Replica analysis of Entanglement Properties and Conditions for Islands

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Outline

1. Entangement Entropy
2. Motivation: Why study EE?
3. Main motivation: Black hole information, islands
4. Conditions for island
5. How to calculate scaling of EE on static spacetimes
6. Results
7. Future work

von Neumann or Entanglement Entropy: Density matrix $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

$$S = -\sum_i p_i \log(p_i) = -\mathrm{Tr}(\rho \log \rho)$$

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Entangled pure state in $\mathcal{H}_{AB}$ imply mixed states in $\mathcal{H}_A$ and $\mathcal{H}_B$

$S_{AB} = 0$, $S_A = S_B$ and is non zero. Hence captures both non local entanglement and local correlations (if mixed in $\mathcal{H}_{AB}$). Also notice $S(U\rho U^{-1}) = S(\rho)$.

Motivation: Why EE?

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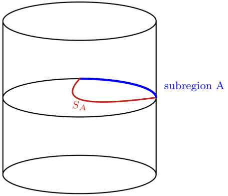
AdS/CFT - Ryu Takayanagi:

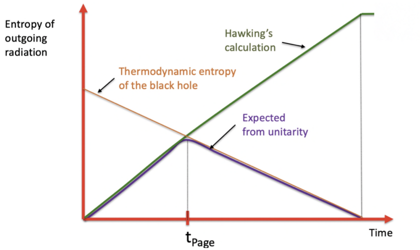
$$S_A = \min_{R \sim A} \frac{Area(R)}{4G}$$

Ryu, Takayanagi PRL 96 (2006)



$$S_{gen} = \frac{\text{Area of horizon}}{4G} + S_{rad}$$

Start with black hole in pure state. As it evaporates it loses area $\frac{dM}{dT} \sim \sigma A T^4 \sim \frac{1}{M^2}$. After a time $S_{rad} > S_{Area}$ and there are not enough dof in BH to purify radiation. Non unitary!



Unitary evolution follows page curve

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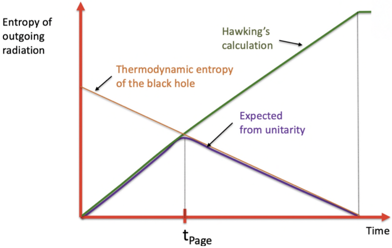

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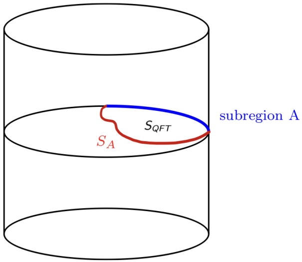
Quantum extremal surface and islands

$$S = \text{min ext}\left(S_{QFT} + S_{area}\right)$$

(Wall, Engelhardt (2014))

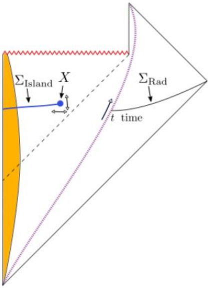


Island

Important in black hole evaporation. We need to minimize generalised entropy for candidate subregions. The presence of a QES not at asymptotic infinity implies that the complementary subregion has a region disconnected from the asymptotic boundary - **Island**. *(Penington, Shenker, Stanford, Yang; Almheri, Mahajan, Maldacena, Zhao...)*

Correlation between island and region between QES and asymptotic boundary brings down EE at later times and preserves unitarity.

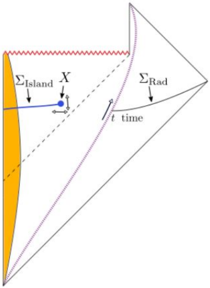
This is a statement about how the EE of the black hole/ radiation (provided we started with a pure black hole) must behave to preserve unitarity. What does this translate to for the QFT spectrum, correlation functions of the radiation?



Conditions for Island

$$ds_d^2 = -g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + r^2 d\Omega_{d-2}^2$$

subregion: $r_1 \leq r \leq r_2 \; \forall \; \Omega$, t constant.



Conditions for island

$$\frac{d}{dr_2} S_{QFT} + \frac{d}{dr_2} S_{area} = 0$$

at finite $r_2 = r_0$

$$\frac{d^2}{dr_2^2} S_{QFT}\Big|_{r_2=r_0} \geq -\frac{d^2}{dr_2^2} S_{area}\Big|_{r_2=r_0}$$

S_{QFT} has information about the QFT spectrum and S_{area} is purely geometric. Hence this imposes constraints on the QFT spectrum in terms of background metric parameters.

Gap in literature: Calculating S_{QFT} and its $r-$scaling is hard! Known results in

- 1+1 dim CFTs *(Holzhey, Larsen, Wilczek (1994); Calabrese, Cardy (2004))*
- 2+1 dim *(Casini, Huerta)*
- Universal features, results restricted to CFTs in flat, vacuum state
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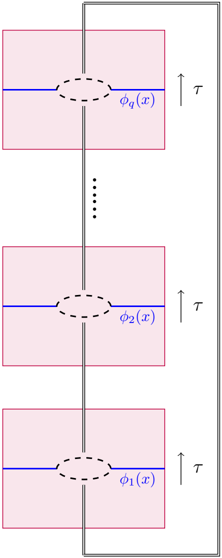
We need EE on static background

$$S_{QFT} = \lim_{q\to 1} S_q = \lim_{q\to 1} \frac{1}{1-q} \log(Tr_A \rho_A^q)$$
$$= \lim_{q\to 1} \frac{1}{1-q} (W_q - qW_1)$$

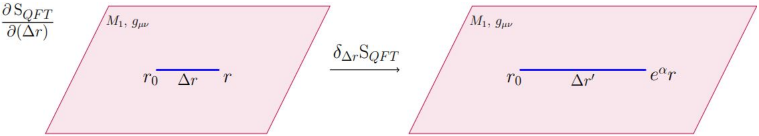
$$Tr(\rho_A^q) = \sum_{i=1}^{q} \sum_{\phi_i(x)}$$



tt

Scaling of EE on static background

We need scaling of EE on static background: Restrict to CFTs



$$\delta(\Delta r)\frac{\delta}{\delta(\Delta r)}S_{QFT} = -\Delta_{\Lambda} S_{QFT}$$

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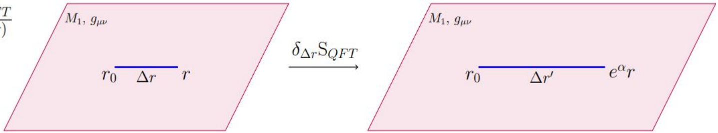
$\frac{\partial S_{QFT}}{\partial(\Delta r)}$



$$\delta(\Delta r)\frac{\delta}{\delta(\Delta r)}S_{QFT} = -\Delta_\Lambda S_{QFT}$$

$$= -\int d^d x \left(\delta_\Lambda g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} + \sigma(x)\beta^i(\lambda)\frac{\delta}{\delta\lambda^i(x)}\right)\left(\lim_{q\to 1}\frac{1}{(1-q)}(W_q - qW_1)\right)$$

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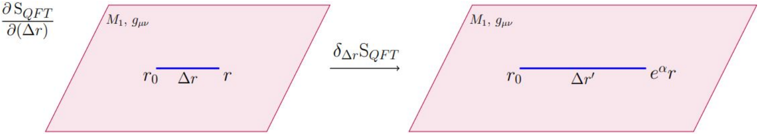

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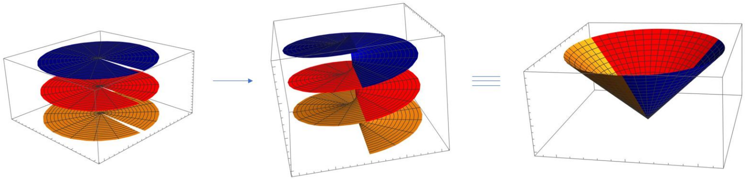
$$= \lim_{q\to1}\frac{1}{(q-1)}\left(\frac{1}{2}\int_{M_q} d^d x\sqrt{g_q}\left(\delta_{\Delta r} g^{(q)}_{\mu\nu}\langle T^{\mu\nu}\rangle_q\right) - \frac{1}{2}q\int_{M_1} d^d x\sqrt{g}\,(\delta_{\Delta r} g_{\mu\nu}\langle T^{\mu\nu}\rangle)\right)$$

Under more general $x^\mu \to x'^\mu = x^\mu + \xi^\mu(x^\mu)$ we have $\delta_{\xi^i} g^{(q)}_{\mu\nu} = \mathcal{L}_{\xi^i} g^{(q)}_{\mu\nu}$

Replica metric $g_{\mu\nu}^{(q)}$

Consider spatial entangling region $x = x_0$ on a $\tau = \tau_0$ slice.



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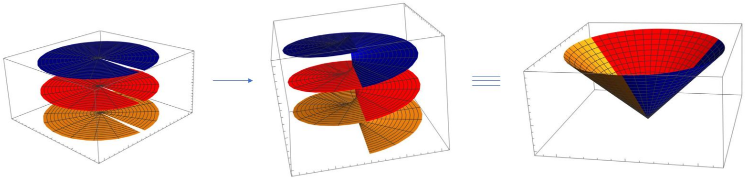

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We use coordinates $x - x_0 = \rho\cos(\psi)$, $\tau - \tau_0 = \rho\sin(\psi)$; $\rho \in [0, \infty)$, $\psi \in [0, 2\pi q)$

$$ds^2_{M_q} \text{ around boundary} = g^q_{\mu\nu} dx^\mu dx^\nu|_{\text{around } \Sigma}$$
$$= U(\rho, a) d\rho^2 + \rho^2 d\psi^2 + (\gamma_{ij} + 2\rho^p c^{1-p}\cos(\psi) K_{1ij} + 2\rho^p c^{1-p}\sin(\psi) K_{2ij}) dy^i dy^j$$
$$+ A_i \varepsilon_{ac} x^a dx^c dy^i + \mathcal{O}\left(x^2\right)$$

$U|_{\rho=0} \to q^2$, $U|_{\rho>>a} \to 1$

Replica Stress Tensor $\langle T^{\mu\nu}\rangle_q$ and (q-1) expansion

The EE is equal to the difference $W_q - qW_1$ at $O(q-1)$. We consider a systematic $O(q-1)$ expansion which reveals the trivial terms in EE scaling. This picks up contributions only due to the localised conical singularity(ies).

$$g^{(q)}_{\mu\nu} = g_{\mu\nu} + (q-1)\left(g^{[1]}_{\mu\nu} = (\partial_q g^{(q)}_{\mu\nu})|_{q=1}\right) + \mathcal{O}\left((q-1)^2\right),$$

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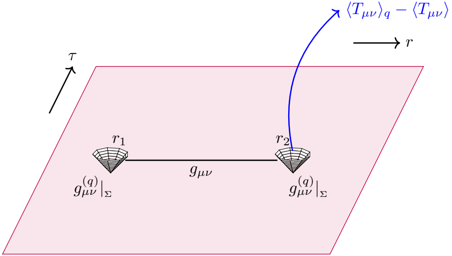
Replica stress tensor

$$\nabla^{(q)}_\mu \langle T_{\mu\nu} \rangle_q = 0$$

$$\langle T^\mu_{\ \mu} \rangle_q = \mathcal{A}[g^{(q)}_{\mu\nu}]$$

The $(d+1)$ equations completely determine $\langle T_{\mu\nu} \rangle^{[1]}$ in terms of $\langle T_{\mu\nu} \rangle$ for states with same symmetry as the static background.



Results for EE scaling in higher dim

EE scaling: General d, subregion, state

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Vacuum and thermal state (odd dim), planar boundary in flat space

$$\langle T^{\mu}{}_{\nu}\rangle^{vacuum}_{q,\ \mathrm{a=0}} = \frac{C_d(q)}{\rho^d}\,\mathrm{diag}((1-d), 1, 1, 1, ..., 1)$$

$$\Delta r\frac{\partial S^{\text{finite, thermal}}_{\text{d dim}}}{\partial \Delta r} = \pi\, Vol(M_{d-2})\left\{\frac{C_d^{[1]}\,2^{d-2}}{(\Delta r)^{d-2}} + \frac{p}{4}(\Delta r)^2\right\}$$

We can now use this to get explicit conditions for islands.

QES, Islands - Some examples

Since scaling of S_{QFT} with region depends on $\langle T_{\mu\nu} \rangle$ for QFTs on static backgrounds, therefore demanding island imposes constraints on $\langle T_{\mu\nu} \rangle$.

Examples: QES in flat!

Thermal state: Odd dimension

$$C_d^{[1]} \left(\frac{2^{d-2}}{(\Delta x)^{d-2}} \right) + \frac{p}{4} (\Delta x)^2 = 0 \implies x_2 = x_1 + 2 \left(\frac{-C_d^{[1]}}{p} \right)^{1/d}$$

Vacuum: Spherical boundary - 4 dim

$$\frac{4a}{\Delta r} + 8\pi r_2 = 0 \implies r_2 = r_1 \pm \frac{1}{2} \left(r_1^2 - \frac{2a}{\pi} \right)^{1/2}$$

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- Explicit conditions on QFT spectrum for islands by making use of the scaling result. Interpret these conditions in the holographic dual to black holes.
- Use our result to test and refine the Quantum focussing conjecture.

Thank you for your attention!