Varieties of four-dimensional gauge theories

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Based on 2409.15430 with Ben Gripaios

A problem

Write down all anomaly-free representations of $\mathfrak{u}(1)$ for four-dimensional spacetime fermions.

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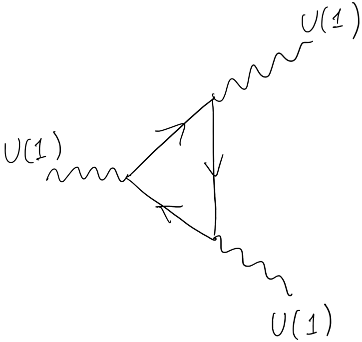
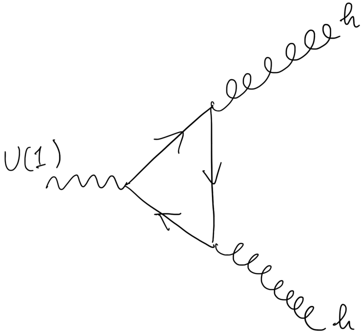
Write down all anomaly-free representations of $\mathfrak{u}(1)$ for four-dimensional spacetime fermions.
Equivalently: List all permissible sets of charges $(Q_1, \ldots, Q_n) \in \mathbb{Z}^n$ of n Weyl fermions in four dimensions.

$\mathfrak{u}(1)$ anomalies





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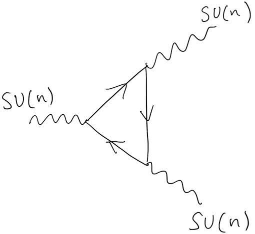
Concession: Consider only irreducible representations.
Irreps $\Rightarrow$ Only need to worry about $\mathfrak{su}(n)$ for $n \geq 3$.

$\mathfrak{su}(n)$ irreps

Dynkin labels: $(m_1, \ldots, m_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$.
Example: $(1, 0, 1)$ is adjoint rep of $\mathfrak{su}(4)$.
$(q_1, \ldots, q_{n-1}) := (m_1 + 1, \ldots, m_{n-1} + 1)$.
Dual rep of $(q_1, \ldots, q_{n-1})$ is $(q_{n-1}, \ldots, q_1)$ and has negative the anomaly.
(Pseudo-)Real irreps are self-dual and so anomaly-free.

$\mathfrak{su}(n)$ anomalies



Banks, Georgi '76: Anomaly $:= A_n = \sum a_{ijk} q_i q_j q_k$.

Which irreps are anomaly-free (ie. have $A_n = 0$)?
Partial answer: All (pseudo-)real irreps.
$\mathfrak{su}(3): A_3 = (q_1 - q_2)(q_1 + 2q_2)(2q_1 + q_2)$
$\Rightarrow$ Only (pseudo-)real irreps of the form (a, a).
$\mathfrak{su}(4): A_4 = (q_1 - q_3)(q_1 + q_3)(q_1 + 2q_2 + q_3)$
$\Rightarrow$ Only (pseudo-)real irreps of the form (a, b, a).

$\mathfrak{su}(n)$ anomalies

$$\mathfrak{su}(5): A_5 = 4q_1^3 + 9q_1^2 q_2 + 3q_1 q_2^2 + 2q_2^3 + 6q_1^2 q_3 + 4q_1 q_2 q_3 + 4q_2^2 q_3 - 2q_1 q_3^2 - 4q_2 q_3^2 - 2q_3^3 + 3q_1^2 q_4 + 2q_1 q_2 q_4 + 2q_2^2 q_4 - 2q_1 q_3 q_4 - 4q_2 q_3 q_4 - 3q_3^2 q_4 - 3q_1 q_4^2 - 6q_2 q_4^2 - 9q_3 q_4^2 - 4q_4^3.$$

$\mathfrak{su}(5)$ has some complex anomaly-free irreps

Eichten, Kang, Koh '82: Dimension below 4×10^9

(m_1, m_2, m_3, m_4)	Dimension
$(0, 7, 3, 3)$	1×10^6
$(1, 8, 1, 5)$	3×10^6
$(7, 7, 15, 1)$	1×10^9

Okubo's new variables

Okubo '77: Transform the $n-1$ positive integers $q_1, \ldots, q_{n-1}$ to the n rationals $\sigma_1, \ldots, \sigma_n$ according to

$$\sigma_i := \frac{1}{n}\left(-\sum_{k=1}^{i-1} kq_k + \sum_{k=i}^{n-1}(n-k)q_k\right).$$

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Example: (Pseudo-)Real irreps of $\mathfrak{su}(5)$ have $(\sigma_1, \ldots, \sigma_5) = (a, b, 0, -b, -a)$ with $a > b > 0$.

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Example: (Pseudo-)Real irreps of $\mathfrak{su}(5)$ have $(\sigma_1, \ldots, \sigma_5) = (a, b, 0, -b, -a)$ with $a > b > 0$.
Constraints:

- $\sum_{i=1}^{n} \sigma_i = 0$.
- $\sigma_1 > \sigma_2 > \cdots > \sigma_n$ (because $\sigma_i - \sigma_{i+1} = q_i$).

Anomaly: $A_n \propto \sum_{i=1}^{n} \sigma_i^3$.

The $\mathfrak{su}(n)$ anomaly cancellation conditions are almost the same as the $\mathfrak{u}(1)$ ones

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Anomaly-free irreps of $\mathfrak{su}(5)$

For $n = 5$, we have to solve a homogeneous cubic equation in four variables, writing $\sigma_5 = -(\sigma_1 + \cdots + \sigma_4)$,

$$\sum_{i=1}^{4} \sigma_i^3 - \left(\sum_{i=1}^{4} \sigma_i \right)^3 = 0.$$

In $\mathbb{k}\mathbb{P}^3$ ($\mathbb{k} \in \{\mathbb{C}, \mathbb{R}, \mathbb{Q}\}$), this defines a projective variety called the Clebsch diagonal cubic surface.

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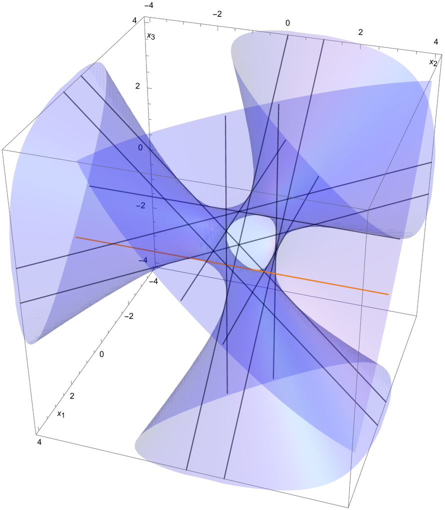
- "Projective" means that given one solution $(\sigma_1, \ldots, \sigma_5)$, infinitely many others are found by scaling.
- "Variety" means that we have the zero locus of a polynomial, and that a fake algebraic geometer such as myself should not say much more lest Zariski starts turning in his grave.

27 lines

Fact 1: Every smooth cubic surface has 27 lines over $\mathbb{C}$.
Fact 2: The Clebsch diagonal cubic surface is the only one with 27 lines over $\mathbb{R}$.
Fact 3: In the σ_i variables, 15 of these lines exist over $\mathbb{Q}$.

Rational lines on the Clebsch cubic



The method of secants in one minute

1. Pick two skew rational lines on the Clebsch cubic.
2. Draw all secants between them.
3. Get all rational points from intersections between secants and Clebsch cubic.

The method of secants in action

With homogeneous coordinates $[\sigma_1 : \cdots : \sigma_5]$,

1. Two skew rational lines on the Clebsch cubic surface are $L_1 = [k_1 : k_2 : 0 : -k_2 : -k_1]$ (the "palindromic" line) and $L_2 = [0 : l_1 : l_2 : -l_2 : -l_1]$.
2. If $p_1 \in L_1$ and $p_2 \in L_2$, then the projective line through them is $L_3 = \alpha_1 p_1 + \alpha_2 p_2$.
3. A point $p_3 \in L_3$ lies on the Clebsch cubic if

$$\sum_{i=1}^{5} p_{3i}^3 = 0 \Leftrightarrow \sum_{i=1}^{5} \alpha_1\alpha_2(\alpha_1 p_{1i}^2 p_{2i} + \alpha_2 p_{1i} p_{2i}^2) = 0.$$

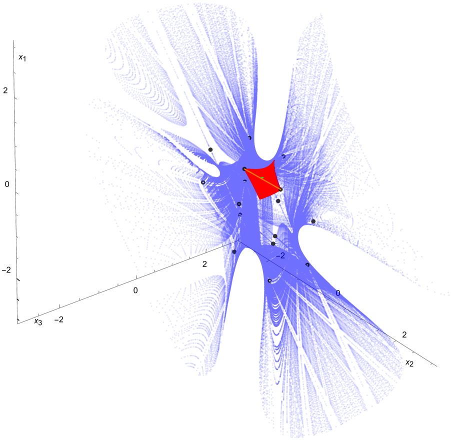
4. "Generically"

$$[\alpha_1 : \alpha_2] = \left[\sum_{i=1}^{5} p_{1i} p_{2i}^2 : -\sum_{i=1}^{5} p_{1i}^2 p_{2i}\right].$$

Caveat?

How about the pesky condition $\sigma_1 > \cdots > \sigma_5$?
Answer: S_5 symmetry, plus the fortunate fact that we essentially never run into trouble.

The solution for $n = 5$



Eichten, Kang, Koh revisited

$A_5 = \sum a_{ijk} q_i q_j q_k$

(m_1, m_2, m_3, m_4)	(q_1, q_2, q_3, q_4)	Dimension
$(0, 7, 3, 3)$	$(1, 8, 4, 4)$	1×10^6
$(1, 8, 1, 5)$	$(2, 9, 2, 6)$	3×10^6
$(7, 7, 15, 1)$	$(8, 8, 16, 2)$	1×10^9
$(0, 17, 12, 5)$	$(1, 18, 13, 6)$	2.5×10^9
$(8, 16, 0, 15)$	$(9, 17, 1, 16)$	2.7×10^9
$(3, 17, 3, 11)$	$(4, 18, 4, 12)$	3×10^9

Concluding remarks

- This all generalizes to higher n (even n are slightly annoying).
- Ongoing projects: Reducible representations, other abelian algebras.

Thank you for listening!