Birational Transformations and Mass Deformations of 2d SUSY gauge theories

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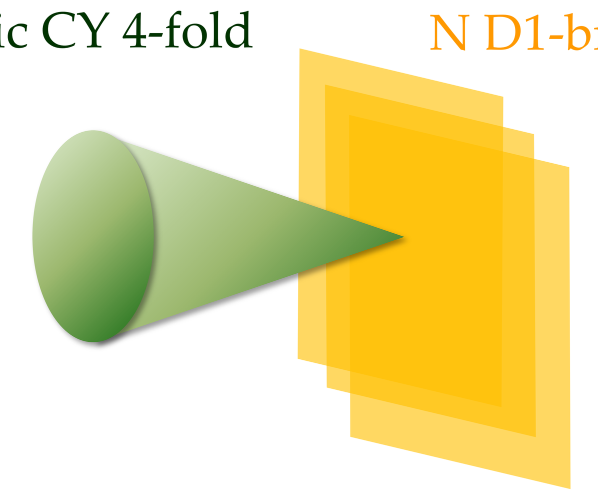
Brane Brick Models : brane construction

- World-volume theory of D1-brane whose transverse geometry is a toric Calabi-Yau 4-fold.
- We call its T-dual Type IIA brane construction as a 'brane brick'



T-dual

NS5 branes wrapping a holomorphic cycle

	0	1	2	3	4	5	6	7	8	9
NS5's	×	×	———			Σ	———		·	·
D4	×	×	×	·	×	·	×	·	·	·

holomorphic surface $\Sigma : P(x,y,z) = 0$

$(2,3) \equiv x$, $(4,5) \equiv y$, $(6,7) \equiv z$

$\{\arg(x), \arg(y), \arg(z)\}$ constitutes a 3 torus

Toric Diagram | Quiver Gauge Theory | Brane Brick

Fig. 1. Brane setup for brane brick models

Birational Transformations ⇄ Mass-Deformation of Gauge Theory

Toric Diagram	Gauge Theory	Brane Brick
combinations of connected edges	(0,2) chiral multiplets	orientable faces
	(0,2) Fermi multiplets	unorientable faces
volume	(0,2) vector multiplets	bricks
	J- and E-terms	edges
vertices	**GLSM fields**	**brick matchings**
edges	phase boundary	

Table 1. Dictionary between gauge theory and combinatorial tools

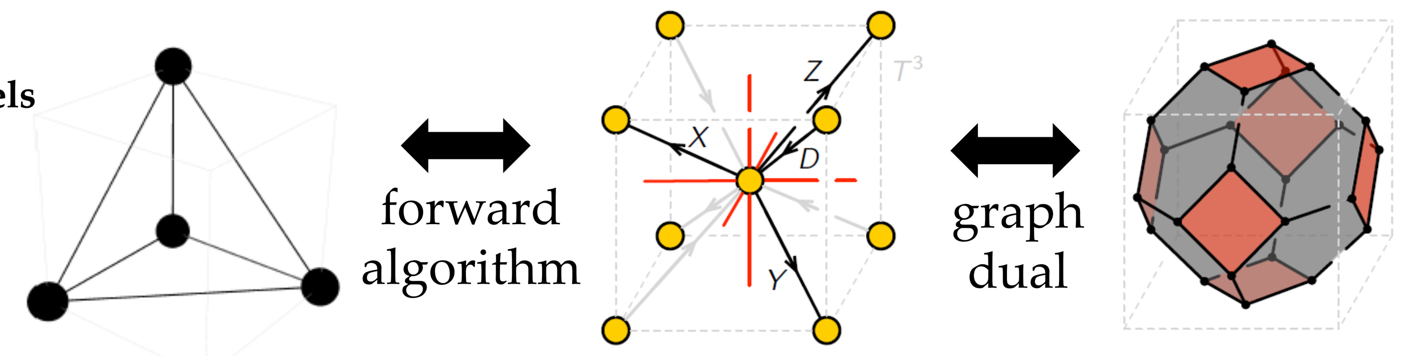


Fig. 2. Toric diagram (left), Periodic quiver (middle) and brane brick (right)

KEY : Mass Deformation of Brane Brick models corresponds to the birational transformation on Newton polynomial of probed CY 4-folds.

Mass deformations of *(0,2)* quiver gauge theories

- Holomorphic interaction of 2d *(0,2)* gauge theories is given by J- and E-terms, each of which is associated with each Fermi multiplet.

$$\Lambda = \lambda_- - \theta^+ G - i\theta^+\bar{\theta}^+ D_+\lambda_- - \bar{\theta}^+ E\,, \quad \bar{\mathcal{D}}\Lambda = E(\Phi_i)$$

$$L_J = -\int d^2x d\theta^+ \sum_a (\Lambda_a\, J_a(\Phi_i)|_{\bar{\theta}^+=0}) - h.c.$$

- J- and E-terms that are linear in chiral multiplet turns on the mass
  - ➔ A chiral-Fermi pair in the same (or opposite) gauge representation can trigger mass deformations.
  - ➔ Massive fields are replaced by binomials of surviving chirals

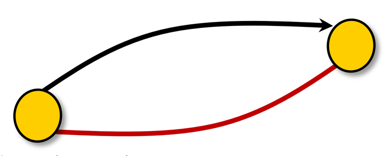


Fig 5. (0,2) Chiral-Fermi pair in a quiver *Q*

$$(\Lambda_{ij}\,, X_{ij}) \in Q : \quad J'_{ji} = J_{ji}\,,\; E'_{ij} = E_{ij} \pm m X_{ij}$$

$$(\overline{\Lambda_{ij}}\,, X_{ji}) \in Q : \quad J'_{ji} = J_{ji} \pm m X_{ji}\,,\; E'_{ij} = E_{ij}$$

$$X_{ij} = \mp\frac{1}{m}E_{ij}$$

$$X_{ji} = \mp\frac{1}{m}J_{ji}$$

Example : Mass deformation of Brane Brick Models

For an example of mass deformation of brane brick models, we consider $\mathcal{C}_{++}$ theory of which quiver is constructed by *3d printing* of *4d N=1* conifold theory. RG flow is triggered by the deformation colored in blue.

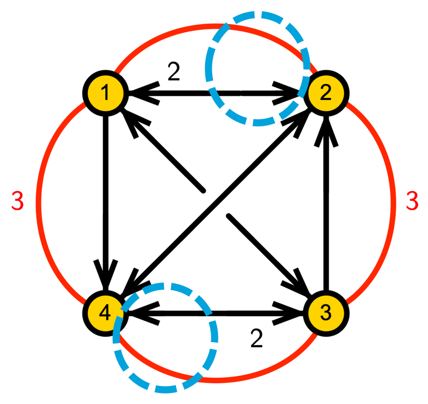


Fig 6. Quiver of C_{++} theory

	J	E
Λ_{12} :	$X_{21}\cdot Y_{12}\cdot Y_{21} - Y_{21}\cdot Y_{12}\cdot X_{21}$	$-Y_{12} + Z_{13}\cdot X_{32} - X_{14}\cdot Z_{42}$
Λ_{34} :	$X_{43}\cdot Y_{34}\cdot Y_{43} - Y_{43}\cdot Y_{34}\cdot X_{43}$	$Y_{34} + Z_{31}\cdot X_{14} - X_{32}\cdot Z_{24}$
Λ_{14} :	$Y_{43}\cdot X_{32}\cdot X_{21} - X_{43}\cdot X_{32}\cdot Y_{21}$	$Z_{13}\cdot Y_{34} - Y_{12}\cdot Z_{24}$
Λ_{32} :	$Y_{21}\cdot X_{14}\cdot X_{43} - X_{21}\cdot X_{14}\cdot Y_{43}$	$Z_{31}\cdot Y_{12} - Y_{34}\cdot Z_{42}$
Λ^1_{23} :	$Y_{34}\cdot Y_{43}\cdot X_{32} - X_{32}\cdot Y_{21}\cdot Y_{12}$	$Z_{24}\cdot X_{43} - X_{21}\cdot Z_{13}$
Λ^2_{23} :	$X_{32}\cdot X_{21}\cdot Y_{12} - Y_{34}\cdot X_{43}\cdot X_{32}$	$Z_{24}\cdot Y_{43} - Y_{21}\cdot Z_{13}$
Λ^1_{41} :	$Y_{12}\cdot Y_{21}\cdot X_{14} - X_{14}\cdot Y_{43}\cdot Y_{34}$	$Z_{42}\cdot X_{21} - X_{43}\cdot Z_{31}$
Λ^2_{41} :	$X_{14}\cdot X_{43}\cdot Y_{34} - Y_{12}\cdot X_{21}\cdot X_{14}$	$Z_{42}\cdot Y_{21} - Y_{43}\cdot Z_{31}$

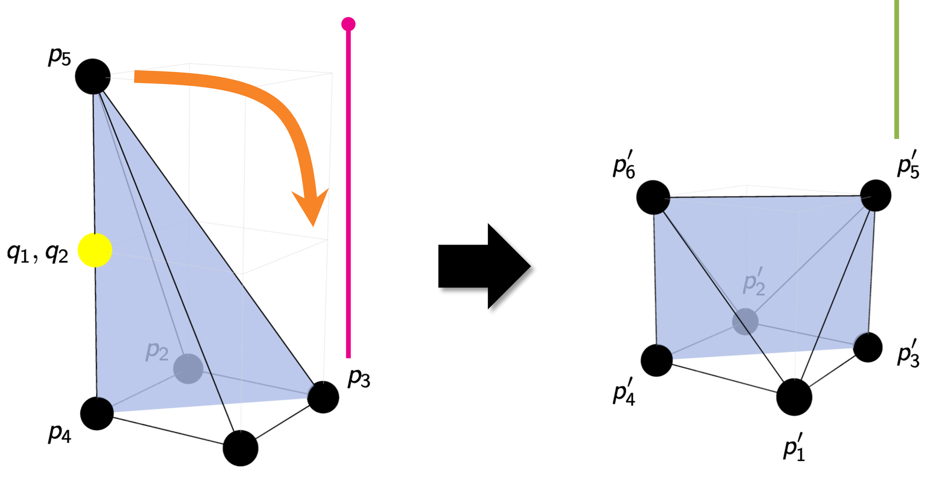
The integration-out of massive fields induce the following substitution of chiral fields in the J- and E-terms.

$$Y_{12} = Z_{13}\cdot X_{32} - X_{14}\cdot Z_{42}$$

$$Y_{34} = -Z_{31}\cdot X_{14} + X_{32}\cdot Z_{24}$$

The resulting quiver gauge theory is given by H_4 theory. The toric diagrams of two gauge theories are described below. You could check

- the vertex in move and where it moves
- the massive vertex which contains chiral fields to become massive



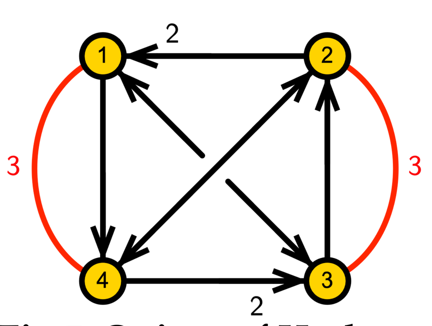


Fig 7. Quiver of H_4 theory

Fig. 8. Move of external vertices of toric diagram under 2d mass deformation

Birational transformation as Algebraic mutation

- Newton polynomial from toric diagram

$$P(x,y,z) = \sum_{\mathbf{v}\in\Delta} c_{\mathbf{v}}\mathbf{x}^{\mathbf{v}} = \frac{1}{y}(1+2z+z^2) + \frac{x}{y} + x + 1$$

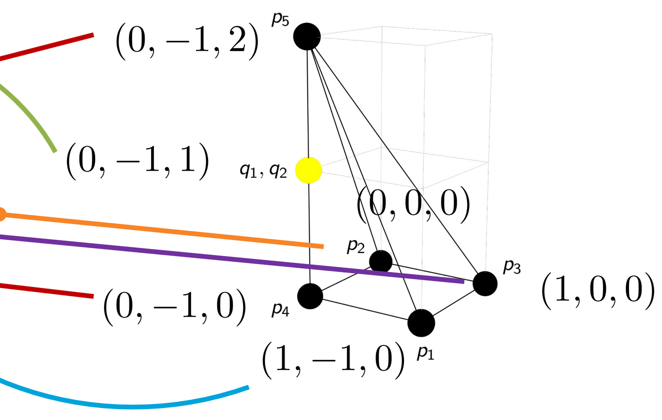


Fig 3. Mapping between terms in Newton polynomial and vertices of toric diagram

- Birational transformation and transform of Newton polynomial

1. SL(3, Z) rotation

$$x \mapsto x^{M_{11}}y^{M_{12}}z^{M_{13}}\,, y \mapsto x^{M_{21}}y^{M_{22}}z^{M_{23}}\,, z \mapsto x^{M_{31}}y^{M_{32}}z^{M_{33}}$$

$$M = (M_{ij}) \in \mathrm{SL}(3,\mathbf{Z})$$

2. rational transformation

$$x \mapsto x\,, \quad y \mapsto y\,, \quad z \mapsto A(x,y)z$$

3. SL (3,Z) rotation

$$x \mapsto x^{N_{11}}y^{N_{12}}z^{N_{13}}\,, y \mapsto x^{N_{21}}y^{N_{22}}z^{N_{23}}\,, z \mapsto x^{N_{31}}y^{N_{32}}z^{N_{33}}$$

$$N = (N_{ij}) \in \mathrm{SL}(3,\mathbf{Z})$$

- Birational transform of Newton polynomial for $\mathcal{C}_{++}$ theory

- step 1. $M = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ $\quad P(x,y,z) = (1+2x+x^2)\,\frac{1}{z} + (1+y) + yz$

- step 2. $A(x,y) = (1+x)$ $\quad P(x,y,z) = (1+x)\,\frac{1}{z} + (1+y) + (y+xy)z$

- step 3. $N = M^{-1}$ $\quad P^{\vee}(x,y,z) = \frac{1}{y}\,(1+z) + 1 + \frac{x}{y} + (1+z)x$

Birational transformation as Combinatorial mutation

- Height assignment and decomposition of toric diagram

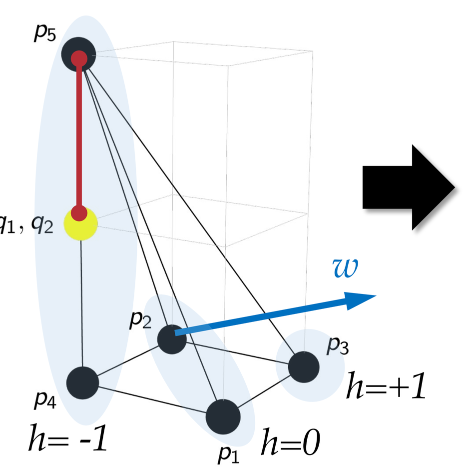
$w = (1,1,0)$

$F = \{(0,0,0)\,,(0,0,1)\}$

$G_{+1} = \{(1,0,0)\}$

$G_0 = \{(0,0,0)\,,(-1,-1,0)\}$

$G_{-1} + F = \{(0,-1,0)\,,(0,-1,1)\} + \{(0,0,0)\,,(0,0,1)\}$



$H_{+1} = G_{+1} + F = \{(1,0,0)\} + \{(0,0,0)\,,(0,0,1)\}$

$H_0 = G_0 = \{(0,0,0)\,,(-1,-1,0)\}$

$H_{-1} = G{-1} = \{(0,-1,0)\,,(0,-1,1)\}$

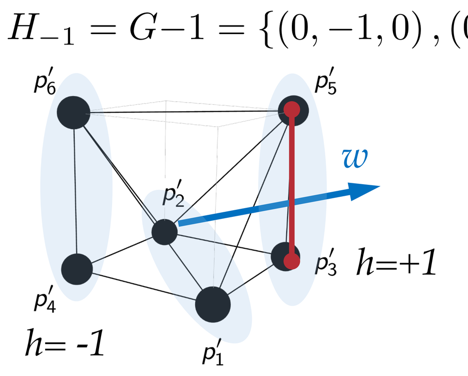


Fig 4. Mutation : moving a factor *F* in underground layers *h < 0* to overground layers *h > 0*

Reference

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[2] D. Ghim, M. Kho and R.-K. Seong, *Combinatorial and algebraic mutations of toric Fano 3-folds and mass deformations of 2d (0,2) quiver gauge theories*, Phys. Rev. D **110** (2024) 8, 08600 [arXiv:2407.19924].

[3] D. Ghim, M. Kho and R.-K. Seong, *Birational Transformations and 2d (0,2) Quiver Gauge Theories beyond Toric Fano 3-folds*, JHEP **06** (2025) 032 [arXiv:2502.08741].

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