Deriving Weyl Double Copies with Sources

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Outline

- Introduction: Gravity and Gauge Theories
- The Classical Double Copy
- Finding a Classical Double Copy for Einstein Maxwell Gravity
- Outlook

Introduction

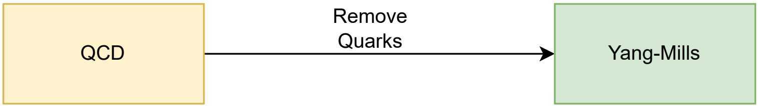
- An open problem in theoretical physics is the relationship between theories of *gravity* and theories of *particle physics*.
- The last 50 years have seen various attempts to resolve this problem: String Theory, Quantum Loop Gravity, etc ...
- Though none of these attempts have completely solidified the relationship between gravity and particle physics, we have learned a great deal from these investigations.

Particle Physics and Gauge Theories

The standard model of particle physics is described by *gauge theories*; quantum field theories which incorporate *local symmetries* defined at every point in spacetime.

The most relevant of these theories for us is the theory of quarks, gluons and the strong nuclear force: **Quantum Chromodynamics**.

We can also remove the quarks from QCD to receive a non-abelian gauge theory known as Yang-Mills theory.

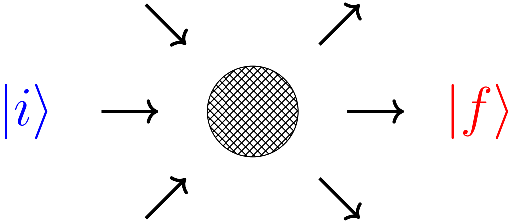


Gravity

- Currently, the best theory we have to describe gravity is *general relativity*.
- Recent experimental results on black holes and gravitational waves has increased interest in the subject.
- Gravity is the only fundamental force that we cannot comfortably combine with quantum mechanics.
- Gravity is an example of a **non-renormalisable** theory.

Quantum Field Theory and Interactions

Scattering amplitudes in quantum field theory are quantities related to the probability for an interaction (also known as a *scattering process*) between particles to happen.



- Number of (external legs) points $\rightarrow$ Number of arrows going in **and** out.
- Number of Loops $\rightarrow$ Number of "self interactions".

Developing a Double Copy for Scattering amplitudes

We can write down (*2203.13013*) for a Yang-Mills like *non-abelian gauge theory*, a scattering amplitude with L loops and m points (in D dimensions).

$$\mathcal{A}_m^{(L)} = i^{L-1} \underbrace{g^{m-2+2L}}_{\text{Coupling Constant}} \sum_i \int \prod_{l=1}^{L} \frac{d^D \ell_l}{(2\pi)^D} \frac{1}{S_i} \frac{c_i n_i}{\prod_{i_j} d_{i_j}} \,, \tag{1}$$

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BCJ Duality

It turns out the kinematic numerators (n_i) can be made to obey (mirroring the colour factors c_i):

$$c_i + c_j + c_k = 0 \tag{2}$$

$$n_i + n_j + n_k = 0 \tag{3}$$

This has become known as BCJ Duality (*0805.3993, 1004.0476*). Importantly, equation (3) implies the existance of structures known as ***kinematic algebras***!

This allows us to write down relevant scattering amplitudes in quantum gravity.

Developing a Double Copy for Scattering amplitudes

We can promote gravity to a quantum field theory (ignoring issues with renormalizability), to write down a scattering amplitude with L loops and m points:

$$\mathcal{M}_m^{(L)} = i^{L-1}\left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_i \int \prod_{l=1}^{L} \frac{d^D \ell_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{\prod_{i_j} d_{i_j}}\,, \qquad (4)$$

For:

$$\kappa = \sqrt{32\pi G_N}$$

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Comparing both expressions:

$$\mathcal{A}_m^{(L)} = i^{L-1} g^{m-2+2L} \sum_i \int \prod_{l=1}^{L} \frac{d^D \ell_l}{(2\pi)^D} \frac{1}{S_i} \frac{c_i n_i}{\prod_{i_j} d_{i_j}} \,,$$

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Double Copy for scattering amplitudes

We can turn the gauge theory amplitude into the gravity amplitude via the following replacements:

$$\mathcal{M}_m^{(L)} = \mathcal{A}_m^{(L)}\Big|_{\substack{c_i \to \tilde{n}_i \\ g \to \kappa/2}} = i^{L-1}\left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_i \int \prod_{l=1}^{L} \frac{d^D \ell_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{\prod_{i_j} d_{i_j}}\,, \tag{5}$$

We then make the choice $n_i \equiv \tilde{n}_i$.

This is known as the **Double Copy**.

The Classical Double Copy

- The double copy can be extended to produce dualities between **exact** classical solutions (*classical yang-mills and general relativity*).
- The first instance of the classical double copy is the so-called Kerr-Schild Double Copy (*1410.0239, 1606.04724*).
- This relates gauge fields in classical yang mills (A_μ^a) with *exact linearized vacuum solutions* to the Einstein equations ($h_{\mu\nu}$). Where:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{6}$$

The Weyl Double Copy

- The Kerr Schild Double Copy **only** applies to Kerr Schild Solutions in General Relativity.
- We can extend this relationship to work for broad class of GR exact solutions, in a framework known as the **Weyl double copy** (*1810.08183*).
- The ***Weyl*** double copy relates spinor quantities in General Relativity to their counterparts in Electromagnetism.
- This relationship has only been seen to work for *Petrov type D and N vacuum Spacetimes*.
- We can **convert** tensorial objects into spinors using the *Infeld-van der Waerden symbols* $\sigma^{\mu}_{A\dot{A}}$.

2-Spinor Formalism

- Normally, field theorists tend to work in the language of *tensors*.
- However, another formalism exists in the language of **2 component** (or *Weyl*)-spinors λ_A and their complex conjugates $\tilde{\lambda}_{\dot{A}}$.
- We can also define their duals λ^A and $\tilde{\lambda}^{\dot{A}}$ respectively.
- We raise and lower spinorial indices using the 2-dimension Levi-Civita symbol (ε_{AB}, ε^{AB}, $\varepsilon_{\dot{A}\dot{B}}$, $\varepsilon^{\dot{A}\dot{B}}$).
- We can also build multi-indexed objects under is formalism (such as $F_{A\dot{A}B\dot{B}}$).

2-Spinor Formalism

For our purposes, we define:

$$\sigma^{\mu}_{A\dot{A}} = (I_{2\times 2}, \sigma_z, \sigma_x, i\sigma_y) \tag{7}$$

Where $\sigma_x, \sigma_y, \sigma_z$ are the standard Pauli matrices and $I_{2\times 2}$ is the 2 dimensional identity matrix.

The advantage of using spinors is that we can see things that otherwise would be ***hidden*** in the tensor language.

Constructing The Weyl Double Copy

Converting the Electromagnetic Field Strength tensor $F_{\mu\nu}$ into spinors we receive:

$$
\begin{aligned}
F_{A\dot{A}B\dot{B}} &= \sigma^{\mu}_{A\dot{A}}\sigma^{\nu}_{B\dot{B}}F_{\mu\nu} \\
&= \phi_{AB}\varepsilon_{\dot{A}\dot{B}} + \tilde{\phi}_{\dot{A}\dot{B}}\varepsilon_{AB}
\end{aligned} \tag{8}
$$

Where ϕ_{AB} is the Maxwell Spinor.

We can obtain ϕ_{AB} directly by:

$$
\phi_{AB} = \frac{1}{2}F_{AB\dot{C}}{}^{\dot{C}} \tag{9}
$$

Constructing The Weyl Double Copy

For *vacuum* solutions ($R_{\mu\nu} = 0$), the Riemann tensor reduces to the **Weyl** tensor:

$$R_{\mu\nu\rho\lambda} = W_{\mu\nu\rho\lambda} \tag{10}$$

Converting $W_{\mu\nu\rho\lambda}$ into a spinor we receive:

$$\begin{aligned} W_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} &= \sigma^{\mu}_{A\dot{A}}\sigma^{\nu}_{B\dot{B}}\sigma^{\rho}_{C\dot{C}}\sigma^{\lambda}_{D\dot{D}}W_{\mu\nu\rho\lambda} \\ &= \Psi_{ABCD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}} + \tilde{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}\varepsilon_{AB}\varepsilon_{CD} \end{aligned} \tag{11}$$

We can obtain Ψ_{ABCD} from $W_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}}$ by:

$$\Psi_{ABCD} = \frac{1}{4}W_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}}\varepsilon^{\dot{A}\dot{B}}\varepsilon^{\dot{C}\dot{D}} \tag{12}$$

Constructing The Weyl Double Copy

Thus for **certain** vacuum solutions in general relativity yields the result:

$$\Psi_{ABCD} = \frac{\phi_{(AB}\phi_{CD)}}{S} \tag{13}$$

Where S is some scalar which is a solution to the equation of motion of Biadjoint Scalar Theory.

This relationship is known as the **Weyl Double Copy**.

The Weyl Double Copy works for **Exact** "Linear" solutions to the Einstein equations.

Einstein-Maxwell Gravity

- Einstein-Maxwell gravity is a theory which allows for the matter inside our spacetime to be electromagnetically charged.
- For **non-vacuum solutions**, the theory admits a Riemann Curvature Spinor $R_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}}$, given in the form:

$$R_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} = \Psi_{ABCD}\epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}} + \bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}\epsilon_{AB}\epsilon_{CD} + \Phi_{AB\dot{C}\dot{D}}\epsilon_{\dot{A}\dot{B}}\epsilon_{CD} + \bar{\Phi}_{\dot{A}\dot{B}CD}\epsilon_{AB}\epsilon_{\dot{C}\dot{D}} + 2\Lambda\left(\epsilon_{AC}\epsilon_{BD}\epsilon_{\dot{A}\dot{C}}\epsilon_{\dot{B}\dot{D}} - \epsilon_{AD}\epsilon_{BC}\epsilon_{\dot{A}\dot{D}}\epsilon_{\dot{B}\dot{C}}\right) \quad (14)$$

Weyl Double Copy for Einstein-Maxwell Gravity

- It was conjectured in (*2210.16339, 2110.02293*) that one could write down a Weyl Double Copy like formula for non-vacuum Einstein-Maxwell gravity:

$$\Psi_{ABCD} = \sum_{n=1}^{m} \frac{1}{\phi^{(n)}} \Phi^{(n)}_{(AB} \Phi^{(n)}_{CD)}, \qquad (15)$$

- The right-hand side contains a **"tower"** of electromagnetic spinors $\Phi^{(n)}_{AB}$ and scalar fields $\phi^{(n)}$.
- The $n = 1$ term corresponds to the traditional **vacuum Weyl double copy**.

Weyl Double Copy for Einstein-Maxwell Gravity

- **Each** $n > 1$ electromagnetic spinor corresponds to a field strength $F_{\mu\nu}^{(n)}$ satisfying a non-vacuum Maxwell equation

$$\partial^\mu F_{\mu\nu}^{(n)} = j_\mu^{(n)}. \tag{16}$$

- **Each** scalar field $n > 1$ $\phi^{(n)}$ satisfies a non-vacuum equation for some charge density $\rho_S^{(n)}$:

$$\partial^2 \phi^{(n)} = \rho_S^{(n)} \tag{17}$$

- How can we systematically prove these results?

The Double Copy: Momentum to Position space

- The Double Copy for classical (*position*) solutions and for scattering amplitudes (*momentum*) are related to each other by special integral transforms. (*2208.08548*)
- Using this relationship, we can systematically derive Weyl Double Copy formulas from scattering amplitudes in a **three-stage process**.
- In particular, the above discussion **must** be carried out in $(2,2)$ signature. (*n.b It is always possible to analytically continue to any choice of signature after one has obtained the final classical fields!*)

Systematically deriving the Weyl Double Copy

(i) Express spinorial solutions of (linearised) classical equations for scalar, gauge and gravity theory as **inverse Fourier transforms** of **momentum-space solutions**.

(ii) One may **transform** the momentum integral to *certain spinor variables*, and carry out some of the integrals to yield an intermediate *twistor-space* representation of each classical solution. This step reproduces the **Twistor Double Copy** (*2012.02479, 2103.16441*).

(iii) The remaining integral can be carried out to yield *position-space* representations of each **classical** solution, with their *principal spinors* clearly identified. These solutions are then found to obey the Weyl double copy.

Stage 1: Inverse Fourier Transforms

- Another way to obtain these inverse Fourier transforms of momentum-space solutions is to consider the sources of these fields (*2112.05111*).
- For our purposes, we choose *spherically symmetric* and *static* sources/currents for each field type (scalar, vector, tensor) :

$$\rho_S = \rho(x), \quad j^\mu(x) = \rho(x) u^\mu, \quad T^{\mu\nu}(x) = \rho(x) u^\mu u^\nu, \tag{18}$$

- Which in momentum space is given by:

$$\tilde{\rho}_S(k) = \delta(u \cdot k)\mathcal{J}(\vec{k}), \quad \tilde{j}^\mu(k) = \delta(u \cdot k)\mathcal{J}(\vec{k}) u^\mu,$$
$$\tilde{T}^{\mu\nu}(k) = \delta(u \cdot k)\mathcal{J}(\vec{k}) u^\mu u^\nu, \tag{19}$$

Stage 1: Inverse Fourier Transforms

The **Scalar Field** pertaining to S:

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} \delta(u \cdot k) \frac{\mathcal{J}(\vec{k})}{k^2} e^{-ik \cdot x}. \tag{20}$$

The **Electromagnetic Field Strength** Tensor:

$$F^{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4} k^{[\mu} u^{\nu]} \delta(u \cdot k) \frac{\mathcal{J}(\vec{k})}{k^2} e^{-ik \cdot x}. \tag{21}$$

The **Riemann Curvature Tensor**:

$$R^{\mu\nu\rho\sigma}(x) = \kappa \int \frac{d^4k}{(2\pi)^4} \left[U^{\rho[\mu} k^{\nu]} k^{\sigma} - U^{\sigma[\mu} k^{\nu]} k^{\rho} \right] \frac{\mathcal{J}(\vec{k})}{k^2} e^{-ik \cdot x}, \tag{22}$$

Stage 1: Inverse Fourier Transform

- Where u^μ is its 4-velocity:

$$u^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (23)$$

- $\rho(x)$ represents the appropriate **charge** density, whereas it will be an **energy** density in gravity.
- k is the 4-momentum $k^\mu = (k^0, \vec{k})$ in a stationary frame, with $k^0 = u \cdot k$.
- Where in (22) we define:

$$U^{\nu\rho} = u^\nu u^\rho - \frac{1}{2}\eta^{\nu\rho}. \quad (24)$$

Recurrence Relations

- By inspection of (20), (21) and (22), we see that our expressions for the electromagnetic and Riemann Curvature tensors can be written **in terms** of a scalar field $\phi^{(n)}(x)$:

$$F^{(n)}_{\mu\nu} = 2u_{[\mu}\partial_{\nu]}\phi^{(n)}. \tag{25}$$

$$R^{(n)}_{\mu\nu\rho\sigma}(x) = -2\left[U_{\rho[\mu}\partial_{\nu]}\partial_{\sigma} - U_{\sigma[\mu}\partial_{\nu]}\partial_{\rho}\right]\phi^{(n)}(r). \tag{26}$$

Stage 2: Integral Transforms and Integration

- Moving to **stage 2** of the process, we now wish to convert our integrals into the language of 2-spinors, and then integrate out some of the degrees of freedom.
- Let's start with the case of the scalar integral.
- Our momentum 4-vector is given by:

$$k^\mu = \omega \ell^\mu + \xi q^\mu, \tag{27}$$

where $\ell^2 = q^2 = 0$; the parameters ω (*on-shell*) and ξ (*off-shell*) have dimensions of energy; with a fixed 4-vector:

$$q^\mu = \frac{1}{2}(0, 0, -1, 1) \tag{28}$$

Momentum Spinorial Translation

- Spinorial translating our 4-momentum we get $k_{A\dot{A}}$:

$$k_{A\dot{A}} = \omega \lambda_A \tilde{\lambda}_{\dot{A}} + \xi q_{A\dot{A}}, \tag{29}$$

- Where λ_A and $\tilde{\lambda}_{\dot{A}}$ are now defined in terms of a complex number z and its complex conjugate $\bar{z}$:

$$\lambda_A = \left(\frac{1}{\sqrt{z}}, \sqrt{z}\right), \quad \tilde{\lambda}_{\dot{A}} = \left(\frac{1}{\sqrt{-\tilde{z}}}, -\sqrt{-\tilde{z}}\right). \tag{30}$$

- In $(2,2)$-signature, complex numbers z and their complex conjugate $\bar{z}$ **decouple**. E.g:

$$\tilde{z} \neq (z)^* \tag{31}$$

Delta Function Transformation

The delta function $\delta(u \cdot k)$ spinorial translation depends on the order in which we perform the integrals over dz and $d\tilde{z}$. The delta function first translates to:

$$\begin{aligned} \delta(u \cdot k) \to \delta\left(u_{A\dot{A}}\lambda^{A}\tilde{\lambda}^{\dot{A}}\right) &= \frac{1}{m}\delta(U^{A\dot{A}}(\omega\lambda_{A}\tilde{\lambda}_{\dot{A}})) \\ &= \frac{2}{\omega}\delta\left(\frac{1+z\tilde{z}}{\sqrt{-z\tilde{z}}}\right) \end{aligned} \tag{32}$$

If we are integrating over dz:

$$\frac{2}{\omega}\delta\left(\frac{1+z\tilde{z}}{\sqrt{-z\tilde{z}}}\right) = \frac{2}{\omega\tilde{z}}\delta\left(z+\frac{1}{\tilde{z}}\right) \tag{33}$$

If we are integrating over $d\tilde{z}$:

$$\frac{2}{\omega}\delta\left(\frac{1+z\tilde{z}}{\sqrt{-z\tilde{z}}}\right) = \frac{2}{\omega z}\delta\left(\tilde{z}+\frac{1}{z}\right) \tag{34}$$

Delta Functions

An important implication of integrating over dz or $d\tilde{z}$ is the replacement rules it generates for λ_A and $\tilde{\lambda}_{\dot{A}}$ respectively:

Integration over dz:

$$\lambda_A = U_A{}^{\dot{A}} \tilde{\lambda}_{\dot{A}} \tag{35}$$

Integration over $d\tilde{z}$:

$$\tilde{\lambda}_{\dot{A}} = -U^A{}_{\dot{A}} \lambda_A \tag{36}$$

Where we define the mixed raised and lowered indexed spinorial translation of the 4-velocity u^μ:

$$U_A{}^{\dot{A}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{37}$$

$$U^A{}_{\dot{A}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{38}$$

Stage 2: Integral Transforms and Integration

We now have a *change of variables* from d^4k to $(\omega, \xi, z, \tilde{z})$:

$$d^4k = \frac{dz d\tilde{z} d\omega d\xi}{4(2\pi)^3}, \tag{39}$$

We can then find:

$$\phi(x) = \int \frac{d\omega d\xi dz d\tilde{z}}{2(2\pi)^4} \frac{\omega\sqrt{-\tilde{z}}}{\tilde{z}\sqrt{z}} \delta\left(\frac{1+z\tilde{z}}{\sqrt{-z\tilde{z}}}\right) \times$$
$$e^{-i\xi q\cdot x} \exp\left[-\frac{i\omega}{2}\tilde{\lambda}_{\dot{A}}\tilde{\lambda}_{\dot{B}} x^{A\dot{A}} u_A{}^{\dot{B}}\right] \mathcal{J}(\vec{k}). \tag{40}$$

It is now useful to consider an example of this integral in practice.

Reissner-Nordstrom black hole

- The **Reissner-Nordstrom black hole** is simply a *Schwarzschild Black hole* that has been allowed to be electrically charged.
- In particular, its Weyl Double Copy is given by:

$$\begin{aligned} \Psi_{ABCD} \\ &= \Psi^{(1)}_{ABCD} + \Psi^{(2)}_{ABCD} \\ &= \frac{1}{\phi^{(1)}}\Phi^{(1)}_{(AB}\Phi^{(1)}_{CD)} + \frac{1}{\phi^{(2)}}\Phi^{(2)}_{(AB}\Phi^{(2)}_{CD)} \end{aligned} \tag{41}$$

- We will be focusing on deriving $\phi^{(2)}$, $\Phi^{(2)}_{AB}$ and $\Psi^{(2)}_{ABCD}$. From the literature, we know that:

$$\phi^{(2)} \propto \frac{1}{r^2} \tag{42}$$

Scalar Field Calculation

- This implies our source current is given by:

$$\mathcal{J}(\vec{k}) = 2\pi^2|\vec{k}|. \tag{43}$$

- Applying this to our integral for $\phi^{(2)}$ and integrating over ξ, ω and z we find:

$$\phi^{(2)}(x) = \frac{1}{\sqrt{32}\pi}\frac{1}{\sqrt{q\cdot x}}\int d\tilde{z}\,\frac{1}{\tilde{z}^{3/2}}\frac{1}{\left(\tilde{\lambda}_{\dot{A}}\tilde{\lambda}_{\dot{B}}x^{A\dot{A}}u_{A}{}^{\dot{B}}\right)^{3/2}}. \tag{44}$$

- Where $q \cdot x$ is given by:

$$q\cdot x = \frac{1}{2}q_{A\dot{A}}x^{A\dot{A}} \tag{45}$$

Scalar Field Calculation

It is now convenient to define the rescaled spinor

$$\tilde{\chi}_{\dot{A}} = \sqrt{-\tilde{z}}\tilde{\lambda}_{\dot{A}} = \begin{pmatrix} 1 \\ \tilde{z} \end{pmatrix}, \tag{46}$$

such that (44) can be written more compactly as

$$\phi^{(2)}(x) = \frac{1}{\sqrt{32}\pi}\frac{1}{\sqrt{q \cdot x}}\int d\tilde{z}\frac{1}{\left(\tilde{\chi}_{\dot{A}}\tilde{\chi}_{\dot{B}}x^{A\dot{A}}u_A{}^{\dot{B}}\right)^{3/2}}. \tag{47}$$

One complication of (47) is that its singularity structure involves *branch cuts* rather than simple poles, and it is not then immediately clear how to choose an appropriate contour.

Scalar Field Calculation

However, we can instead carry out the **inverse Fourier transform** of our original expression for $\phi^{(2)}$ in (20) to obtain:

$$\begin{aligned}\phi^{(2)}(x) &= \int \frac{d^4 k}{(2\pi)^4} \frac{2\pi^2 \delta(u \cdot k)|\vec{k}|}{k^2} e^{-ik\cdot x} \\ &= \frac{1}{(x^2 - (u \cdot x)^2)} = \frac{1}{r^2}.\end{aligned} \tag{48}$$

Comparison with (47) then yields

$$\int d\tilde{z} \frac{1}{\left(\tilde{\chi}_{\dot{A}}\tilde{\chi}_{\dot{B}} x^{A\dot{A}} u_A{}^{\dot{B}}\right)^{3/2}} = \frac{\sqrt{32}\pi\sqrt{q \cdot x}}{r^2}. \tag{49}$$

With denominator being give as:

$$\tilde{\chi}_{\dot{A}}\tilde{\chi}_{\dot{B}} x^{A\dot{A}} u_A{}^{\dot{B}} = \tilde{z}^2(y + \mathrm{z}) + 2x\tilde{z} + \mathrm{z} - y. \tag{50}$$

Recurrance Relations in Practice

Recalling that $\phi^{(2)}$ can be used to find the electromagnetic and curvature fields respectively, we define a family of integrals:

$$I_{m,n} = (-1)^{m+1} \int d\tilde{z} \frac{\tilde{z}^n}{[\tilde{z}^2(y+z) + 2x\tilde{z} + z - y]^{3/2+m}}, \qquad (51)$$

where the scalar field in (49) is given by:

$$\phi^{(2)}(x) = \frac{1}{\sqrt{32}\pi} \frac{1}{\sqrt{q \cdot x}} I_{0,0}. \qquad (52)$$

with

$$I_{0,0} = \frac{\sqrt{32}\pi\sqrt{q \cdot x}}{r^2} = \frac{4\pi\sqrt{y+z}}{x^2 + y^2 - z^2}. \qquad (53)$$

Recurrance Relations in Practice

By differentiating (51) with respect to the spatial coordinates (x, y, z), one can derive the recurrence relations

$$\begin{aligned} \frac{\partial}{\partial x} I_{m,n} &= -(3+2m) I_{m+1,n+1}, \\ \left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) I_{m,n} &= -(3+2m) I_{m+1,n+2}, \\ \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right) I_{m,n} &= (3+2m) I_{m+1,n}, \end{aligned} \tag{54}$$

which we will use **repeatedly** in what follows.

Deriving Maxwell Fields

Spinorially translating the Fourier transform of the electromagnetic field strength tensor in (21), then applying (9) to obtain the Maxwell Spinor. Then, following the **same** process as done for the scalar field previously we find:

$$\Phi^{(2)}_{\dot{A}\dot{B}} = -\frac{3}{4\pi}\frac{1}{(x^{0\dot{1}})^{1/2}}\int d\tilde{z}\,\frac{\tilde{\chi}_{\dot{A}}\tilde{\chi}_{\dot{B}}}{(\tilde{\chi}_{\dot{A}}\tilde{\chi}_{\dot{B}}x^{A\dot{A}}u_A{}^{\dot{B}})^{5/2}} + \frac{1}{4\pi}\frac{1}{(x^{0\dot{1}})^{3/2}}\int d\tilde{z}\,\frac{u_{\dot{A}}{}^{A}q_A\tilde{q}_{\dot{B}} + u_{\dot{B}}{}^{B}q_B\tilde{q}_{\dot{A}}}{(\tilde{\chi}_{\dot{A}}\tilde{\chi}_{\dot{B}}x^{A\dot{A}}u_A{}^{\dot{B}})^{3/2}}. \tag{55}$$

Using our integral relations, we can write:

$$\Phi^{(2)}_{\dot{A}\dot{B}} = \frac{1}{4\pi}\left[\frac{1}{(q\cdot x)^{3/2}}\begin{pmatrix} 0 & 0 \\ 0 & -I_{0,0}\end{pmatrix} - \frac{6}{\sqrt{q\cdot x}}\begin{pmatrix} I_{1,0} & I_{1,1} \\ I_{1,1} & I_{1,2}\end{pmatrix}\right], \tag{56}$$

Deriving Maxwell Fields

$$I_{1,0} = -\frac{32\pi(q\cdot x)^{3/2}}{3r^4},$$
$$I_{1,1} = \frac{16\pi x(q\cdot x)}{3r^4},$$
$$I_{1,2} = \frac{16\pi}{3}(y - \mathtt{z})\frac{\sqrt{q\cdot x}}{r^4} - \frac{1}{6}\frac{I_{00}}{q\cdot x}. \quad (57)$$

Thus, the Maxwell spinor becomes

$$\Phi^{(2)}_{AB} = -\frac{2}{r^4}\begin{pmatrix} -y-\mathtt{z} & x \\ x & -y+\mathtt{z} \end{pmatrix}, \quad (58)$$

$$\Phi^{(2)}_{AB} = 2\frac{u_{(\dot{A}}{}^{B} r_{\dot{B})B}}{r^4}, \quad (59)$$

where

$$r_{B\dot{B}} = (x - (u\cdot x)u)\cdot\sigma_{B\dot{B}} = \begin{pmatrix} -x & \mathtt{z}-y \\ -\mathtt{z}-y & x \end{pmatrix}. \quad (60)$$

Weyl Spinor Calculation

The Weyl Spinor can be obtained from the Riemann Curvature spinor via:

$$\Psi_{ABCD} = \frac{1}{4} R_{(A\dot{X}B}{}^{\dot{X}}{}_{C\dot{Y}D)}{}^{\dot{Y}} \tag{61}$$

Then applying the same procedure as before to (22):

$$\Psi^{(2)}_{\dot{A}\dot{B}\dot{C}\dot{D}} = \frac{3\kappa}{2}\int d\tilde{z}\left[\frac{5\tilde{\chi}_{\dot{A}}\tilde{\chi}_{\dot{B}}\tilde{\chi}_{\dot{C}}\tilde{\chi}_{\dot{D}}}{(\tilde{\chi}_{\dot{A}}\tilde{\chi}_{\dot{B}}x^{A\dot{A}}u_A{}^{\dot{B}})^{7/2}(q\cdot x)^{1/2}} + \frac{2\tilde{\chi}_{(\dot{A}}\tilde{\chi}_{\dot{B}}(u\cdot q)_{\dot{C}}\tilde{q}_{\dot{D})}}{(\tilde{\chi}_{\dot{A}}\tilde{\chi}_{\dot{B}}x^{A\dot{A}}u_A{}^{\dot{B}})^{5/2}(q\cdot x)^{3/2}} + \frac{(u\cdot q)_{(\dot{A}}\tilde{q}_{\dot{B}}(u\cdot q)_{\dot{C}}\tilde{q}_{\dot{D})}}{(\tilde{\chi}_{\dot{A}}\tilde{\chi}_{\dot{B}}x^{A\dot{A}}u_A{}^{\dot{B}})^{3/2}(q\cdot x)^{5/2}}\right]. \tag{62}$$

Weyl Spinor Calculation

Using our recurrence relations, the Weyl spinor becomes:

$$\Psi_{(2)}^{\dot{A}\dot{B}\dot{C}\dot{D}} = \frac{1}{6r^6}\Big[u_C^{(\dot{A}} u_B^{\dot{D})} r^{\dot{B}B} r^{\dot{C}C} - u_B^{(\dot{C}} u_A^{\dot{D})} r^{\dot{A}A} r^{\dot{B}B} - u_C^{(\dot{B}} u_A^{\dot{D})} r^{\dot{A}A} r^{\dot{C}C}$$
$$- \left(u_D^{\dot{C}}\left(u_A^{\dot{B}} r^{\dot{A}A} + u_B^{\dot{A}} r^{\dot{B}B}\right) + u_D^{\dot{B}}\left(u_A^{\dot{C}} r^{\dot{A}A} + u_C^{\dot{A}} r^{\dot{C}C}\right)\right.$$
$$\left.+ u_D^{\dot{A}}\left(u_B^{\dot{C}} r^{\dot{B}B} + u_C^{\dot{B}} r^{\dot{C}C}\right)\right) r^{\dot{D}D}\Big], \qquad (63)$$

The various terms combine in such a way as to yield the combination

$$\Psi^{(2)}_{\dot{A}\dot{B}\dot{C}\dot{D}} = -2\frac{\Phi^{(2)}_{(\dot{A}\dot{B}}\Phi^{(2)}_{\dot{C}\dot{D})}}{\phi^{(2)}}, \qquad (64)$$

in terms of the electromagnetic spinor of (59). Thus, a Weyl double copy does indeed hold for the **integrated Weyl spinor**, even in the presence of a non-trivial source!

Sourced Weyl Double Copy from Tensor Methods

The Weyl Double for Einstein Maxwell Gravity can also be expressed in the **language of tensors**:

$$W^{\mu\nu\rho\sigma} = \sum_{n=1}^{m} \frac{2+n}{n\phi^{(n)}} \mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega} F^{(n),\tau\lambda} F^{(n),\eta\omega}. \tag{65}$$

where $\mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega}$ is some *projector* given by:

$$\mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega} = \delta^{\mu}_{\tau}\delta^{\nu}_{\lambda}\delta^{\rho}_{\eta}\delta^{\sigma}_{\omega} + \frac{1}{2} g_{\tau\eta}\delta^{[\mu}_{\lambda} g^{\nu][\rho}\delta^{\sigma]}_{\omega} + \frac{1}{6} g_{\tau\eta} g_{\lambda\omega} g^{\mu[\rho} g^{\sigma]\nu}, \tag{66}$$

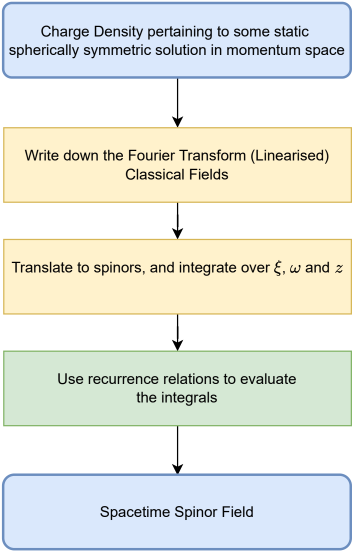
Using this language, we can prove that the Double Copy holds for **arbitrary** n:

$$\phi^{(n)} = \frac{1}{r^n}, \tag{67}$$

Spinning Sources

- So far, we have proven that there exists a Weyl Double Copy for the case of non-vacuum solutions in Einstein-Maxwell gravity sourced by static spherically symmetric sources.
- It was also conjectured the Weyl Double Copy holds for ***spinning*** sources.
- Performing the ***Janis Newman Shift*** on our results for the Reissner-Nordstrom black hole, we obtain the Kerr-Newman Black Hole (a rotating black hole sourced by a charged disk).

Overview


Charge Density pertaining to some static spherically symmetric solution in momentum space
Write down the Fourier Transform (Linearised) Classical Fields
Translate to spinors, and integrate over ξ, ω and z
Use recurrence relations to evaluate the integrals
Spacetime Spinor Field

Conclusion and Further Work

- The Weyl Double Copy **relates** Petrov Vacuum Type D and N solutions in General relativity to phenomena in electromagnetism.
- We **prove** conjectures of a Weyl Double Copy for Einstein-Maxwell gravity for scalar fields in arbitrary of r^{-n}.
- Further work should probe whether we can extend our discussion of the **Kerr-Newman black hole** to arbitrary r^{-n}.
- It would be interesting to whether our results could have applications to ***astrophysical settings***.