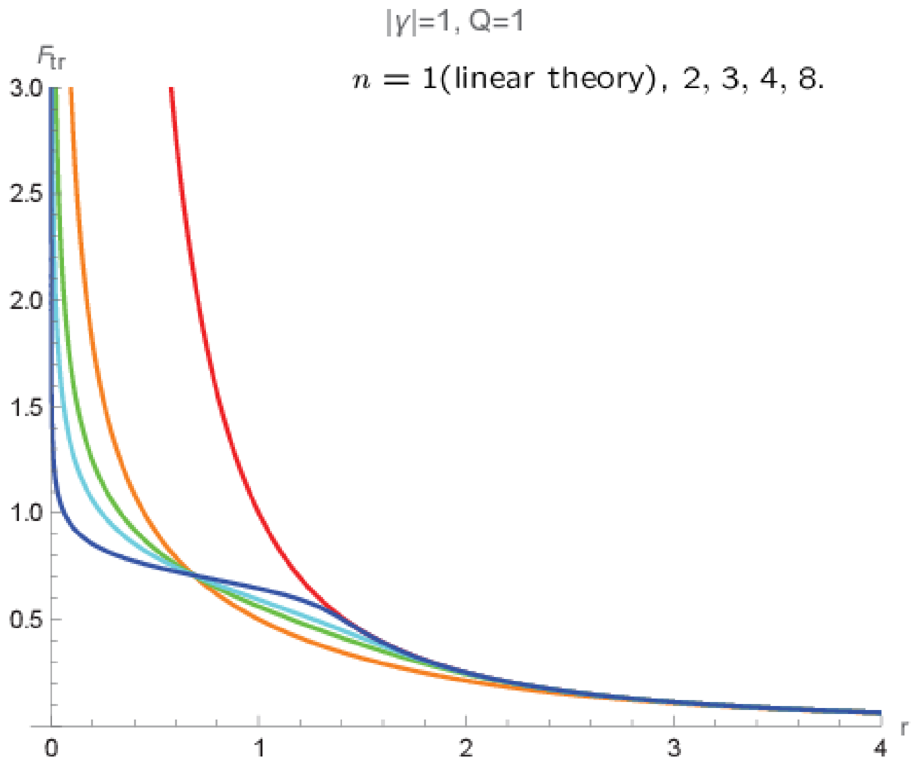
No explicit expression for $F_{tr}(r)$ for all n, but analytic treatment still possible.

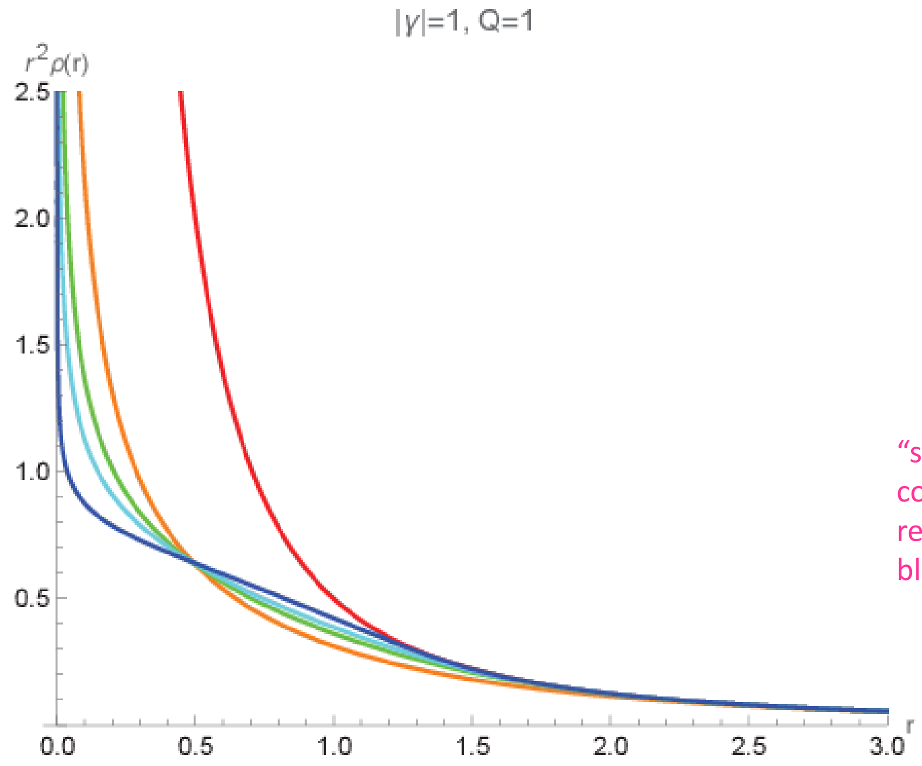
Notice: “healthy” solutions for: $(-1)^n \gamma > 0$.

Electric Field and Energy density Profiles

Fields are singular at the origin, but field energy is finite.

near $r = 0$: $F_{tr}(r) \simeq 1/r^{2/(2n-1)}$; asymptotically: $F_{tr}(r) \simeq 1/r^2$





"spectroscopic" color code: red-small, blue-large

Energy and $T_{\mu\nu}$ for spherically-symmetric electric field:

$$T_t^t = T_r^r = \frac{1}{2}F_{tr}^2 + \frac{(2n-1)(-2)^n\gamma}{2n}F_{tr}^{2n}$$

$$T_\theta^\theta = T_\phi^\phi = -\frac{1}{2}F_{tr}^2 - \frac{(-2)^n\gamma}{n}F_{tr}^{2n}$$

WEC OK

SEC OK

Total Energy – integrate over $F_{tr} = \mathcal{E}$:

$$E = 4\pi\int_0^\infty r^2 T_t^t dr = -4\pi\int_0^\infty r^2(\mathcal{E})\left[\frac{1}{2}\mathcal{E}^2 + \frac{(2n-1)2^n|\gamma|}{2n}\mathcal{E}^{2n}\right]\frac{dr}{d\mathcal{E}}d\mathcal{E}\ .$$

and explicitly:

$$E(|\gamma|, n, Q) = \frac{8\sqrt{\pi}Q^{3/2}}{3\,(2^n|\gamma|)^{1/4(n-1)}}\frac{\Gamma\left(\frac{1}{4(n-1)}\right)\Gamma\left(\frac{6n-7}{4(n-1)}\right)}{n-\frac{3}{2}}\ .$$

(E decreases with n)

Electrostatic BHs in the GEH Lagrangian

$$\mathcal{L} = \frac{1}{2\kappa} R - \frac{X}{4} + \frac{\gamma}{2n} X^n \; .$$

$$ds^2 = h(r)dt^2 - dr^2/f(r) - r^2 d\Omega^2 \; .$$

$T_t^t = T_r^r \;\Rightarrow\; h(r) = f(r) \;\Rightarrow$ a single Einstein Eq.+ flat space Maxwell:

$$\frac{1}{r^2}\frac{d}{dr} r(1-f) \;=\; \kappa \left(\frac{1}{2} F_{tr}^2 + \frac{(2n-1)2^n|\gamma|}{2n} F_{tr}^{2n} \right)$$

$$f(r) = 1 - \frac{2M(r)}{r} \quad \Rightarrow \quad \frac{dM}{dr} \;=\; \frac{\kappa r^2}{2} \left(\frac{1}{2} F_{tr}^2 + \frac{(2n-1)2^n|\gamma|}{2n} F_{tr}^{2n} \right) \; ,$$

Coulomb-Maxwell: $F_{tr} + (-2)^n \gamma F_{tr}^{2n-1} = \frac{Q}{r^2}$.

Analytic solution in terms of $\mathcal{E}$ instead of r:

$$\frac{dM}{d\mathcal{E}} = \frac{\kappa r^2(\mathcal{E})}{2}\left(\frac{1}{2}\mathcal{E}^2 + \frac{(2n-1)2^n|\gamma|}{2n}\mathcal{E}^{2n}\right)\frac{dr}{d\mathcal{E}}.$$

Solution in terms of hypergeometric function (M_0 integr. const.):

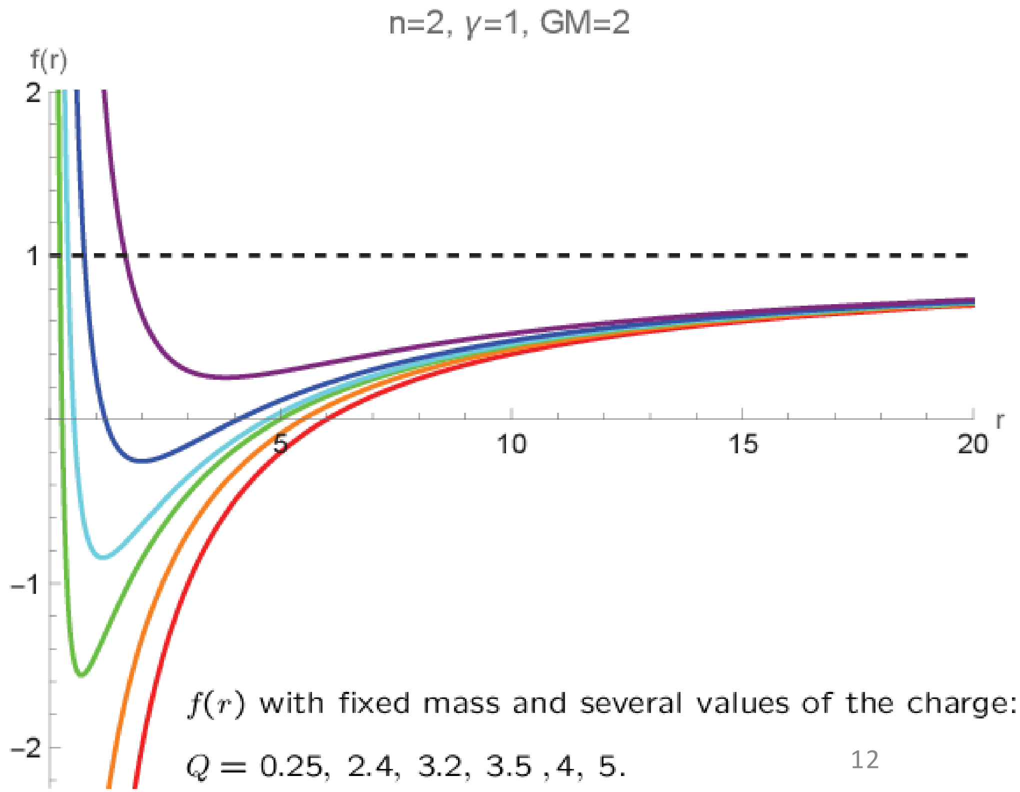
$$M(\mathcal{E}) = \kappa Q^{3/2}\left(\frac{5n + (6n-1)2^n|\gamma|\mathcal{E}^{2(n-1)}}{12n\left(1+2^n|\gamma|\mathcal{E}^{2(n-1)}\right)^{3/2}}\mathcal{E}^{1/2} + \right.$$

$$\left. + \frac{2}{3(2n-3)\sqrt{2^n|\gamma|}\,\mathcal{E}^{n-\frac{3}{2}}}F\left(\frac{1}{2}, \frac{2n-3}{4(n-1)}, \frac{6n-7}{4(n-1)}, -\frac{1}{2^n|\gamma|\mathcal{E}^{2(n-1)}}\right)\right) + M_0$$

Total Mass $M(\mathcal{E} \to 0)$ is sum of M_0 + field energy (calculated in flat space above): $M(|\gamma|, n, Q) = M_0 + E(|\gamma|, n, Q)$.

Metric component $f(r)$ in parametric form:

$$\begin{cases} f(\mathcal{E}) = 1 - \frac{2}{Q^{1/2}} \left(\mathcal{E} + 2^n|\gamma|\mathcal{E}^{2n-1}\right)^{1/2} M(\mathcal{E}) \\ r(\mathcal{E}) = \frac{Q^{1/2}}{\left(\mathcal{E} + 2^n|\gamma|\mathcal{E}^{2n-1}\right)^{1/2}} \, . \end{cases}$$



$f(r)$ with fixed mass and several values of the charge: $Q =$ 0.25, 2.4, 3.2, 3.5 ,4, 5.

Notice 2 branches:

- S-like (1 horizon) for small Q
- RN-like (2 horizons) for larger Q up to a maximal charge.

Poor man's way to NLED in first order formalism

Recall 1st order Maxwell Lagrangian; Rewritten with $F_{\mu\nu} \to P_{\mu\nu}$,

keeping $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$:

$$\mathcal{L}_{Max-Pal}(A_\lambda, P_{\mu\nu}) \;=\; \frac{1}{4}P^{\mu\nu}P_{\mu\nu} - \frac{1}{2}P^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu \;.$$

Generalize only the NL term as in 2nd order: $P^{\mu\nu}P_{\mu\nu} \to -h(P^{\mu\nu}P_{\mu\nu}, P_{\mu\nu}{}^*P^{\mu\nu})$:

$$\mathcal{L}^{(1)}_{NLED} = -\frac{1}{4}h(P^{\mu\nu}P_{\mu\nu}, P_{\mu\nu}{}^*P^{\mu\nu}) - \frac{1}{2}P^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu = -\frac{1}{4}h(Z,\Omega) - \frac{1}{2}Y - J^\mu A_\mu \,,$$

$Z = P_{\mu\nu}P^{\mu\nu}$, $Y = P^{\mu\nu}F_{\mu\nu}$, $\Omega = P_{\mu\nu}{}^*P^{\mu\nu}$.

Maxwell limit: for weak fields: $h(Z,\Omega) \simeq -Z$.

FEqs by independent variations of A_μ and $P_{\mu\nu}$ (respectively):

$$\nabla_\mu P^{\mu\nu} = J^\nu \quad , \quad h_Z P_{\mu\nu} + h_\Omega {}^* P^{\mu\nu} + F_{\mu\nu} = 0 \ ,$$

or more similar to Maxwell's, in terms of $P_{\mu\nu}$ only:

$$\nabla_\mu P^{\mu\nu} = J^\nu \quad , \quad \nabla_\mu [h_Z {}^* P^{\mu\nu} - h_\Omega P^{\mu\nu}] = 0 \ .$$

Expressing FEqs in term of $F_{\mu\nu}$ - invert P-F relation

$$P^{\mu\nu} = \frac{h_\Omega {}^* F^{\mu\nu} - h_Z F^{\mu\nu}}{h_Z^2 + h_\Omega^2} \ .$$

$$\nabla_\mu \left(f_X F^{\mu\nu} + f_\Xi {}^* F^{\mu\nu} \right) = J^\nu$$

∴ 1st order FEqs are equivalent to 2nd order (up to invertibility).

Plebanski's way to first order NLED ($\sim$1970): "Legendre transformation"

- Start with the 2nd order NLED Lagrangian $\mathcal{L}^{(2)}(A_\mu, F_{\mu\nu})$
- Define covariant "conjugate momenta" by $\Pi^{\mu\nu} = \partial\mathcal{L}^{(2)}/\partial F_{\mu\nu}$
- Define a "Legendrian" $\mathcal{H} = \Pi^{\mu\nu} F_{\mu\nu} - \mathcal{L}^{(2)}$
- Express the "velocities" $F_{\mu\nu}$ in terms of the "momenta" $\Pi^{\mu\nu}$
- Field equations become Hamiltonian-like:

$$F_{\mu\nu} = \frac{\partial\mathcal{H}}{\partial\Pi^{\mu\nu}} \quad , \quad 2\partial_\mu \Pi^{\mu\nu} = -\frac{\partial\mathcal{H}}{\partial A_\mu} = -J^\nu$$

Last step: define 1st order Lagrangian from which the "Hamilton Eqs" are derived:

$$\mathcal{L}^{(1)}(A_\lambda, F_{\mu\nu}, \Pi^{\rho\sigma}) = \Pi^{\mu\nu} F_{\mu\nu} - \mathcal{H}(A_\lambda, \Pi^{\mu\nu}).$$

Relationship: $\Pi^{\mu\nu} = -P^{\mu\nu}/2$, $\mathcal{H} = h(Z, \Omega)/4 + J^\mu A_\mu$.

Notice: **Equivalence is not automatic**. (Example: Ayon-Beato & Garcia, 1998)

Simple Example: Polynomial Electrodynamics in 1st Order

$$h = -Z + \frac{2\gamma}{n} Z^n - \alpha\Omega + \frac{2\beta}{n}\Omega^n$$

ONLY IF TIME PERMITS

Purely electric fields:

$$\nabla_\mu P^{\mu\nu} = J^\nu \quad , \quad \nabla_\mu [(1 - 2\gamma Z^{n-1})\ {}^*P^{\mu\nu}] = 0$$

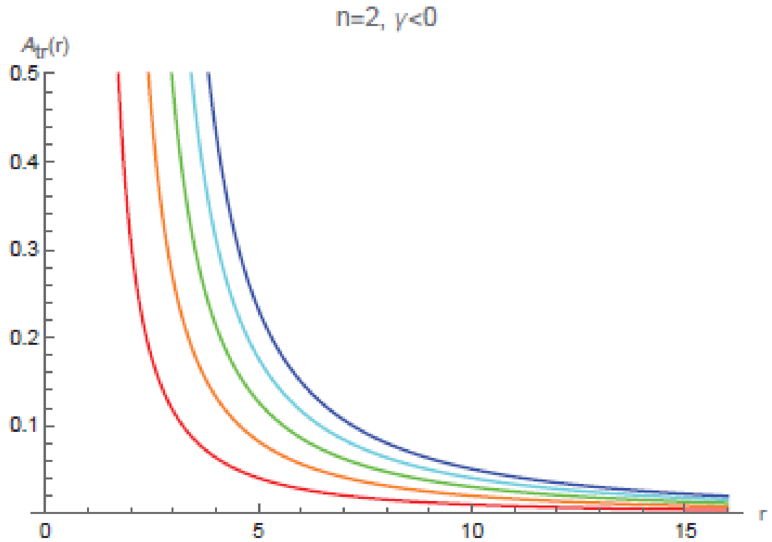
Spherically-symmetric solutions:

$$P_{tr} = \frac{Q}{r^2} \quad , \quad F_{tr} = \frac{Q}{r^2} + (-2)^n \gamma \left(\frac{Q}{r^2}\right)^{2n-1}$$

$$\rho = \frac{Q^2}{2r^4} + \frac{(-2)^n \gamma}{2n} \frac{Q^{2n}}{r^{4n}}$$

"healthy" solutions: $(-1)^n \gamma > 0$

Total field energy diverges.



Spherical electric Black Holes:

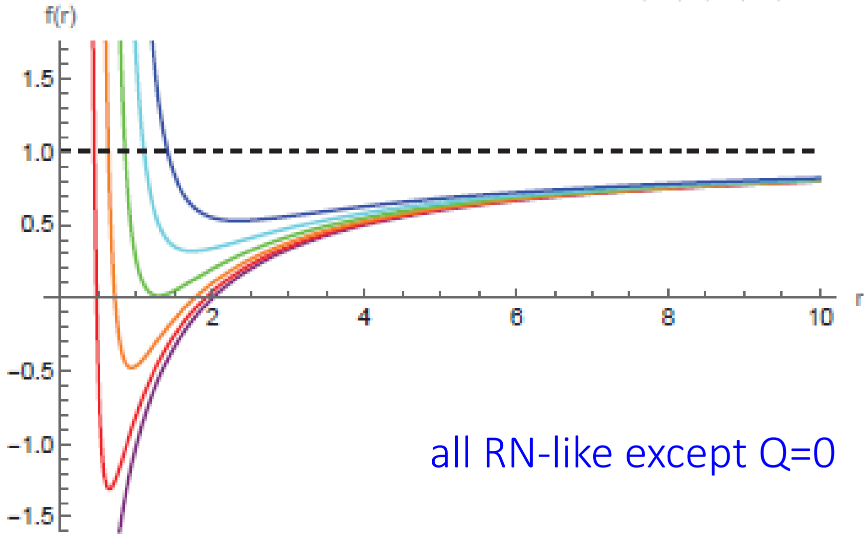
$$ds^2 = h(r)dt^2 - dr^2/f(r) - r^2 d\Omega^2 \ .$$

WEC OK

SEC OK

$$T_t^t = T_r^r = \frac{1}{2}P_{tr}^2 + \frac{(-2)^n \gamma}{2n} P_{tr}^{2n}$$

$$T_\theta^\theta = T_\phi^\phi = -\frac{1}{2}P_{tr}^2 - \frac{(2n-1)(-2)^n \gamma}{2n} P_{tr}^{2n}$$



$$\frac{1}{r^2}\frac{d}{dr} r(1-f) = \kappa \left(\frac{Q^2}{2r^4} + \frac{(-2)^n \gamma}{2n} \frac{Q^{2n}}{r^{4n}} \right) \quad \Rightarrow \quad \frac{dM}{dr} = \frac{\kappa}{2} \left(\frac{Q^2}{2r^2} + \cdot \frac{(-2)^n \gamma}{2n} \frac{Q^{2n}}{r^{2(2n-1)}} \right)$$

$$f(r) = 1 - \frac{2M}{r} + \frac{\kappa Q^2}{2r^2} + \frac{(-2)^n \gamma}{(4n-3)n} \frac{\kappa Q^{2n}}{r^{2(2n-1)}}$$

Palatini Nonlinear Electrodynamics (PNLED)

Allow an arbitrary defining function of all 4 quadratic Lorentz scalars containing up to 1st derivatives of the dynamical variables A_λ and $P_{\mu\nu}$: $Z = P_{\mu\nu}P^{\mu\nu}$, $\Omega = P_{\mu\nu}{}^{*}P^{\mu\nu}$, $Y = P^{\mu\nu}F_{\mu\nu}$ and $\Upsilon = F_{\mu\nu}{}^{*}P^{\mu\nu}$.

Lagrangian: $\mathcal{L}^{(1)}_{PNLED} = \mathcal{K}(Z,\Omega,Y,\Upsilon) - J^{\mu}A_{\mu}$,

Field Equations:

$$-2\nabla_{\mu}\left(\mathcal{K}_{Y}P^{\mu\nu} + \mathcal{K}_{\Upsilon}{}^{*}P^{\mu\nu}\right) = J^{\nu}$$

$$2\left(\mathcal{K}_{Z}P^{\mu\nu} + \mathcal{K}_{\Omega}{}^{*}P^{\mu\nu}\right) + \mathcal{K}_{Y}F^{\mu\nu} + \mathcal{K}_{\Upsilon}{}^{*}F^{\mu\nu} = 0 \ .$$

Simple PNLED Model: Y^n

$$\mathcal{K}(Z,\Omega,Y,\Upsilon) = Z/4 - Y/2 + \gamma Y^n/(2n) \Rightarrow$$

$$\mathcal{L}_{Y^n} = \frac{1}{4}P^{\mu\nu}P_{\mu\nu} - \frac{1}{2}P^{\mu\nu}F_{\mu\nu} + \frac{\gamma}{2n}(P^{\mu\nu}F_{\mu\nu})^n - J^\mu A_\mu$$

Not of the Plebanski form: $\mathcal{L}^{(1)}_{NLED} = -\frac{1}{4}h(Z,\Omega) - \frac{1}{2}Y - J^\mu A_\mu$

Field Equations:

$$\nabla_\mu\left[\left(1-\gamma Y^{n-1}\right)P^{\mu\nu}\right] = J^\nu \quad , \quad P_{\mu\nu} = \left(1-\gamma Y^{n-1}\right)F_{\mu\nu}$$

Decouple Eqs by P-F relation which gives $Y(Z)$ and $Y(X)$:

$$\gamma X Y^{n-1} + Y - X = 0 \quad , \quad \gamma Y^n - Y + Z = 0$$

Also: $1-\gamma Y^{n-1} = Y/X = Z/Y$.

Simplified FEqs and $T_{\mu\nu}$ in terms of $F_{\mu\nu}$ (easier):

$$\nabla_\mu \left[\left(\frac{Y}{X}\right)^2 F^{\mu\nu} \right] = J^\nu \quad , \quad P_{\mu\nu} = \left(\frac{Y}{X}\right) F_{\mu\nu} \ .$$

$$T_{\mu\nu} = -\left(\frac{Y}{X}\right)^2 F_{\mu\alpha} F_\nu{}^\alpha - \frac{g_{\mu\nu}}{2} Y \left[\frac{n-2}{2n} \frac{Y}{X} - \frac{n-1}{n} \right] \ .$$

$n=2:\ Y = X/(1+\gamma X) \qquad , \qquad n=3:\ Y = \frac{-1\pm\sqrt{1+4\gamma(X)^2}}{2\gamma X}$.

Electric point charge in the Y^n Model: $n = 2$

Field Equations integrated easily due to spherical symmetry:

$$\frac{F_{tr}(r)}{\left[1 - 2\gamma F_{tr}(r)^2\right]^2} = \frac{Q}{r^2} \ ,$$

Physical solutions (defined for all r, decreasing with r) for $\gamma > 0$.

near $r = 0$: $F_{tr}(r) \simeq (2\gamma)^{-1/2} \left(1 - \frac{(2\gamma)^{-1/4}}{2|Q|} r\right)$;

asymptotically: $F_{tr}(r) \simeq 1/r^2$.

$\therefore$ As good as BI: finite $F_{tr}(0)$ and finite energy.

$$E = -4\pi \int_0^\infty r^2(\mathcal{E})\, \rho(\mathcal{E})\, \frac{dr}{d\mathcal{E}}\, d\mathcal{E} = \frac{32 \cdot 2^{3/4}\pi}{15} \frac{Q^{3/2}}{\gamma^{1/4}} \ .$$

Electric Black Holes in the Y^n Model: $n = 2$

$T_{\mu\nu}$ components:

$$T_0^0 = T_1^1 = -\frac{F_{tr}^2(1 + 2\gamma F_{tr}^2)}{2(1 - 2\gamma F_{tr}^2)^2} \quad , \; T_2^2 = T_3^3 = \frac{F_{tr}^2}{2(1 - 2\gamma F_{tr}^2)^2} \; ,$$

the single Einstein equation:

$$\frac{1}{r^2}\frac{d}{dr}r(1 - f) = \kappa\frac{F_{tr}^2(1 + 2\gamma F_{tr}^2)}{2(1 - 2\gamma F_{tr}^2)^2} \quad \Rightarrow \quad \frac{dM}{dr} = \kappa r^2\frac{F_{tr}^2(1 + 2\gamma F_{tr}^2)}{4(1 - 2\gamma F_{tr}^2)^2} \; .$$

Change variables to $\mathcal{E}$ gives:

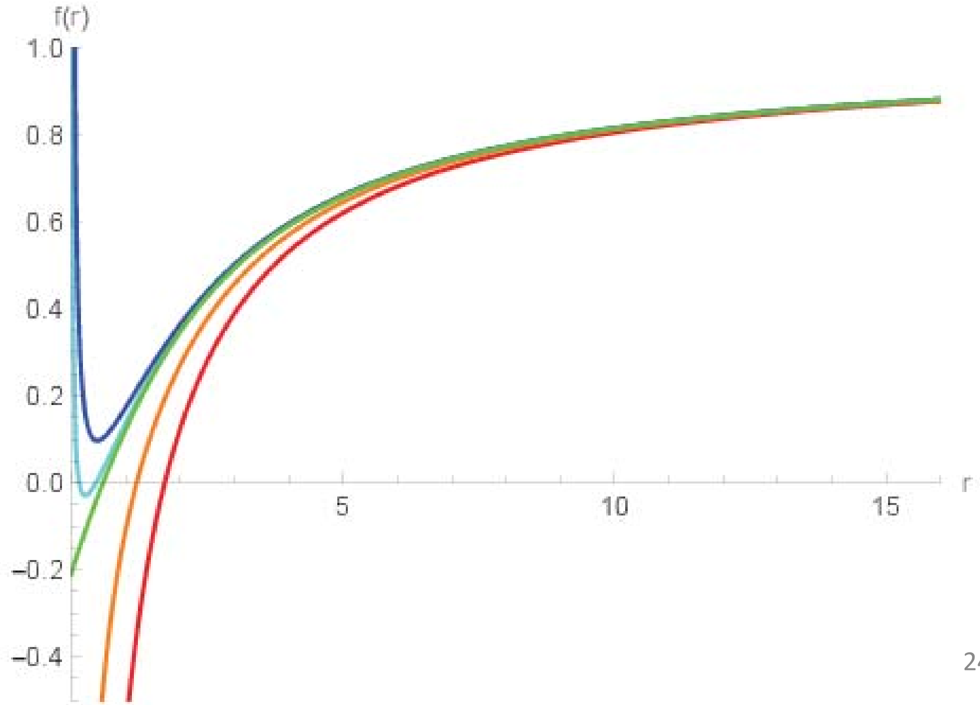
$$\frac{dM}{d\mathcal{E}} = -\frac{\kappa Q^{3/2}(1 + 2\gamma\mathcal{E}^2)(1 + 6\gamma\mathcal{E}^2))}{8\mathcal{E}^{1/2}} \; .$$

Parametric representation of $f(r)$:

$$\begin{cases} f(\mathcal{E}) = 1 - \dfrac{2M\mathcal{E}^{1/2}}{Q^{1/2}\left(1-2\gamma\mathcal{E}^2\right)} + \dfrac{\kappa Q\,\mathcal{E}\left(15+24\gamma\mathcal{E}^2+20\gamma^2\mathcal{E}^4\right)}{30\left(1-2\gamma\mathcal{E}^2\right)} \\[2ex] r(\mathcal{E}) = Q^{1/2}\left(1/\mathcal{E}^{1/2} - 2\gamma\mathcal{E}^{3/2}\right) \ . \end{cases}$$

Gravitational profiles: fixed M, varying Q.

- S-like and RN-like **solutions**.
- Notice the solution with finite $f(0)$. It is NOT regular BH.



$n \geq 3$ point charge not more difficult - and covers $n = 2$

Write field Eqs using $W(Y) = 1 - \gamma Y^{n-1}$ as:

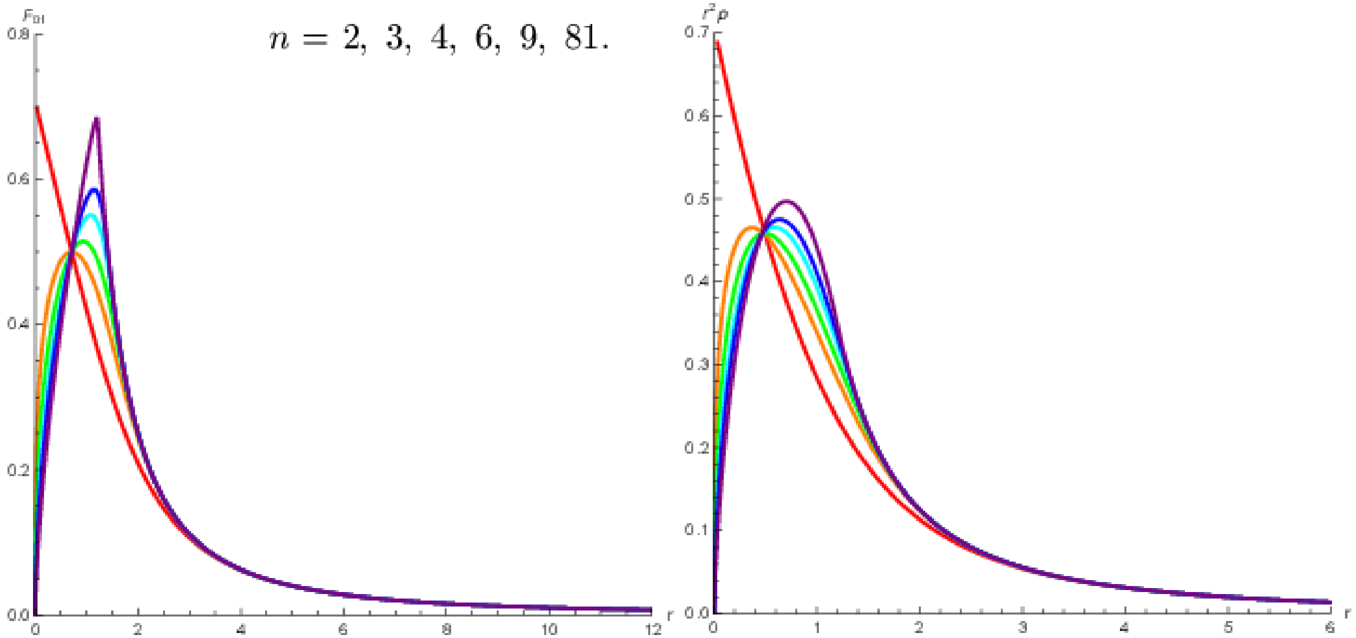
$$\nabla_\mu \left[W(Y) P^{\mu\nu}\right] = J^\nu \quad , \quad P_{\mu\nu} = W(Y) F_{\mu\nu} \ .$$

Now use Y for parametric representation of point charge fields:

$$\begin{aligned} P_{tr} &= \sqrt{-W(Y)Y/2} \quad , \quad F_{tr} = \sqrt{-Y/2W(Y)} \\ r &= \left(\frac{2Q^2}{-YW^3(Y)}\right)^{1/4} \ . \end{aligned}$$

... and energy density:

$$\rho = T^0_0 = -\frac{3n-2}{4n} Z + \frac{n-1}{2n} Y = -\frac{1}{4} Y + \frac{3n-2}{4n} \gamma Y^n$$



Again: "healthy" solutions for: $(-1)^n\gamma > 0$.

Asymptotic behavior: $F_{tr}(r) \simeq Q/r^2$; $P_{tr}(r) \simeq Q/r^2$.

near $r = 0$:

$$F_{tr}(r) \simeq \frac{1}{(2^{n-1}|\gamma|)^{\frac{2}{3n-2}}} \left(\frac{r^2}{Q}\right)^{\frac{n-2}{3n-2}} \left(1 - \frac{1}{(2^{4n-3}|\gamma|)^{\frac{1}{3n-2}}} \left(\frac{r^2}{Q}\right)^{\frac{2(n-1)}{3n-2}}\right)$$

$$P_{tr}(r) \simeq \frac{1}{(2^{n-1}|\gamma|)^{\frac{1}{3n-2}}} \left(\frac{Q}{r^2}\right)^{\frac{n}{3n-2}}$$

Total field energy: $E = 4\pi \int_0^\infty r^2 \rho dr = 4\pi \int_{-\infty}^0 dY \frac{dr}{dY} r^2(Y) \left[-\frac{1}{4}Y + \frac{3n-2}{4n} \gamma Y^n\right]$

Integration: $E(|\gamma|, n, Q) = \frac{8 \cdot 2^{3/4} \pi Q^{3/2}}{3|\gamma|^{1/4(n-1)}} \frac{\Gamma\left(\frac{4n-3}{4(n-1)}\right) \Gamma\left(\frac{5n-6}{4(n-1)}\right)}{\Gamma\left(\frac{9}{4}\right)}$

Electric Black Holes in the Y^n Model

Metric tensor and T_μ^ν:

$$ds^2 = h(r)dt^2 - dr^2/f(r) - r^2 d\Omega^2 \,.$$

$$T_0^0 = T_r^r = -\frac{1}{4}Y + \frac{3n-2}{4n}\gamma Y^n$$

WEC OK

$$T_\theta^\theta = T_\varphi^\varphi = \frac{1}{4}Y + \frac{n-2}{4n}\gamma Y^n$$

SEC NO

$T_t^t = T_r^r \;\Rightarrow\; h(r) = f(r) \;\Rightarrow$ a single Einstein Eq.+ flat space "Maxwell":

$$\begin{aligned} \frac{1}{r^2}\frac{d}{dr}r(1-f) &= \kappa\left(-\frac{1}{4}Y + \frac{3n-2}{4n}\gamma Y^n\right) \\ \Rightarrow \quad \frac{dM}{dr} &= \frac{\kappa r^2}{2}\left(-\frac{1}{4}Y + \frac{3n-2}{4n}\gamma Y^n\right) \end{aligned}$$

Change variables to Y gives:

$$\frac{dM}{dY} = -\frac{\kappa r^2}{2}\frac{dr}{dY}\left(-\frac{1}{4}Y + \frac{3n-2}{4n}\,\gamma Y^n\right)$$

with: $$r = \left(\frac{2Q^2}{-YW^3(Y)}\right)^{1/4}$$

Integration gives the mass function:

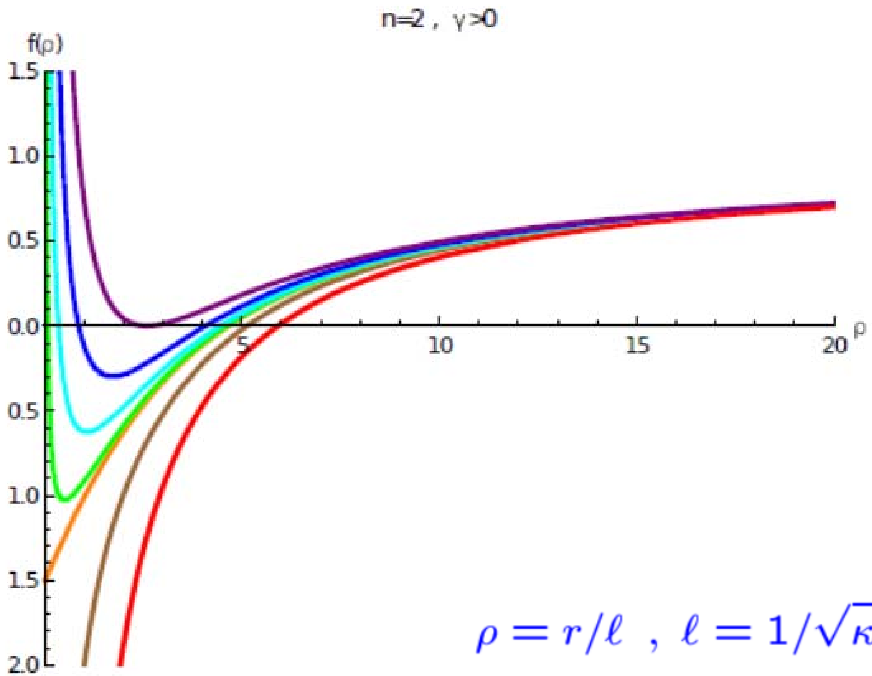
$$M(Y) = \frac{\kappa Q^{3/2}(-Y)^{1/4}}{60\cdot 2^{1/4} n(n-1)\left(1+|\gamma|(-Y)^{n-1}\right)^{9/4}}$$

$$\Big[-32n|\gamma|^2(-Y)^{2(n-1)} + \left(n(27n-101)+10\right)|\gamma|(-Y)^{n-1} + n(17n-49)$$

$$+\frac{32n\left(1+|\gamma|(-Y)^{n-1}\right)^3}{|\gamma|(-Y)^{n-1}} F\left(1, \frac{4n-5}{4(n-1)}, \frac{5n-6}{4(n-1)}, -\frac{1}{|\gamma|(-Y)^{n-1}}\right)\Big] + M_0$$

Parametric representation of $f(r)$:

$$
\begin{cases}
f(Y) = 1 - \dfrac{2M(Y)}{r(Y)} \\[2ex]
r(Y) = \left(\dfrac{2Q^2}{-YW^3(Y)}\right)^{1/4}
\end{cases} .
$$

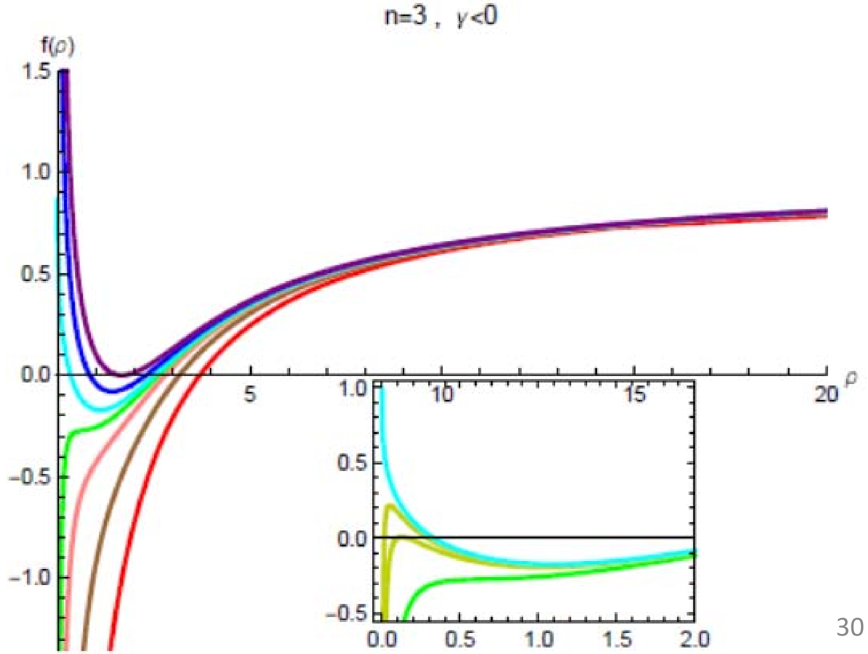


$\rho = r/\ell \;,\; \ell = 1/\sqrt{\kappa \mathfrak{E}^2}$

Gravitational profiles: fixed M, varying Q.

- S-like and RN-like solutions. And more:

- For n=2, solution with finite $f(0)$. It is NOT a regular BH.
- For n=3, a new intermediate family exists with 2 inner horizons



BH Mass-Charge-Horizon Relations

Define also: $y = -Y/\mathfrak{E}^2$ such $\varrho = \left(\frac{2q^2}{y(1+y^{n-1})^3}\right)^{1/4}$

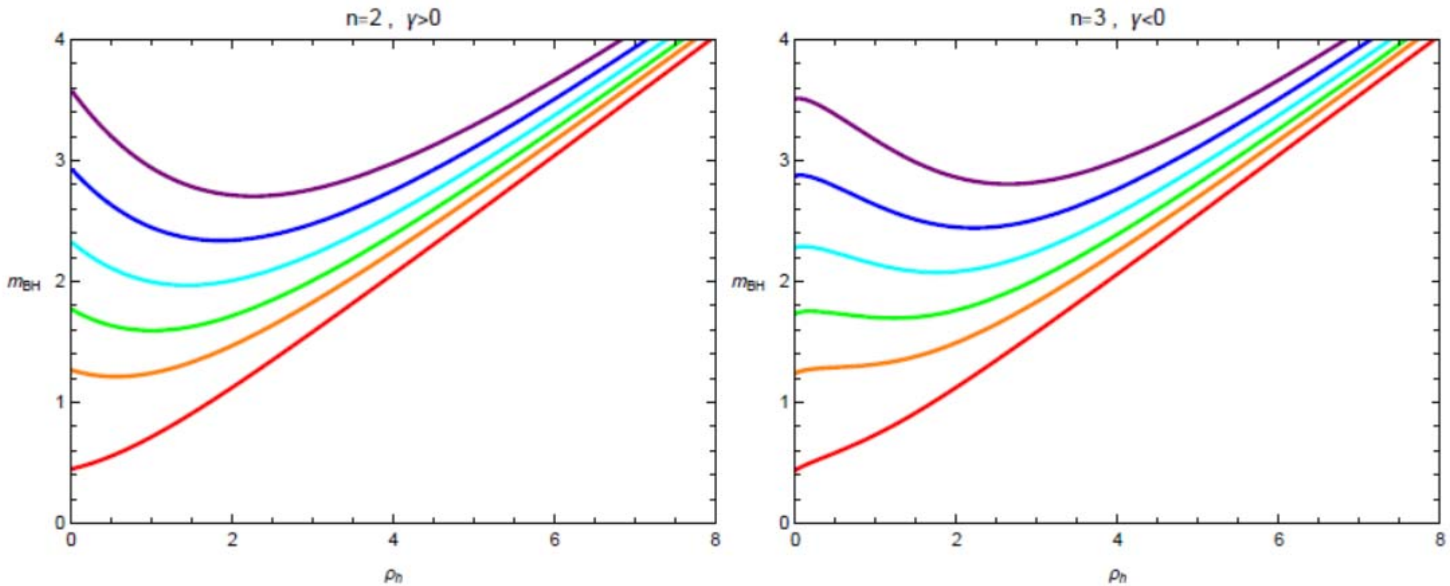
and:

$$m_{field}(y) = \frac{q^{3/2}}{2^{1/4}\cdot 15(n-1)}\left[\frac{n(17n-49)+(10+n(27n-101))y^{n-1}-32ny^{2(n-1)}}{4n\,(1+y^{n-1})^{9/4}}y^{1/4}+\frac{8}{y^{(n-2)/4}}F\left(\frac{1}{4},\frac{n-2}{4(n-1)},\frac{5n-6}{4(n-1)},-\frac{1}{y^{n-1}}\right)\right]$$

such that: $m_{BH} = \overline{m}_{field} + \frac{1}{2}\varrho(y_h) - m_{field}(y_h)$

and m_{BH} given as a function o (q, ϱ_h) $(\varrho(q,y_h), m_{BH}(q,y_h))$

BH Mass-Charge-Horizon plots

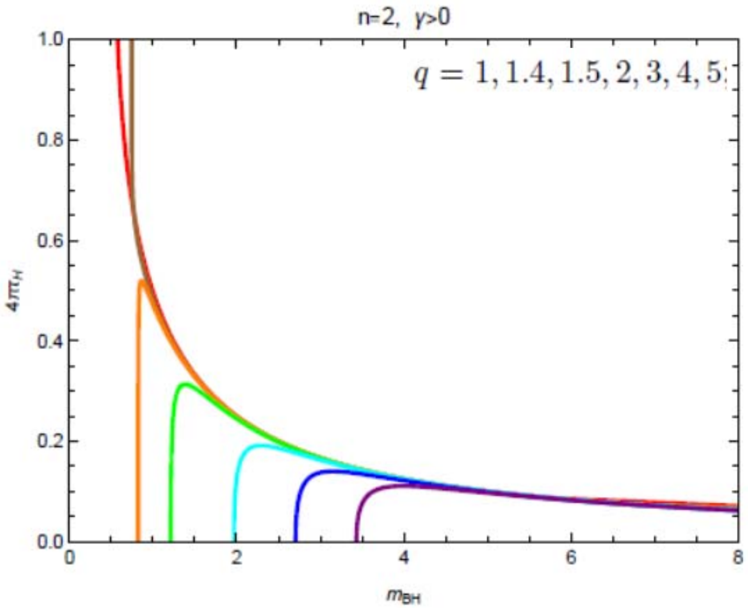


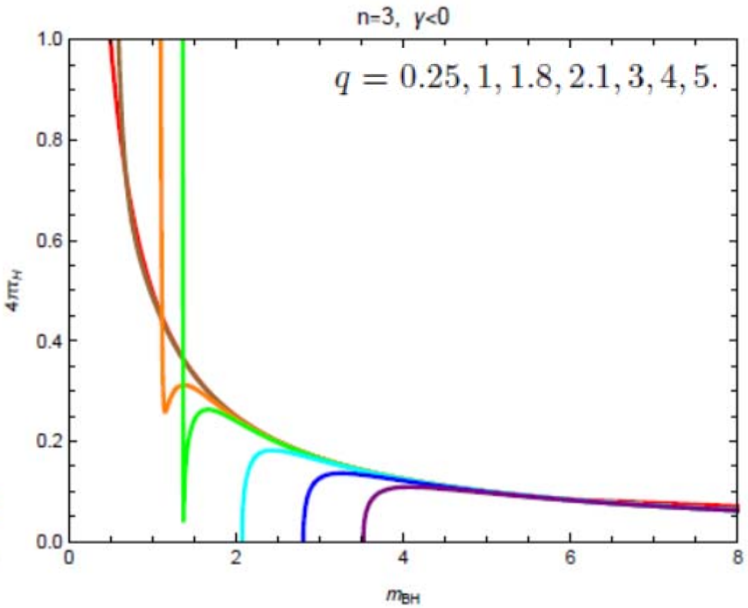
$q = 1,\ 2,\ 2.5,\ 3,\ 3.5,\ 4$

Thermodynamics: temperature and entropy

Spherical symmetry: $T_H = \frac{\hbar}{4\pi k_B} f'(r)|_{r=r_h}$

Dimensionless temperature: $\frac{\ell k_B}{\hbar} T_H = \tau_H = \frac{1}{4\pi}\left[\frac{1}{\varrho_h} - \varrho_h\left(\frac{1}{4}y_h + \frac{3n-2}{4n}\, y_h^n\right)\right]$





Entropy: $S_{BH} = \frac{k_B}{4l_P^2} A_h \Rightarrow s = \frac{S_{BH}}{k_B} \left(\frac{l_P}{\ell}\right)^2 = \pi \varrho_h^2$

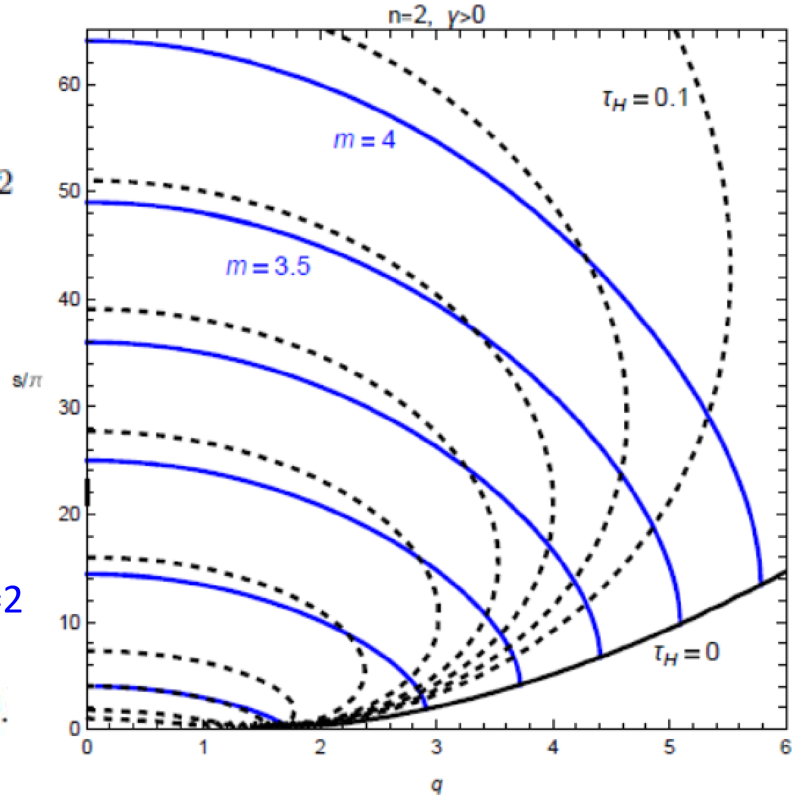
Y^n Model

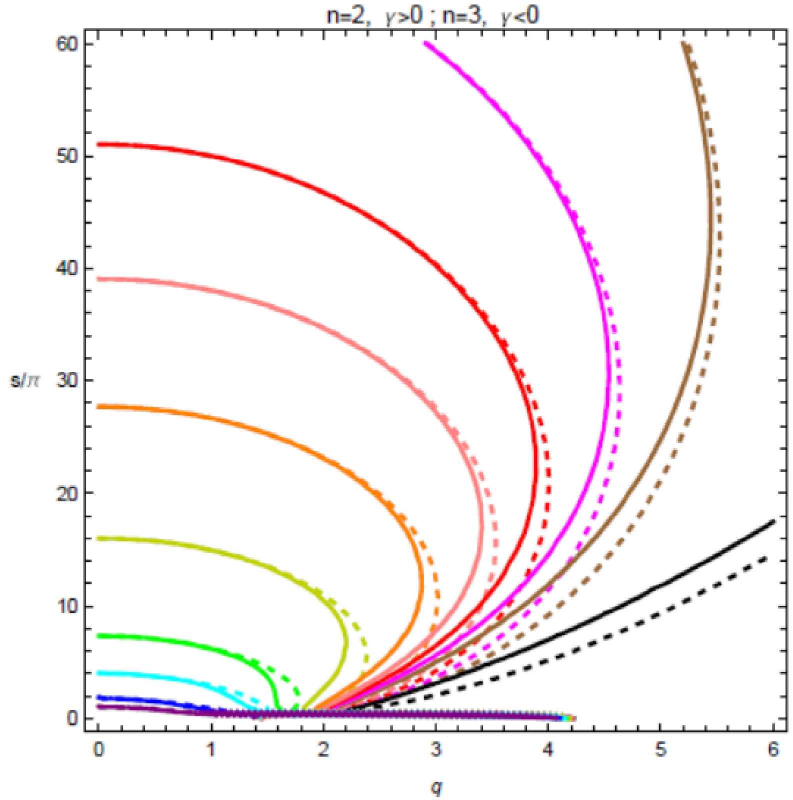
$$s = \pi \left(\frac{2q^2}{y_h (1 + y_h^{n-1})^3} \right)^{1/2}$$

Lines in *s*-*q* plane

Lines of constant mass for n=2 only (left):

$m_{BH} = 1.0, 1.9, 2.5, 3.0, 3.5, 4.0.$





Lines of constant *T* for *n*=2 (solid) and *n*=3 (dashed): $4\pi\tau_H = 0, 0.10, 0.12, 0.14, 0.16, 0.19, 0.25, 0.37, 0.50, 0.75, 1.00.$

What Next?

- Systematic study of PNLED beyond Y^n .
- Causality constraints, ghosts?
- Birefringence and light propagation
- Black hole solutions and their main properties:
  electric, magnetic, dyonic, (non-) rotating,
  mass-charge-horizon relations, temperature, entropy…
  particle trajectories and light rays
- Regular black holes?
- Cosmology – beyond Rassanen&Verbin, Open Journal of Astrophysics, **6** 2023 [2211.15584]


Thank you for your attention

Additional Slides

Maxwell Lagrangian and Field Equations – Second Order

$$\mathcal{L}_{Max} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - J^{\mu} A_{\mu} \quad , \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \ .$$

Dynamical variables (for performing variation): A_ν .

Variation with respect to A_ν gives the inhomogeneous FEqs:

$$\nabla_\mu F^{\mu\nu} = J^\nu \ .$$

The definition $F \equiv dA$ gives the homogeneous (sourceless) Eqs.:

$$\nabla_\mu {}^*F^{\mu\nu} = 0 \quad , \quad {}^*F^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} (\partial_\rho A_\sigma - \partial_\sigma A_\rho)/2\sqrt{-g} \ .$$

BI - Electric Field of point charge

B-I-Coulomb law: $\nabla_i \left(\frac{F^{i0}}{\sqrt{1+X/2b^2}} \right) = J^0$

point charge Q in flat space: $F_{tr} = \mathcal{E}(r) \Rightarrow \frac{r^2\mathcal{E}}{\sqrt{1-\mathcal{E}^2/b^2}} = Q.$

$$\mathcal{E} = \frac{Q}{\sqrt{Q^2/b^2 + r^4}}$$

- Electric field of point particle is finite: $\mathcal{E}(0) = b$
- Total electrostatic energy also finite

BUT: $\vec{\mathcal{E}}(\vec{r})$ still discontinuous at $r = 0$.