Bounds on complex structure moduli for perturbative control

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Perturbative control in string compactifications

In string compactification, features of 4-dimensional theory is explained in terms of the size (Kähler moduli) and shape (complex structure moduli) of 6-dimensional internal manifold.

Dynamics of the moduli can be explicitly studied in the framework of effective supergravity, which is obtained by taking two limits,

1. Large (string frame) volume limit ($\mathcal{V} \to \infty$)

$$\alpha' = \ell_s^2/(2\pi)^2 \ll \mathcal{V}^{1/3}\ell_s^2$$

2. Weak coupling limit : $g_s \to 0$ or equivalently, $\mathrm{Im}(\tau) \to \infty$ ($\tau = C_0 + i\, e^{-\Phi}$)

: asymptotic limit in the field space of the Kähler moduli and the axio-dilaton.

- However, the field values cannot be arbitrarily large as a tower of states descend from UV :

1. As $\mathcal{V} \to \infty$, KK modes become light, 4-dim EFT no linger holds.
2. As $\mathcal{V} \to \infty$ and $g_s \to 0$, string mass scale becomes low:

$$M_s^2 = \frac{1}{\ell_s^2} = \frac{g_s^2}{4\pi\mathcal{V}} M_{\rm Pl}^2$$

: Distance conjecture H. Ooguri, C. Vafa, Nucl. Phys. B**766** (2007) 21 [hep-th/0605264] $m_{\rm tower} \sim M_{\rm Pl} e^{-\text{geodesic distance of modulus}}$

3. In the strict asymptotic regime, the flux is not large enough to stabilize all the complex structure moduli (tadpole conjecture) :

$$\frac{1}{2}\int_{Y_4} G_4 \wedge G_4 \sim \frac{\chi(Y_4)}{24} \lesssim \frac{h^{3,1}}{4}, \qquad \text{but} \qquad \frac{1}{2}\int_{Y_4} G_4 \wedge G_4\Big|_{\text{whole CSM stabilized}} \gtrsim \frac{h^{3,1}}{3}$$

I. Bena, J. Blåbäck, M. Graña, S. Lüst, JHEP **11** (2021) 223 [2010.10519],

see also T. Coudarchet, F. Marchesano, D. Pietro, M. A. Urkiola, JHEP **08** (2023) 016 [2304.04789]

- If we insist on the models in which all the moduli are stabilized, the strict asymptotic regime is not appropriate, but still perturbative control must not be spoiled.

: the value of the Kähler moduli and the axio-dilaton are still sizeable so deep interior of the moduli space is not appropriate.

- Moreover, quantum corrections to the Kähler potential contain the mixing between

Kähler, complex structure moduli, as well as the dilaton.

If the correction to the Kähler potential diverges along the large value of the complex structure moduli, the perturbative control requires that the values of the complex structure moduli cannot be arbitrarily large (cf. tadpole conjecture) but restricted by the values of $\mathcal{V}$ and g_s, hence the KK/string mass scale.

M.-S. Seo, 2504.01268

Analogy : Dilaton value constraints from perturbative control

The 4-loop correction to the world-sheet β-function generates $\mathcal{O}(\alpha'^4)R^4$ term,

$$e^{-2\Phi_4} = e^{-2\Phi}\mathcal{V}_s \to e^{-2\Phi}\left(\mathcal{V}_s + \frac{1}{2}\xi\right)$$

$$= e^{-2\Phi}\left(e^{6\times(\Phi/4)}\mathcal{V}_E + \frac{1}{2}\xi\right) = e^{-\Phi/2}\left(\mathcal{V}_E + \frac{\xi}{2}e^{-\frac{3}{2}\Phi}\right)$$

$$G^{(E)}_{MN} = e^{-\Phi/2}G^{(s)}_{MN}$$

$$\xi = -\frac{\chi(X_6)\zeta(3)}{2(2\pi)^3}$$

Resulting in

$$\boxed{K_{\mathrm{K}} = -2\log\left[\mathcal{V}_E + \frac{\xi}{2g_s^{3/2}}\right]}$$

K. Becker, M. Becker, M. Haak, J. Jouis, JHEP **06** (2002) 060 [hep-th/0204254]

See also D. J. Gross, E. Witten, Nucl. Phys. B**277** (1986) 1

I. Antoniadis, S. Ferrara, R. Minasian, K. S. Narain, Nucl. Phys. B**507** (1997) 571 [hep-th/9707013]

Therefore, the correction to the Kähler potential diverges in the limit $g_s \to 0$ ($\mathrm{Im}(\tau) \to \infty$)

: Perturbative control can be lost in the strict asymptotic regime.

Such a behavior may be a part of more generic term allowed by the SL(2, $\mathbb{Z}$) symmetry of the Kähler potential for the axio-dilaton.

: Axio-dilaton shows a similar behavior to the complex structure modulus of torus

Periods of the holomorphic 1-form $\Omega_1 = Z^0(\alpha - \tau\,\beta)$ are given by

$$Z^0 = \int_A \Omega_1, \qquad Z^0\tau = \int_B \Omega_1$$

Then the Kähler potential can be written as

$$K_\tau = -\log\left[-i\int_{\mathbb{T}^2}\Omega_1\wedge\overline{\Omega}_1\right] = -\log\left[-i|Z^0|^2(\tau-\overline{\tau})\right]$$

which is invariant under SL(2, $\mathbb{Z}$) $\cong$ Sp(2, $\mathbb{Z}$) transformation,

$$\boxed{\tau \to \frac{a\tau+b}{c\tau+d}, \qquad ad-bc=1}$$

or

$$\boxed{\begin{gathered}\begin{pmatrix} Z^0 \\ Z^0\tau \end{pmatrix} \to \begin{pmatrix} d & c \\ b & a \end{pmatrix}\begin{pmatrix} Z^0 \\ Z^0\tau \end{pmatrix} \\ \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\end{gathered}}$$

As SL(2, ℤ) ≅ Sp(2, ℤ) functions, we may consider the Eisenstein series,

$$E_s(\tau) = \sum_{(n,m)\neq(0,0)} \frac{\left(-\frac{i}{2}(\overline{Z}^0(Z^0\tau) - Z^0\overline{(Z^0\tau)})\right)^s}{|nZ^0 + mZ^0\tau|^{2s}} = \sum_{(n,m)\neq(0,0)} \frac{(\mathrm{Im}\tau)^s}{|n+m\tau|^{2s}},$$

the leading term of which in the limit $g_s \to 0$ $(\mathrm{Im}(\tau) \to \infty)$ becomes

$$\sum_{m\in\mathbb{Z}-\{0\}} \frac{(\mathrm{Im}\tau)^s}{n^{2s}} = 2(\mathrm{Im}\tau)^s \sum_{n\in\mathbb{N}} \frac{1}{n^{2s}} = 2(\mathrm{Im}\tau)^s \zeta(2s)$$

$$K_{\mathrm{K}} = -2\log\left[\mathcal{V} + \frac{\xi}{2g_s^{3/2}}\right]$$

Then the correction to the Kähler potential may be regarded as a part of

$$\frac{1}{2}E_{3/2}(\tau,\overline{\tau}) = \frac{1}{2}\sum_{(n,m)\neq(0,0)} \frac{(\mathrm{Im}\tau)^{3/2}}{|n+m\tau|^3}$$

$$= \zeta(3)(\mathrm{Im}\tau)^{3/2} + \frac{\pi^{5/2}}{6(\mathrm{Im}\tau)^{1/2}} + (\mathrm{Im}\tau)^{3/2}\frac{\pi^{1/2}}{2}\sum_{(n,m)\neq(0,0)} \int dy e^{-\pi^2\frac{n^2}{y} + 2\pi i m n \mathrm{Re}\tau - y m^2 \mathrm{Im}\tau^2}.$$

(D-brane instanton effects)

M. B. Green, M. Gutperle, Nucl. Phys. B498 (1997) 195 [hep-th/9701093]

Now, the perturbative control requires $\mathcal{V} \gg 1$

$$K_{\rm K} = -2\log\left[\mathcal{V}_E + \frac{\xi}{2g_s^{3/2}}\right]$$

$$\mathcal{V}_E g_s^{3/2} \gg 1$$

From $M_{\rm KK} \sim \frac{M_s}{\mathcal{V}^{1/6}} \sim \frac{g_s}{\mathcal{V}^{2/3}} M_{\rm Pl}, \sim \frac{M_s}{g_s^{1/4}\mathcal{V}_E^{1/6}} \sim \frac{M_{\rm Pl}}{\mathcal{V}_E^{2/3}}$ this condition reads

$$\mathrm{Im}\tau = \frac{1}{g_s} \ll \mathcal{V}_E^{2/3} \sim \frac{M_{\rm Pl}}{M_{\rm KK}}.$$

Since $\mathrm{Im}(\tau) = e^{-\Phi} = e^{|\Phi|}$, the condition above corresponds to the distance conjecture-like bound :

$$|\Phi| \ll \log\left(\frac{M_{\rm Pl}}{M_{\rm KK}}\right), \qquad \text{or} \qquad M_{\rm KK} \ll M_{\rm Pl}e^{-|\Phi|}.$$

: Since the KK scale must be high enough to decouple from the 4-dimensilnal low energy EFT, the value of Φ cannot be too large.

Moreover, the perturbative control condition can be rewritten in terms of the string mass scale as

$$\frac{1}{g_s} \ll \mathcal{V}_E^{2/3} \sim g_s^{1/3}\left(\frac{M_{\rm Pl}}{M_s}\right)^{4/3}$$

$$\Rightarrow \quad {\rm Im}\tau = \frac{1}{g_s} \ll \frac{M_{\rm Pl}}{M_s}$$

$$|\Phi| \ll \log\left(\frac{M_{\rm Pl}}{M_s}\right), \qquad \text{or} \qquad M_s \ll M_{\rm Pl}e^{-|\Phi|}.$$

The case of the complex structure moduli

- The tree level the Kähler potential respects the $\mathrm{Sp}(2(h^{2,1}+1))$ symmetry

In terms of periods of holomorphic 3-form over 3-cycles (homology basis)

$$Z^I = \int_{A^I} \Omega_3, \qquad \mathcal{F}_I = \int_{B_I} \Omega_3$$

$$z^a = Z^a/Z^I \ (a = 1, \cdots, h^{2,1}(X_6))$$

$$\mathcal{F}_I = \partial \mathcal{F}/\partial Z^I \equiv \partial_I \mathcal{F}$$

$$\boxed{K_{\mathrm{cs}} = -\log\left[i\int_{X_6} \Omega_3 \wedge \overline{\Omega}_3\right] = -\log[i(\overline{Z}^I \mathcal{F}_I - Z^I \overline{\mathcal{F}}_I)]}$$

$$\Pi = (Z^I, \mathcal{F}_I)^T$$

$$\Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

the combination $\overline{Z}^I \mathcal{F}_I - Z^I \overline{\mathcal{F}}_I = \overline{\Pi}^T \cdot \Sigma \cdot \Pi$ is invariant under $\Pi \to M \cdot \Pi$, where M is the real matrix satisfying $M^T \cdot \Sigma \cdot M = \Sigma$.

Then similarly to the case of the dilaton, one may construct the Sp$(2(h^{2,1}+1))$ invariant function

$$E_s(Z^I,\overline{Z}^I) = \sum_{(n_I,m^I)\neq(0,0)} \frac{\left(-\frac{i}{2}\overline{\Pi}^T\cdot\Sigma\cdot\Pi\right)^s}{|L^T\cdot\Pi|^{2s}} = \sum_{(n_I,m^I)\neq(0,0)} \frac{\left(-\frac{i}{2}(\overline{Z}^I\mathcal{F}_I - Z^I\overline{\mathcal{F}}_I)\right)^s}{|n_I Z^I + m^I\mathcal{F}_I|^{2s}}$$

$$= \sum_{(n_I,m^I)\neq(0,0)} \frac{\left(-\frac{i}{2}[2(F-\overline{F})-(z^a-\overline{z}^a)(\partial_a F+\overline{\partial_a F})]\right)^s}{|n_0+n_a z^a+m^0(2F-z^aF_a)+m^aF_a|^{2s}}$$

By considering the lattice

$$\Lambda = \{L=(n_I,m^I)^T | n_I, m^I \in \mathbb{Z}, I=0,\cdots,h^{2,1}\}$$

Such that the sum of $L^T\cdot\Pi = n_I Z^I + m^I\mathcal{F}_I$ over the lattice is Sp$(2(h^{2,1}+1))$ invariant.

$$E_s(Z^I, \overline{Z}^I) = \sum_{(n_I, m^I)\neq(0,0)} \frac{\left(-\frac{i}{2}\overline{\Pi}^T \cdot \Sigma \cdot \Pi\right)^s}{|L^T \cdot \Pi|^{2s}} = \sum_{(n_I, m^I)\neq(0,0)} \frac{\left(-\frac{i}{2}(\overline{Z}^I \mathcal{F}_I - Z^I \overline{\mathcal{F}}_I)\right)^s}{|n_I Z^I + m^I \mathcal{F}_I|^{2s}}$$

$$= \sum_{(n_I, m^I)\neq(0,0)} \frac{\left(-\frac{i}{2}[2(F - \overline{F}) - (z^a - \overline{z}^a)(\partial_a F + \overline{\partial_a F})]\right)^s}{|n_0 + n_a z^a + m^0(2F - z^a F_a) + m^a F_a|^{2s}}$$

Properties :

1. The exponents of the numerator (s) and the denominator ($2s$) are chosen such that it is independent of Z^0
2. In the presence of the complex structure modulus z such that $|\overline{\Pi}^T \cdot \Sigma \cdot \Pi| \to \infty$ in the limit $|z| \to \infty$, it diverges due to the $n_0 \neq 0,\ n_a, m^I = 0$ part of the sum : $E_s \sim |z|^s$

Even if the explicit dependence of the correction to the Kähler potential on the complex structure moduli is in general unknown, we may find some calculable examples in which the correction to the Kähler potential contains the combination above.

- Example : Type IIB orientifold compactification with D3- and D7-branes

The string loop corrections at

$\mathcal{O}(g_s^2\alpha'^2)$ $\mathcal{O}(g_s^2\alpha'^4)$ (string frame)

$\mathcal{O}(g_s\mathcal{V}_E^{-2/3})$ $\mathcal{O}(\mathcal{V}_E^{-4/3})$ (Einstein frame)

$$\delta K_{\mathrm{KK}} = g_s \sum_i c_i^{\mathrm{KK}}(z^a,\bar{z}^a)\frac{t_i^{\perp}}{\mathcal{V}_E}, \qquad \delta K_{\mathrm{W}} = \sum_j c_j^{\mathrm{W}}(z^a,\bar{z}^a)\frac{1}{t_j^{\cap}\mathcal{V}_E}$$

M. Berg, M. Haack, B. Kors, JHEP **11** (2005) 030 [hep-th/0508043], Phys. Rev. Lett **96** (2006) 021601 [hep-th/0508171]

The dependence on the complex structure moduli can be explicitly calculable for the complex structure modulus U of the $\mathbb{T}^2$ factor :

$$E_2(U,\overline{U};A) = \sum_{(n^4,n^5)\neq(0,0)} e^{2\pi i n^p a_p} \frac{U_2^2}{|n^4+Un^5|^4}$$

$$A = Ua_4 - a_5$$

$$\begin{aligned}
&= 2\pi^4\left(\frac{U_2^2}{90} - \frac{1}{3}A_2^2 + \frac{2}{3}\frac{A_2^3}{U_2} - \frac{1}{3}\frac{A_2^4}{U_2^2}\right) \\
&+ \frac{\pi}{2U_2}\left[\mathrm{Li}_3(e^{2\pi i A}) + 2\pi A_2 \mathrm{Li}_3(e^{2\pi i A}) + \mathrm{c.c.}\right] \\
&+ \frac{\pi^2}{U_2}\sum_{m>0}\left[(mU_2 - A_2)\mathrm{Li}_2(e^{2\pi i(mU-A)}) + (mU_2 + A_2)\mathrm{Li}_2(e^{2\pi i(mU+A)})\right) \\
&\qquad + \mathrm{c.c.}\Big] \\
&+ \frac{\pi}{2U_2}\sum_{m>0}\left[\mathrm{Li}_3(e^{2\pi i(mU-A)}) + \mathrm{Li}_3(e^{2\pi i(mU+A)}) + \mathrm{c.c.}\right],
\end{aligned}$$

Except for the phase in the sum, it has the same structure as the $\mathrm{Sp}(2(h^{2,1}+1))$ invariant function $E_2(U,\overline{U})$.

The phase depends on the gauge fields which breaks the $\mathrm{Sp}(2(h^{2,1}+1))$ symmetry, but suppressed in the large field limit :

$a_4 = A_2/U_2$ $\longrightarrow$ the phase $2\pi i(n^4a_4 + n^5a_5)$ in the $n_4 \neq 0, n_5 = 0$ part of the sum $\to 0$ as $U_2 \to \infty$

- Perturbative control : since

$$-2\log \mathcal{V}_E + \delta K \simeq -2\log\left(\mathcal{V}_E - \frac{\mathcal{V}_E}{2}\delta K\right)$$

we require that $\delta K \ll 1$

$$\delta K_{\rm KK} \sim g_s E_s(z^a, \bar{z}^a; A)\frac{t_i^\perp}{\mathcal{V}_E}$$

the condition $\delta K_{\rm KK} \ll 1$ becomes

$$E_s(z^a, \bar{z}^a; A) \sim |z|^s \ll \frac{1}{g_s}\frac{\mathcal{V}_E}{t_i^\perp}$$

Since $t_i^\perp \lesssim \mathcal{V}_E^{1/3}$,

$$|z|^s \ll \frac{\mathcal{V}_E^{2/3}}{g_s} \sim \frac{1}{g_s}\frac{M_{\rm Pl}}{M_{\rm KK}} \ll \left(\frac{M_{\rm Pl}}{M_{\rm KK}}\right)^2$$

Ignoring the potential, and if the kinetic term is given by $\alpha^2|\partial_\mu z|^2/({\rm Im} z)^2$, the geodesic distance of Im(z) ~ $\varphi = \alpha\log(|{\rm Im} z|)$ then the above bound becomes

$$\varphi \ll 2\frac{\alpha}{s}\log\left(\frac{M_{\rm Pl}}{M_{\rm KK}}\right), \qquad \text{or} \qquad M_{\rm KK} \ll M_{\rm Pl}e^{-\frac{s}{2\alpha}\varphi}.$$

Moreover,

$$|z|^s \ll \frac{\mathcal{V}_E^{2/3}}{g_s} \sim \frac{1}{g_s^{2/3}}\left(\frac{M_{\rm Pl}}{M_s}\right)^{4/3} \ll \left(\frac{M_{\rm Pl}}{M_s}\right)^2$$

$$\varphi \ll 2\frac{\alpha}{s}\log\left(\frac{M_{\rm Pl}}{M_s}\right), \qquad \text{or} \qquad M_s \ll M_{\rm Pl}e^{-\frac{s}{2\alpha}\varphi}$$

Meanwhile,

$$\delta K_{\rm W}/\delta K_{\rm KK} \sim 1/(g_s\mathcal{V}_E^{2/3})$$

So far as the perturbative control $\mathcal{V} \gg 1$ is automatically achieved,

$$\delta K_{KK} \gg \delta K_W$$

Summary

- The models like the KKLT or the large volume scenario require that all the moduli are stabilized and perturbative control is achieved.
- Some corrections to the Kähler potential tend to diverge in the large field limit of the dilaton and the complex structure moduli, spoiling the perturbative control in the strict asymptotic regime.
- Such a behavior reflects the symmetry of the Kähler potential, SL(2, $\mathbb{Z}$) for the dilaton and Sp($2(h^{2,1}+1)$) for the complex structure moduli.
- For the consistency with the perturbative control, the values of the dilaton and complex structure moduli are bounded from above, and the bounds are determined by the volume of the internal manifold or the string coupling constant, therefore the tower scales like the KK and the string mass scale.
- From this, we obtain the distance conjecture like bound for the dilaton and the complex structure moduli.