Kaluza-Klein Theory with Fuzzy Sphere Fibre

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Outline

- Fuzzy sphere geometry
- Dimensional reduction for Ricci scalar and scalar field
- Spinor space on fuzzy sphere
- Dimensional reduction for Dirac field

Fuzzy Sphere

Fuzzy Sphere Algebra $\mathbb{C}_\lambda[S^2]$:
$\mathbb{C}_\lambda[S^2]$ generated by y^i satisfying:

$$[y^i, y^j] = 2\imath\lambda\epsilon_{ijk}y^k, \quad \sum_i (y^i)^2 = 1 - \lambda^2.$$

Decomposition as su_2 representations: $\mathbb{C}_\lambda[S^2] = \oplus_{l=0} A_l$. Therefore, any $f \in \mathbb{C}_\lambda[S^2]$ can be written as:

$$f = \sum_{l=0}^{\infty} (f_l)_{i_1 \ldots i_l} y^{i_1} y^{i_2} \ldots y^{i_l}$$
$$= f_0 + (f_1)_i y^i + (f_2)_{ij} y^i y^j + \ldots + (f_l)_{i_1 \ldots i_l} y^{i_1} \ldots y^{i_l} + \ldots$$

Reduced Fuzzy Sphere:

$$\lambda = \frac{1}{2j+1}: \quad \mathbb{C}_\lambda[S^2] \cong M_{2j+1}(\mathbb{C}), \quad f = \sum_{l=0}^{2j} (f_l)_{i_1 \ldots i_l} y^{i_1} y^{i_2} \ldots y^{i_l}$$

As an example, $j = \frac{1}{2}: \quad \mathbb{C}_\lambda[S^2] \cong M_2(\mathbb{C}), \quad f = f_0 + f_i y^i$

Curvature on fuzzy sphere

- Differential calculus:

$$\mathrm{d}y^i = \epsilon_{ijk} y^j s^k, \quad \mathrm{d}s^i = -\frac{1}{2}\epsilon_{ijk} s^j \wedge s^k, \quad [y^i, s^j] = 0$$

- Metric:

$$g = h_{ij} s^i \otimes s^j$$

Due to the inverse of the metric g^{-1} as a bilinear map, we have

$$a.g^{-1}(s^i \otimes s^j) = g^{-1}(a.s^i \otimes s^j) = g^{-1}(s^i \otimes s^j.a) = g^{-1}(s^i \otimes s^j).a$$

or

$$a.h^{ij} = h^{ij}.a$$

which *forces* $h^{ij} \in \mathbb{C}.1$ (so as $h_{ij} \in \mathbb{C}.1$)

- Connection

$$\nabla s^i = H^i_{jk} s^j \otimes s^k$$

For simplicity, we assume $H^i_{jk} \in \mathbb{C}.1$

Curvature and scalar field on fuzzy sphere

- Ricci Scalar ($\Phi = \ln(\underline{h})$)

$$R_h = \frac{1}{2\det(\underline{h})}\left(\mathrm{Tr}(\underline{h}^2) - \frac{1}{2}\mathrm{Tr}(\underline{h})^2\right) = \frac{e^{-\mathrm{Tr}(\Phi)}}{2}\left(\mathrm{Tr}(e^{2\Phi}) - \frac{1}{2}\mathrm{Tr}(e^{\Phi})^2\right)$$

Scalar field on fuzzy sphere:

$$S = \int f \Box f$$

Here $\Box = g^{-1}\nabla \mathrm{d}$ and d obeys

$$\mathrm{d}(ab) = a\mathrm{d}b + (\mathrm{d}a)b, \quad \mathrm{d}^2 = 0$$

and the integral $\int \mathbb{C}_\lambda[S^2] \to \mathbb{C}$ is

$$\int f = f_0$$

We are mostly concerned with the round metric $h_{ij} = h\delta_{ij}$, therefore

$$S = \int \frac{1}{h} f \partial_i \partial_i f = \sum_{l=0} -\frac{l(l+1)}{h}\alpha_l (f_l)_{i_1 \dots i_l}(f_l)_{i_1 \dots i_l}$$

KK theory with fuzzy sphere fibre

Now we consider a tensor product algebra

$$A = C^\infty(M) \otimes \mathbb{C}_\lambda[S^2]$$

on which

$$[x^\mu, y^i] = 0$$

Consider a generic metric

$$g = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + A_{\mu i}(\mathrm{d}x^\mu \otimes s^i + s^i \otimes \mathrm{d}x^\mu) + h_{ij}s^i \otimes s^j$$

g^{-1} as a bilinear map forces $g_{\mu\nu}, A_{\mu i}, h_{ij} \in C^\infty(M)$. This derives the cylinder ansatz in the usual KK theory!

The dimensional reduction of Ricci scalar:

$$R = R_M + \frac{1}{8}h_{ij}F^i_{\mu\nu}F^{j\mu\nu} + \frac{1}{8}\left(\mathrm{Tr}(\Phi_\alpha\Phi^\alpha) + \mathrm{Tr}(\Phi_\alpha)\mathrm{Tr}(\Phi^\alpha)\right) + R_h(\Phi)$$

where $\Phi^i_{\alpha j} := h^{ik}\nabla_{A\alpha}h_{kj}$ is the matrix valued Liouville Field.

Dimensional reduction of a scalar field

In the round constant metric case $h_{ij} = h\delta_{ij}, \partial_\alpha h = 0$, the dimensional reduction of a scalar field ϕ generates KK tower of scalar **multiplets**

$$\begin{aligned} S_\phi &= \int_M \mathrm{d}^n x \sqrt{-|g|} \int_h \phi \Box \phi \\ &= \int_M \mathrm{d}^n x \sqrt{-|g|} \sum_{l=0} (\phi_l)_{i_1 \ldots i_l} \left(\left(\Box_A - m_l^2 \right) \phi_l \right)_{i_1 \ldots i_l} \\ &= \int_M \mathrm{d}^n x \sqrt{-|g|} \sum_{l=0} \phi_l^\intercal \left(\Box_A - m_l^2 \right) \phi_l \end{aligned}$$

where for the last step we suppress the multiplet indices. Here

$$m_l = \sqrt{\frac{l(l+1)}{h}}$$

As an example in the reduced fuzzy sphere case, we consider $j = \frac{1}{2}$, then the KK tower obtains a **natural truncation**

$$S_\phi = \int_M \mathrm{d}^n x \sqrt{-|g|} \left(\phi_0 \Box \phi_0 + \phi_1^\intercal \left(\Box_A - \frac{2}{h} \right) \phi_1 \right)$$

Dimensional reduction of a Dirac field

For a massless Dirac field in curved spacetime, the action is

$$S_\psi = \int \bar{\psi}\gamma^\mu D_\mu \psi, \quad D_\mu = \partial_\mu + \frac{1}{2}(\omega_{\nu\rho})_\mu \sigma^{\nu\rho}, \quad \sigma^{ab} \equiv -\frac{\imath}{4}[\gamma^a, \gamma^b]$$

where ω is the spin connection.
The **spinor space** on the fuzzy sphere is

$$S_h = \mathbb{C}_\lambda[S^2] \otimes \mathbb{C}^2 = \bigoplus_{l=0} A_l \otimes \mathbb{C}^2 = \bigoplus_{l=0} (S_l^+ \oplus S_l^-)$$

Here $S_l^\pm$ are the eigenspaces of Dirac operator on the fuzzy sphere with $\imath\gamma^\mu D_\mu|_{S_l^\pm} = \frac{1}{4} \pm (l + \frac{1}{2})$. For example,

$$j = 0: \quad S_h = \mathbb{C} \otimes \mathbb{C}^2 = \mathbb{C}^2 = 2$$

$$j = \frac{1}{2}: \quad S_h = M_2 \otimes \mathbb{C}^2 = S_0^+ \oplus S_1^- \oplus S_1^+ = 2 \oplus 2 \oplus 4,$$

$$j = 1; \quad S_h = M_3 \otimes \mathbb{C}^2 = S_0^+ \oplus S_1^- \oplus S_1^+ \oplus S_2^- \oplus S_2^+ = 2 \oplus 2 \oplus 4 \oplus 4 \oplus 6$$

where we put the $2j+1$-dimensional irreps of $SU(2)$.

Dimensional reduction of a Dirac field

- On the tensor product spinor bundle

$$S = S_M \otimes S_h = \bigoplus_{l=0} S_M \otimes (S_l^+ \oplus S_l^-)$$

Therefore, a Dirac spinor $\psi_{A(l,\pm)j_3} \in S$ has three kinds of spinor indices:

$$A = 1,2,3,4 \quad \text{from 4d-spacetime}$$
$$(l,\pm) \quad \text{or } j = l \pm \frac{1}{2} \quad \text{from 3d-fuzzy sphere}$$
$$-j \leq j_3 \leq j \quad \text{from 3d-fuzzy sphere}$$

- By using the Clifford algebra $\{\gamma^I, \gamma^J\} = 2g^{IJ}\mathrm{id}$, and the definition of D_μ, one can perform the dimensional reduction of Dirac field

Dimensional reduction of a Dirac field

For a massless Dirac field, we obtain

$$S_\psi = \int \bar{\psi} \not{D} \psi$$
$$= \sum_{l \geq 0, \pm} \int_M \overline{\psi}_{l,\pm} \left(\not{D}_A - m_{l,\pm} + \frac{\imath \sqrt{h}}{4(2l+1)} F^a_{\mu\nu} T^a \gamma^{\mu\nu} \right) \psi_{l,\pm}$$

where

$$m_{l,\pm} = \frac{1}{\sqrt{h}} \left(l + \frac{1}{2} \pm \frac{1}{4} \right)$$

and we suppress the spinor indices A from spacetime and j_3 indices. The summation is understood that when $l = 0$ one only has the $+$ case. Here $\psi_{l,\pm}$ are the $2j + 1$ multiplets. Hence, **the spinor index j_3 from the fuzzy sphere part becomes multiplet index**.

Three generations of fermions

As an example on the reduced fuzzy sphere case $j = \frac{1}{2}$,

$$S_\psi = \int_M \overline{\psi}_{0,+} \left(\not{D}_A - m_{0,+} + \frac{\imath\sqrt{h}}{4} F^a_{\mu\nu} T^a \gamma^{\mu\nu} \right) \psi_{0,+}$$

$$+ \int_M \overline{\psi}_{1,-} \left(\not{D}_A - m_{1,-} + \frac{\imath\sqrt{h}}{12} F^a_{\mu\nu} T^a \gamma^{\mu\nu} \right) \psi_{1,-}$$

$$+ \int_M \overline{\psi}_{1,+} \left(\not{D}_A - m_{1,+} + \frac{\imath\sqrt{h}}{12} F^a_{\mu\nu} T^a \gamma^{\mu\nu} \right) \psi_{1,+}$$

where $\psi_{0,+}$ is a doublet (i.e. $j = 1/2$), ψ_- is another doublet ($j = 1/2$), and $\psi_{1,+}$ is a quadruplet ($j = 3/2$) with masses in ratio

$$1 : 5/3 : 7/3$$

After considering the internal structure of spacetime (described by a fuzzy sphere), one fermionic field becomes three fermionic fields, exhibiting the phenomenon of **'three generations of fermions'.**

Generating fermion masses through interactions with fibers

Another interesting example is the 0-radius fuzzy sphere case where $j = 0$. In this case, the KK tower is truncated to a single layer

$$S_\psi = \int \bar{\psi} \not{D} \psi = \int_M \overline{\psi}_{0,+} \left(\not{D}_A - m_{0,+} + \frac{\imath\sqrt{h}}{4} F^a_{\mu\nu} T^a \gamma^{\mu\nu} \right) \psi_{0,+}$$

where

$$m_{0,+} = \frac{3}{4\sqrt{h}}$$

This shows how a massless fermionic singlet becomes a massive fermionic doublet. **Note that the fermion obtains the mass without involving the Higgs mechanism.**

One can think a fermion can obtain its mass by interacting with fuzzy sphere fibre.
This is different from the usual KK theory where the field on the first layer keeps massless.

Conclusion and Future Directions

Conclusion

- One can unify gravity and YM fields nicely from KK theory on fuzzy sphere fibre and **derive cylinder ansatz**.
- the KK tower obtains a **natural truncation** from fuzzy sphere fibre.
- there exists the phenomenon of **three generations for fermions** in a reduced fuzzy sphere case.
- a fermion can **obtain mass** by interacting with the fuzzy sphere fibre.

Future Directions

- Extend the model to more general noncommutative fibres.
- KK masses are at the Planckian level, as in other KK towers. To overcome this drawback, one possible method is to replace the tensor product with a semidirect product.

Thank You!