Double copy in AdS_3 from Minitwistor Space

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Overview

BCJ Amplitude Double Copy

- Write Yang-Mills as $A_{YM} \sim \sum_i \frac{n_i c_i}{d_i}$ such that Colour-Kinematic (CK) duality is satisfied $\sum_i n_i = 0 \Leftrightarrow \sum_i c_i = 0$

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- Replace "colour factor" c_i by a second copy of "kinematic factor" n_i
  Leave propagator d_i unchanged
- Obtain gravitational amplitudes $M_{grav} \sim \sum_i \frac{n_i n_i}{d_i}$ around flat space $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

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  Generalise background?
  Generalise perturbations?

Twistor Classical Double Copy

- The solutions to some equations are very simple in twistor space
- Take a spin s solution $\Psi_{\alpha_1 \ldots \alpha_{2s}}(x)$ and write it as a "Penrose transform":

$$\Psi_{\alpha_1 \ldots \alpha_{2s}}(X) = \oint_\gamma \langle \lambda \mathrm{d} \lambda \rangle \; \lambda_{\alpha_1} \ldots \lambda_{\alpha_{2s}} \, f_s$$

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- f is a twistor representative, whose homogeneity determines the specific solution
- The Double Copy is a simple consequence of that homoegeneity:

$$\boxed{f_{s=2} = \frac{f_{s=1} f_{s=1}}{f_{s=0}}}$$

Relation between the Double Copies

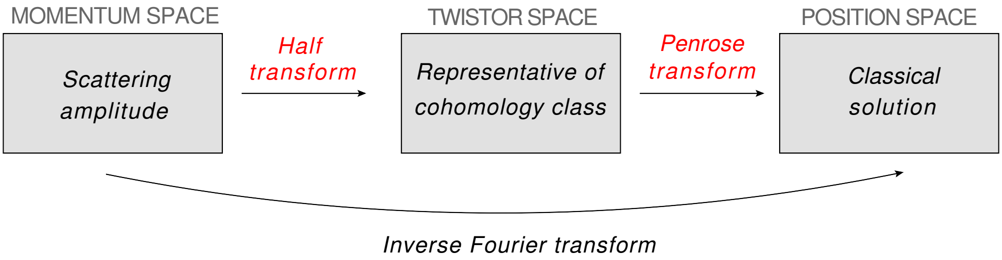

Figure: Figure taken from Luna, Moynihan, White summarising the scheme for obtaining position-space classical solutions from momentum-space scattering amplitudes.

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- Yes, do both thanks to twistors and going to 3d!

Overview

Topologically Massive Theories in 3d

- No local dof in 3d Einstein $\to$ too trivial for a local double copy
- Adding (topological) Chern-Simons term surprisingly adds local dofs to the theory
- Parity-violating and massive
- Teach lessons about the double copy with mass?

Topologically Massive Theories in 3d

- Topologically Massive Yang Mills (TMYM)

$$S_{TMYM} = \frac{1}{\tilde{g}^2} \int d^3x \sqrt{-g}\, \mathrm{tr} \left(-\frac{1}{2} F^{\mu\nu} F_{\mu\nu} + 2\tilde{g} A^{\mu} J_{\mu} - \frac{i\mu_V}{2} \varepsilon^{\mu\nu\rho} \left(F_{\mu\nu} - \frac{2}{3} A_{\mu} A_{\nu} \right) A_{\rho} \right)$$

Bianchi: $D_{[\mu} F_{\nu\rho]} = 0$ EOM: $D_{\mu} F^{\mu\nu} - i\frac{\mu_V}{2} \varepsilon^{\nu\rho\gamma} F_{\rho\gamma} = \tilde{g} J^{\mu}$

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- Spinors:

$$T^{\alpha\dot{\alpha}\dots}{}_{\beta\dot{\beta}\dots} = \sigma^{\alpha\dot{\alpha}}_{A}\dots\sigma^{B}_{\beta\dot{\beta}}T^{A\dots}{}_{B\dots}$$

$$\langle\lambda B\rangle = \lambda^{\alpha}B_{\alpha} = \lambda_1 B_0 - \lambda_0 B_1$$

$$[\mu A] = \mu^{\dot{\alpha}}A_{\dot{\alpha}} = \mu_{\dot{1}}A_{\dot{0}} - \mu_{\dot{0}}A_{\dot{1}}.$$

- Vacuum EOM + Bianchi $\nabla_{\alpha}{}^{\gamma}f_{\beta\gamma} = -m_V f_{\alpha\beta}$

Topologically Massive Theories in 3d

- TMYM $\nabla_{\alpha}{}^{\gamma} f_{\beta\gamma} = -m_V f_{\alpha\beta}$
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- Topologically Massive Gravity (TMG) $\nabla_\alpha{}^\epsilon C_{\beta\gamma\delta\epsilon} = -m_G C_{\alpha\beta\gamma\delta}$
- Topologically Massive Higher Spin: $\nabla^\beta{}_{(\alpha_1}\Psi_{\alpha_2\ldots\alpha_{2s})\beta} = -m_{(s)}\Psi_{\alpha_1\ldots\alpha_{2s}}$

Generalising the background: Minitwistors of AdS_3

- Defined through double fibration

$$
\begin{array}{ccc}
 & \mathbb{F} & \\
\swarrow & & \searrow \\
\mathbb{MT} & & AdS_3
\end{array}
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$$([\mu^{\dot{\alpha}}], [\lambda_\alpha]) \in \mathbb{MT} \simeq \mathbb{CP}^1 \times \mathbb{CP}^1 \qquad [X^{\alpha\dot{\alpha}}] \in AdS_3$$

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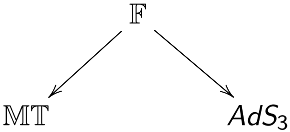

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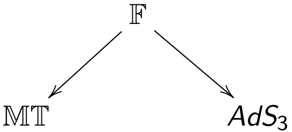

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- Non-local correspondence

Point $\leftrightarrow$ Null geodesic
Sphere $\leftrightarrow$ Point

Generalising the perturbation: Minitwistors and the Petrov Classification

- Unique decomposition of Cotton Tensor up to a constant factor

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\begin{aligned}
\text{Type I} \quad & C_{\alpha\beta\gamma\delta} = \psi(x) r_{(\alpha} s_\beta t_\gamma u_{\delta)}, \\
\text{Type II} \quad & C_{\alpha\beta\gamma\delta} = \psi(x) r_{(\alpha} r_\beta s_\gamma t_{\delta)}, \\
\text{Type D} \sim \text{Black Hole} \quad & C_{\alpha\beta\gamma\delta} = \psi(x) r_{(\alpha} r_\beta t_\gamma t_{\delta)}, \\
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Overview

Mass Prescription

- Double Copy relation is not enough to fix all the masses

$$f_{s=2,\widetilde{m}_G}(\mu^{\dot{\alpha}},\lambda_\alpha)=\frac{f_{s=1,\widetilde{m}_V}(\mu^{\dot{\alpha}},\lambda_\alpha)f'_{s=1,\widetilde{m}'_V}(\mu^{\dot{\alpha}},\lambda_\alpha)}{f_{s=0,\widetilde{m}_\phi}(\mu^{\dot{\alpha}},\lambda_\alpha)} \implies \widetilde{m}_G=\widetilde{m}_V+\widetilde{m}'_V-\widetilde{m}_\phi$$

- Try $\widetilde{m}_G=\widetilde{m}_V=\widetilde{m}_\phi$? After all, the propagator (and therefore the mass) is left untouched in the usual double copy

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- $\widetilde{m}_G=\widetilde{m}_V=\widetilde{m}_\phi$ (incorrect prescription)
- Correct prescription

$$\boxed{\widetilde{m}_G=\widetilde{m}_V+\mathrm{sign}(\widetilde{m}_G)=\widetilde{m}_\phi+2\ \mathrm{sign}(\widetilde{m}_G)\ ,}$$

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- $\tilde{m}_G = \tilde{m}_V = \tilde{m}_\phi$ (incorrect prescription).
- Correct prescription

$$\boxed{\tilde{m}_G = \tilde{m}_V + \mathrm{sign}(\tilde{m}_G) = \tilde{m}_\phi + 2\ \mathrm{sign}(\tilde{m}_G)\ ,}$$

- Simpler + more physical
- Why? Read our paper!

Overview

Gravitational Waves Double Copy

- Type N $\implies$ 1 simple pole
- Correspond to bulk-to-boundary propagator $\Psi_{\alpha_1 \ldots \alpha_{2s}}(x) \propto \frac{A_{\alpha_1} \ldots A_{\alpha_{2s}}}{(v \cdot x)^{s+\tilde{m}+1}}$

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- Applying mass prescription we get

$$f_{s=0,\widetilde{m}_\phi} = \frac{\langle \lambda B \rangle^{\widetilde{m}_\phi}}{\langle \lambda A \rangle \, [\mu C]^{\widetilde{m}_\phi + 1}} \qquad \phi_0 \sim \frac{\langle AB \rangle^{\widetilde{m}_\phi}}{(v \cdot x)^{\widetilde{m}_\phi + 1}}$$

$$f_{s=1,\widetilde{m}_\phi+1} = \frac{\langle \lambda B \rangle^{\widetilde{m}_\phi}}{\langle \lambda A \rangle \, [\mu C]^{\widetilde{m}_\phi + 3}} \qquad f_{\alpha\beta} \sim A_\alpha A_\beta \frac{\langle AB \rangle^{\widetilde{m}_\phi}}{(v \cdot x)^{\widetilde{m}_\phi + 3}}$$

$$f_{s=2,\widetilde{m}_\phi+2} = \frac{\langle \lambda B \rangle^{\widetilde{m}_\phi}}{\langle \lambda A \rangle \, [\mu C]^{\widetilde{m}_\phi + 5}} \qquad C_{\alpha\beta\gamma\delta} \sim A_\alpha A_\beta A_\gamma A_\delta \frac{\langle AB \rangle^{\widetilde{m}_\phi}}{(v \cdot x)^{\widetilde{m}_\phi + 5}}$$

- Petrov type conserved (as it should): waves double copy to waves

Black Holes Double Copy

- Type D $\implies$ 2 triple poles $\implies$ $\tilde{m} = \pm 3$
- Like BTZ Black Holes, have identification of points
  In TMG, BH are squashed AdS spacetimes in either timelike or spacelike direction with AdS_2 base
- Locally $ds^2 = \frac{9}{m_G^2 - 27\Lambda}\left[\frac{dt^2 - dx^2}{x^2} - \frac{4m_G^2}{m_G^2 - 27\Lambda}\left(dz + \frac{dt}{x}\right)^2\right]$
- Obtain all masses by varying the length of the AdS background

Black Holes Double Copy

- Applying mass prescription we get

$$f_{s=0,\widetilde{m}_\phi=1} = \frac{1}{[\mu A][\mu B]} \qquad \phi = \frac{1}{[BA]}$$

$$f_{s=1,\widetilde{m}_V=2} = \frac{1}{[\mu A]^2[\mu B]^2} \qquad f_{\alpha\delta} = 2\frac{A(x)_{(\alpha}B(x)_{\delta)}}{[AB]^3}$$

$$f_{s=2,\widetilde{m}_G=3} = \frac{1}{[\mu A]^3[\mu B]^3} \qquad C_{\alpha\beta\gamma\delta} = 6\frac{A(x)_{(\alpha}A(x)_\beta B(x)_\gamma B(x)_{\delta)}}{[BA]^5}$$

- In position space, we get $C_{\alpha\beta\gamma\delta} = \frac{3}{2}\frac{f_{(\alpha\beta}f_{\gamma\delta)}}{\phi}$
- Again Petrov type naturally preserved

Conclusion

- Generalised fluctuations and background
- Curvature splits the masses
- Double Copy of Black Holes in TMG can finally be done
- Interesting features I skipped:
  - Double Copy of CFT currents on the boundary
  - Type III works better than expected
  - The Double Copy is topological in the massless limit
- Does a corresponding correlators Double Copy exist?

Thanks for listening!