Defects in Conformal Field Theory

Their symmetries, fusion, and cusps

Based on arXiv:2406.04561 with Petr Kravchuk & Ritam Sinha
and upcoming work with Petr Kravchuk

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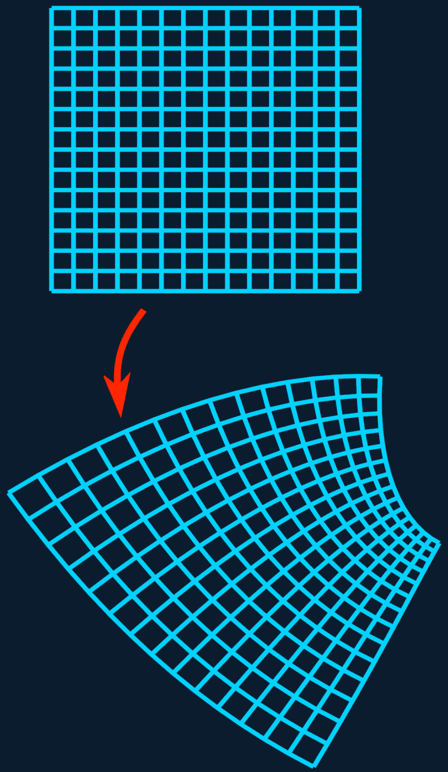
Conformal Field Theory

- A CFT is a QFT that is invariant under the conformal group, $SO_0(d+1,1)$ (if $d \geq 3$).
- A variety of applications:
  - Second-order phase transitions
  - AdS/CFT correspondence
  - String theory on the worldsheet
  - Pure maths (e.g. monstrous moonshine, Langlands program, spectral geometry, automorphic forms…)



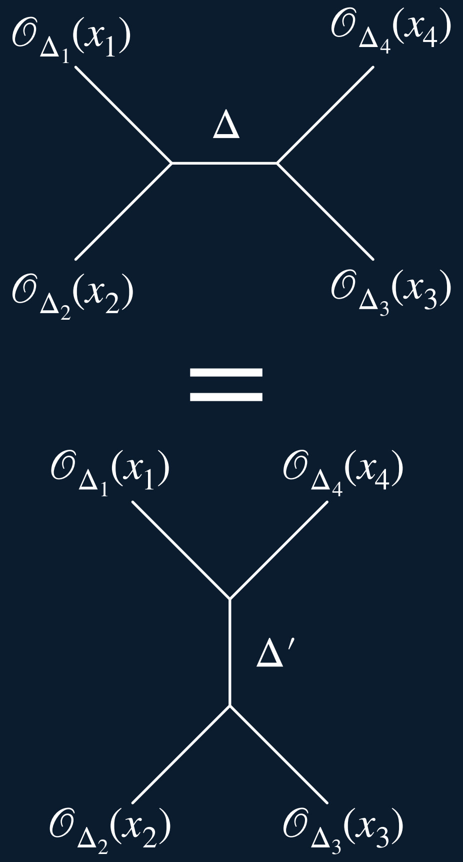
[Alexandrov, Conformal map, Wikipedia]

Constraints of Conformal Symmetry

- Scale–invariance!
- Can find a set of operators $\mathcal{O}_\Delta(x)$ such that:
    - $\mathcal{O}_\Delta(\lambda x) = \lambda^{-\Delta}\mathcal{O}_\Delta(x)$,
    - Every operator in our theory can be written in a basis of $\partial^{\mu_1}\ldots\partial^{\mu_p}\mathcal{O}_\Delta(x)$.
- These are called primary operators.
- Scale–invariance tells us that
    - $\langle \mathcal{O}_\Delta(x)\mathcal{O}_{\Delta'}(x')\rangle = \dfrac{c\delta_{\Delta\Delta'}}{|x-x'|^{\Delta}}$

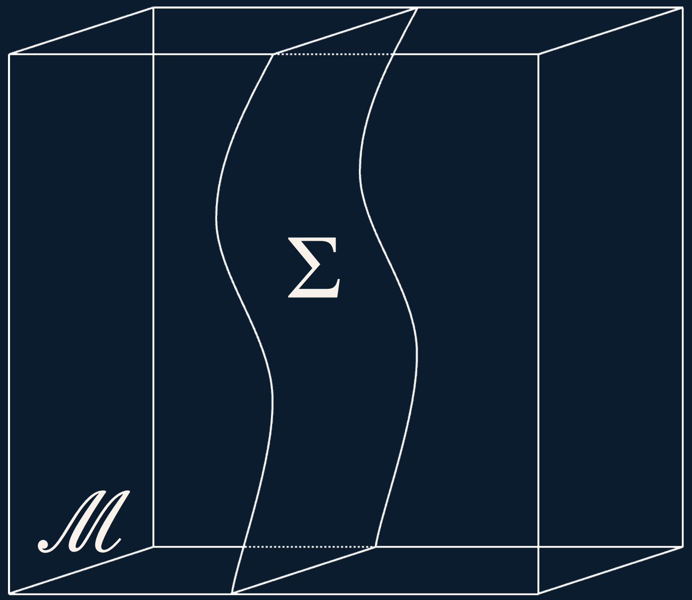
Operator–Product Expansion

- $\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2) = \sum_{\Delta_3} f_{\Delta_1\Delta_2\Delta_3} C_{\Delta_3}(x_1, x_2, x_3, \partial_{x_3})\mathcal{O}_{\Delta_3}(x_3)$ is called an operator product expansion.
- We can use this in a correlator to write an n−point function as a sum over $(n-1)$ − point functions.



Defects in Conformal Field Theory

- $S = S_{\text{CFT}} + S_{\mathscr{D}}[\Sigma]$,
  where
- $S_{\mathscr{D}}[\Sigma] = \int_{\Sigma} d^p\sigma \sqrt{\gamma}\, \mathscr{L}_{\mathscr{D}}$
- $\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle_{\mathscr{D}}$
  $= \int [D\Phi] e^{-S_{CFT}-S_{\mathscr{D}}[\Sigma]} \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)$
  $= \langle e^{-S_{\mathscr{D}}[\Sigma]}\, \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle$
- Non−local operator $e^{-S_{\mathscr{D}}[\Sigma]}$!



Why care about defects?

CFT Application	Defect Significance
Second-order phase transition	Defects can be used to model impurities in materials.
AdS/CFT	Supersymmetric Wilson loops in N=4 SYM correspond to boundary conditions to strings in AdS.
Pure maths	Wilson loops in Chern-Simons are related to knot polynomials.

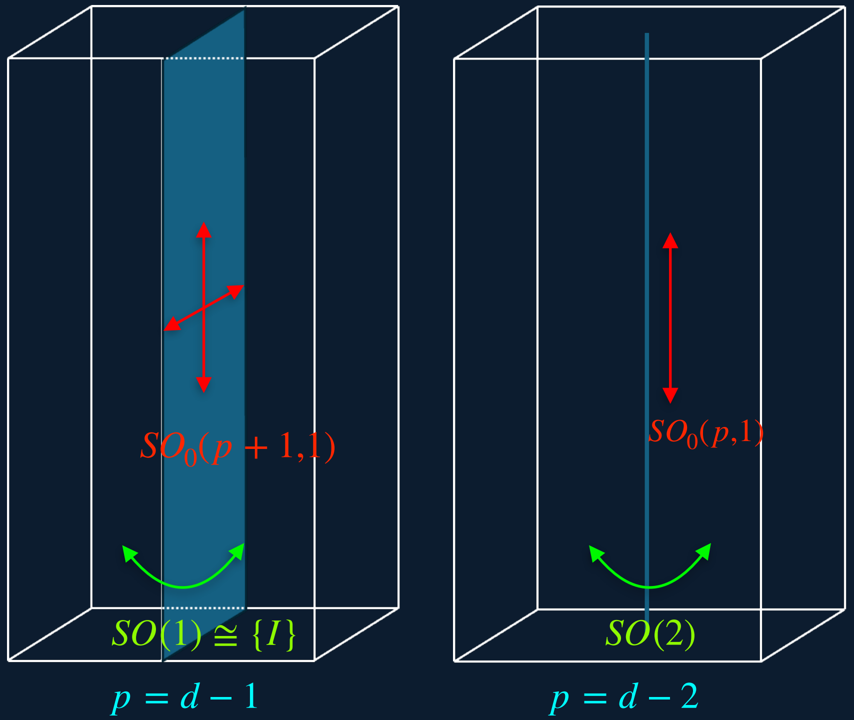
Symmetries and Conformal Defects

- Insertion of a $p-$dimensional conformal defect breaks conformal symmetry,

$$SO_0(d+1,1)$$

$$\downarrow$$

$$SO_0(p+1,1) \times SO(d-p)$$

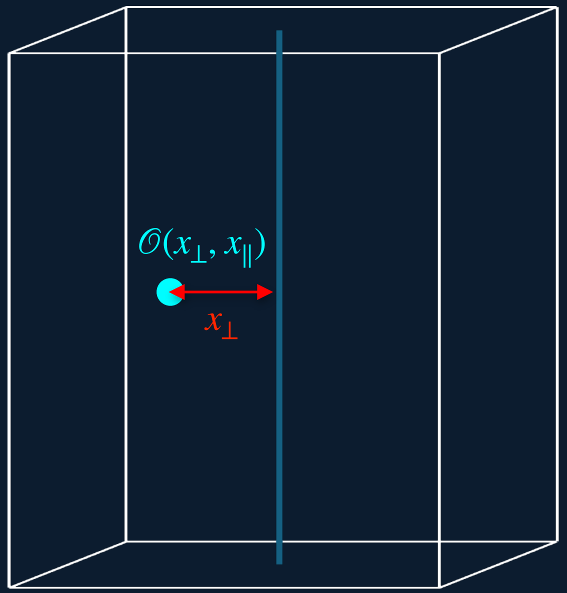


Properties of conformal defects.

- As translation symmetry is broken, we introduce a length scale into correlation functions, $x_\perp$.
- For example, $\langle \mathcal{O}(x_\perp, x_\parallel) \rangle = \dfrac{a}{|x_\perp|^\Delta}$
- Stress tensor is no longer conserved,

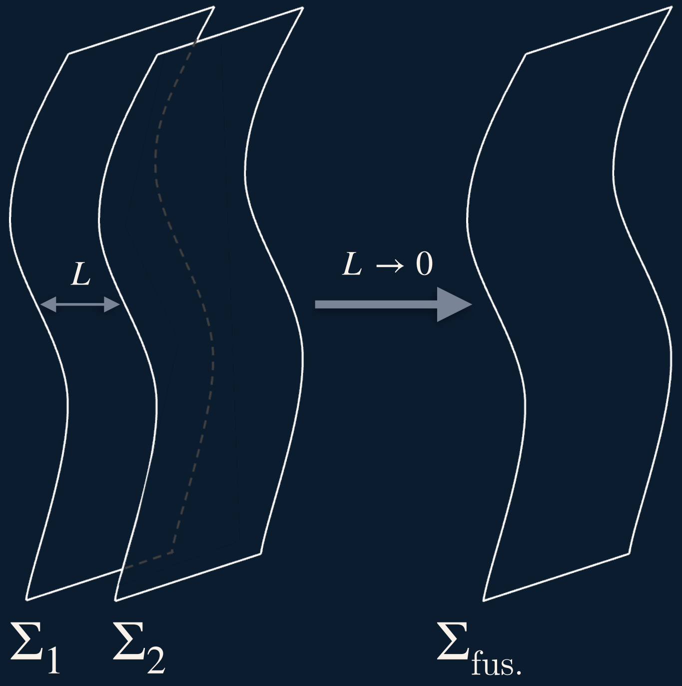
$$\partial_\mu T^{\mu i}(x) = \delta^{(d-p)}(x_\perp) D^i(x_\parallel), \qquad i = p+1, \dots, d$$

- The displacement operator $D^i(x_\parallel)$ generates translations of the defect.



Fusion of Conformal Defects

- $e^{-S_{\mathscr{D}_1}[\Sigma_1]}e^{-S_{\mathscr{D}_2}[\Sigma_2]} \xrightarrow{L\to 0} e^{-S_{\text{eff}}[\Sigma_{\text{fus.}}]}$
- Like an OPE for defects
- The effective description will be valid at scales $R \gg L$.
- View this as an RG flow, with S_{eff} providing an effective IR description of the theory



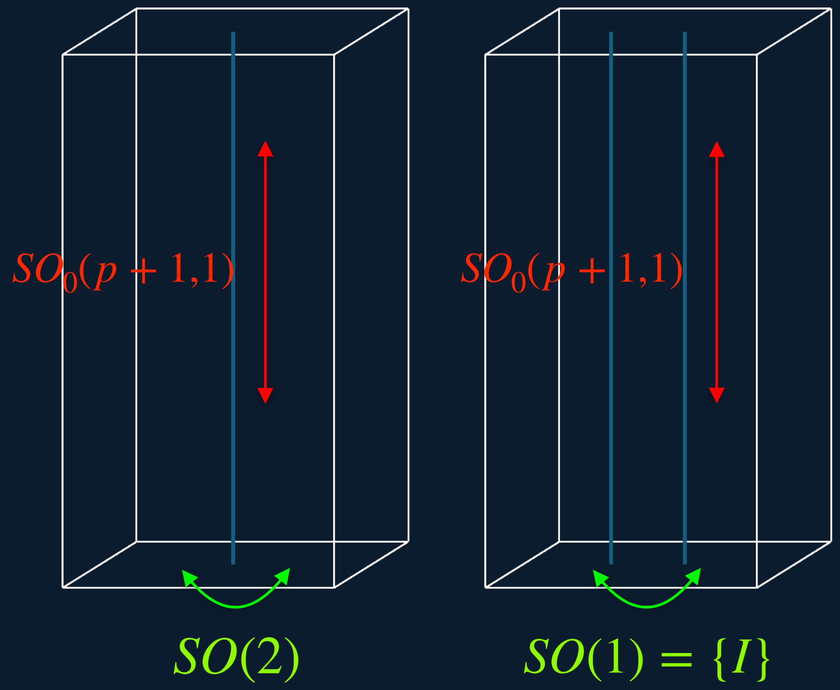
Symmetries and Conformal Defect Fusion

- Insertion of another p −dimensional conformal defect breaks the symmetry further

$$SO_0(p+1,1) \times SO(d-p)$$

$$\downarrow$$

$$SO_0(p+1,1) \times SO(d-p-1)$$



Lagrangian = bad

But sometimes,

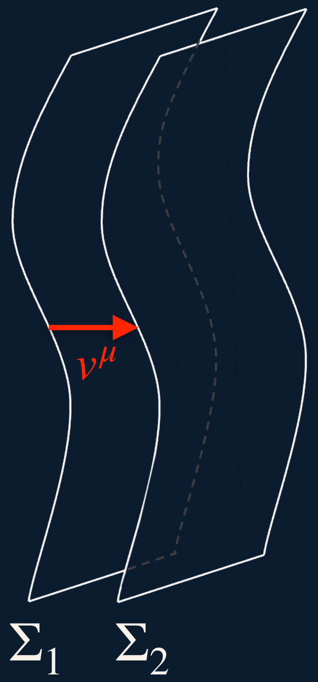
Lagrangian = ~~bad~~ necessary

Finding the effective action

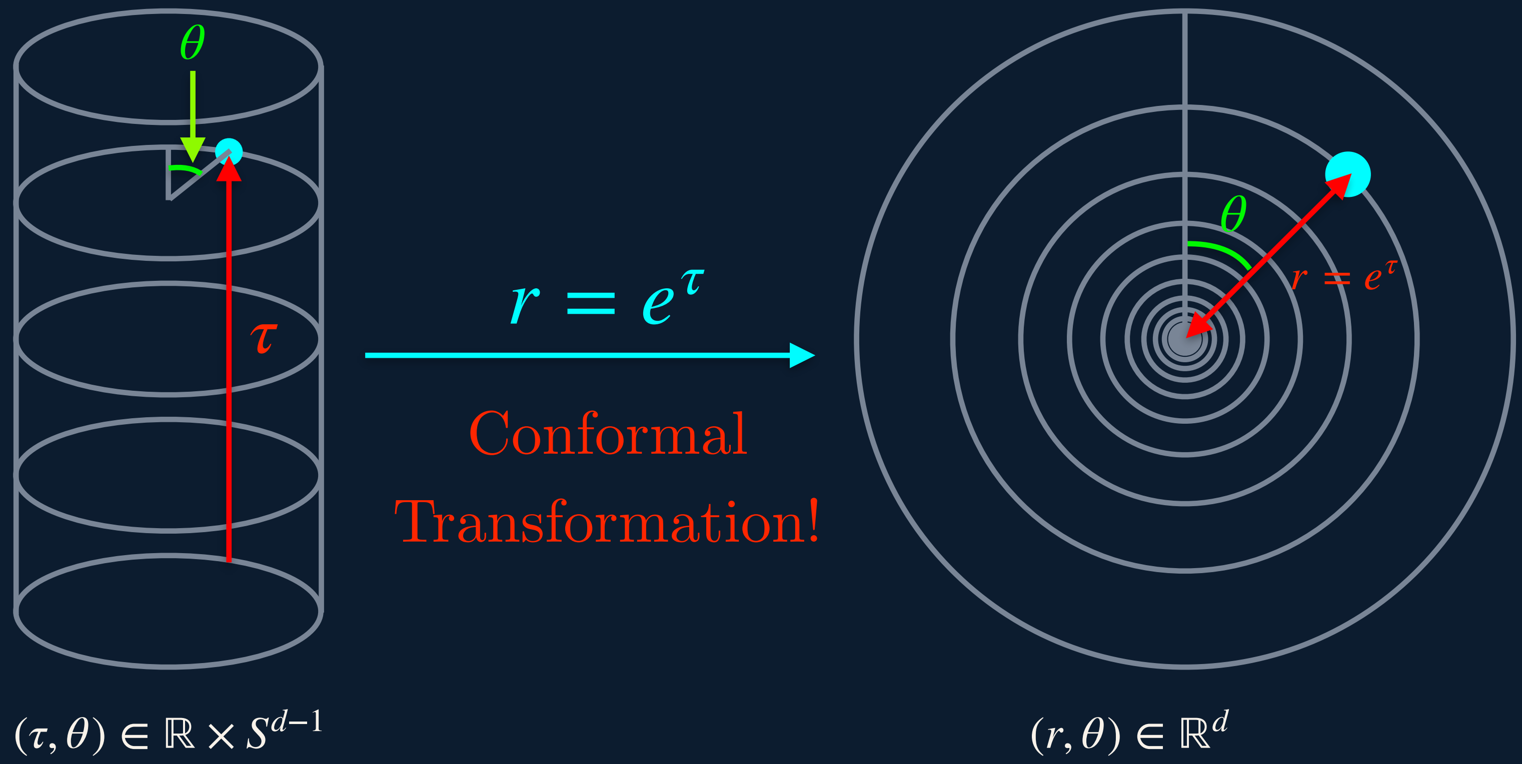
- We rescale the metric, $g^{\mu\nu}(x) \to \hat{g}^{\mu\nu}(x) = \Omega(x)^2 g^{\mu\nu}(x)$
- We choose the rescaling such that:
    - $v^\mu \hat{g}_{\mu\nu} v^\nu = 1$
    - $\hat{I\!I}^\mu = 0$, $\hat{R}^{ij} - \frac{\hat{R}}{2(d-1)}\hat{g}^{ij} = 0$, where $i, j \in \{p+1, \ldots, d\}$
- Write down the most general local effective action, compatible with symmetry properties

$$S_{\text{eff}} = \int d^p\sigma \sqrt{\hat{\gamma}} \sum_\Delta \lambda_\Delta \mathcal{O}_\Delta$$

- Use theory-specific techniques to calculate the undetermined coefficients (e.g. quantum spectral curve, fuzzy sphere regularization).



Exponential Map



Defect Fusion and cusps

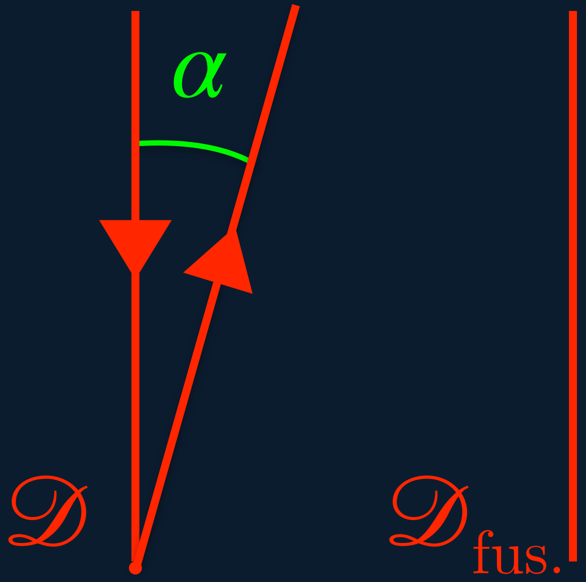
Parallel Lines
on $\mathbb{R} \times S^{d-1}$

Line with a
cusp on $\mathbb{R}^d$

An Example — Cusped Wilson Line in $\mathscr{N} = 4$ SYM

- $W[\mathscr{D}] = P\left[\exp\left(\left(\int d\tau i(A_\mu \dot{x}^\mu + \hat{n}\cdot\phi)\right)\right)\right]$
- $W[\mathscr{D}] \xrightarrow{\alpha\to 0} \exp(S_{\text{eff}}[\mathscr{D}_{\text{fus.}}])$, where
- $S_{\text{eff}}[\mathscr{D}_{\text{fus}}] = \int d\sigma\sqrt{\hat{\gamma}}\left(a_0 + a_1\hat{R}\ldots\right)\mathbf{1}(\sigma) + \ldots$

$$= \int d\sigma\left(-\frac{a}{\alpha} - b\alpha + \ldots\right)\mathbf{1}(\sigma) + \ldots$$



Conclusions

- Can take a theory with 2 conformal defects, and view it as a theory with 1 conformal defect.
- Cool consequences of this work include:
  - Can do calculations in DCFTs with 2 defects, or a defect with a cusp
  - Operator asymptotics in DCFTs with one defect
  - Some cool geometry
  - Some applications to "spinning DCFTs"

Thank you for listening!

Any questions?

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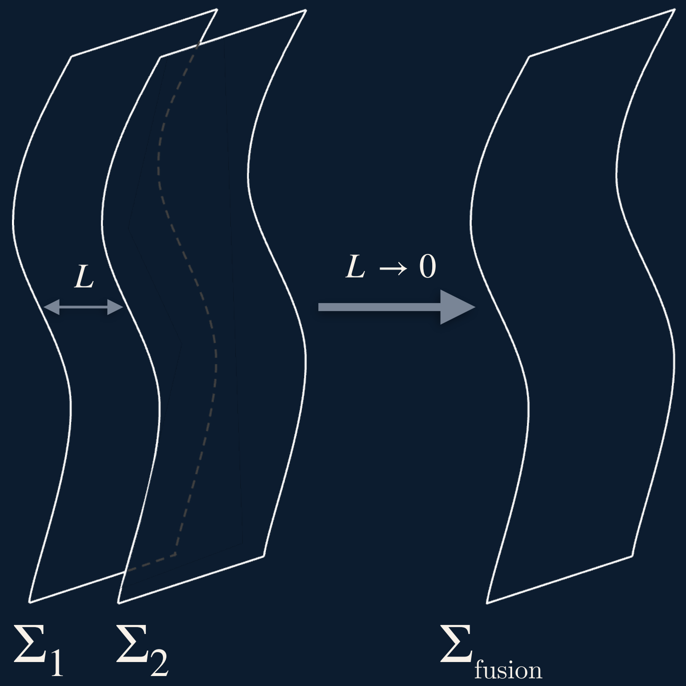

Classifying Weyl Anomalies

- The Weyl anomaly is the variation of the effective action $W = -\log Z$ under an infinitesimal Weyl transformation $g_{\mu\nu}(x) \to (1 + 2\omega(x))g_{\mu\nu}(x)$.
- We use the following algorithm to find the most general possible Weyl anomaly:
    1. Propose the most general Weyl anomaly $\delta_\omega W$ that is consistent with dimensional analysis; $SO(d-p)$
    2. Impose Wess–Zumino consistency, i.e. since Weyl transformations commute, $\delta_{\omega_1}\delta_{\omega_2}W = \delta_{\omega_2}\delta_{\omega_1}W$;
    3. Find the part of this anomaly that is scheme-independent.
- [Chalabi, Herzog, O'Bannon, Robinson, Sisti (2022)]

Weyl Anomalies for a Spinning Line Defect in 3D

- We take a line defect Σ in a 3-dimensional bulk $\mathscr{M}_3$, and along the defect, we introduce a unit vector field normal to the defect, $v^\mu(\sigma)$ to parameterize the breaking of SO(2) to the trivial subgroup.
- We also introduce the unit vector field tangent to the defect $x^\mu(\sigma)$, and $w_\mu(\sigma) = \sqrt{g}\varepsilon_{\mu\nu\rho}v^\nu(\sigma)x^\rho(\sigma)$ (Fig. 5.).
- We find that the only scheme-independent term that can appear in a Weyl anomaly is $\delta_\omega W = a\int_\Sigma d\sigma\sqrt{\bar{g}}\,\left(x^\nu(\sigma)w_\mu(\sigma)\,\bar{\nabla}_\nu v^\mu(\sigma)\right)\,\omega(\sigma).$
- If we consider the Weyl anomaly associated with uniform rescalings, we find that the Weyl anomaly is proportional to the self-linking number of the defect,
  - $\dfrac{1}{4\pi}\int_\Sigma d\sigma\sqrt{\bar{g}}\,\left(x^\nu(\sigma)w_\mu(\sigma)\,\bar{\nabla}_\nu v^\mu(\sigma)\right).$
- For a surface defect in a 4-dimensional bulk, and we find 12 scheme-independent terms.