Compact Two-Loop QCD Corrections for Vjj Production in pp Collisions

Giuseppe De Laurentis
University of Edinburgh

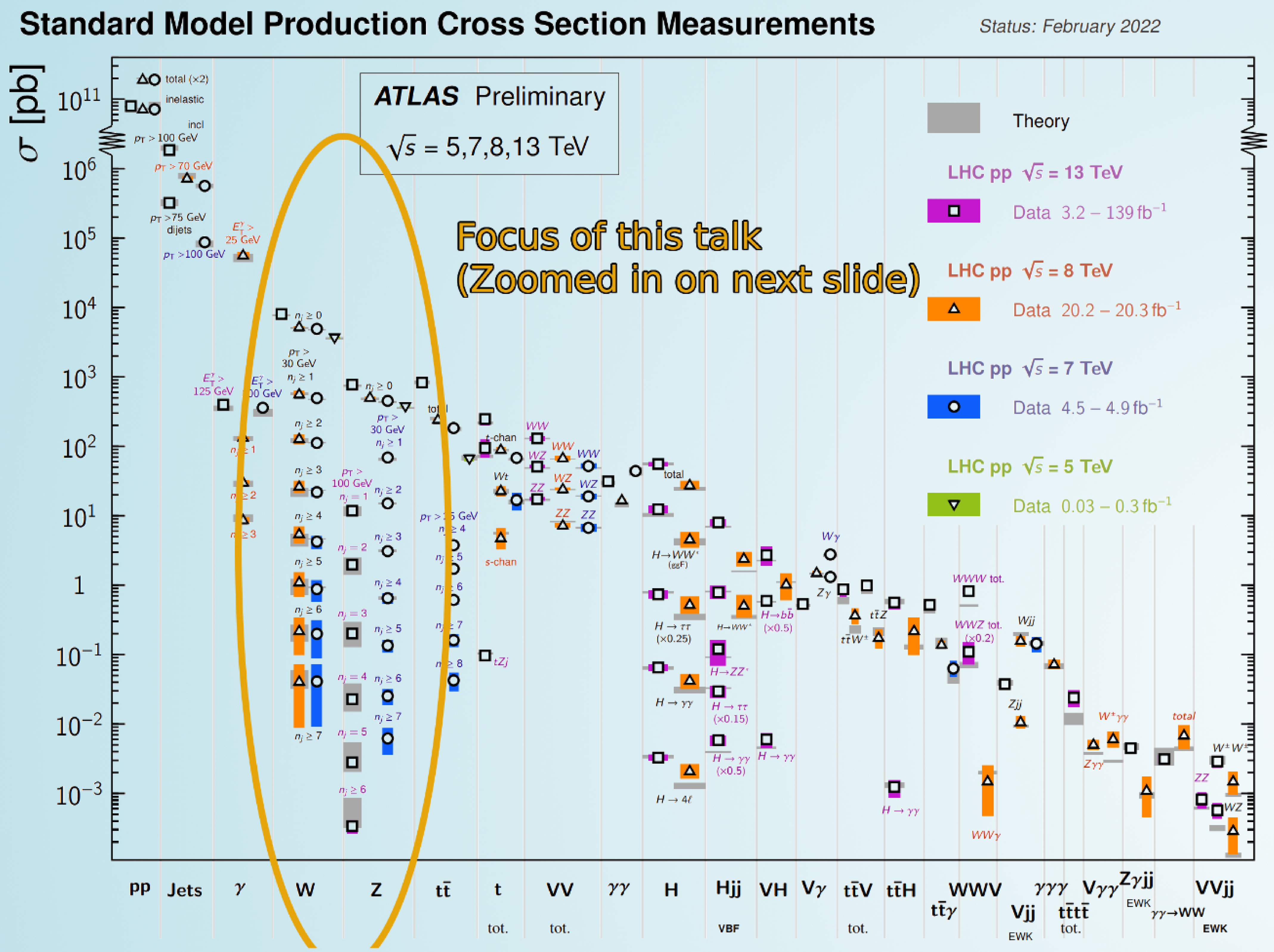
arXiv:2503.10595
(GDL, H. Ita, B. Page, V. Sotnikov)

SM@LHC 2025
Durham

Find these slides at gdelaurentis.github.io/slides/sm@lhc_apr2025

INTRODUCTION

$V + n$-jet Cross Sections at the LHC



3	2023 [6]	?	?
2	2007 [4]	2021 [5]	?
1	1981 [1]	1997 [2]	2008 [3]
Loops ↑ **Jets →**	1	2	≥ 3

Analytic | Numeric | Analytic (LCA) | Unknown

[1] Ellis, Ross, Terrano; [2] Bern, Dixon, Kosower; [3] BlackHat; OpenLoops; [4] Gehrmann-De Ridder, Gehrmann, Glover, Heinrich; [5] Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov + This talk; [6] Gehrmann, Jakubčík, Mella, Syrrakos, Tancredi

Observations at the LHC are beautifully predicted by the Standard Model

$$\sigma_{2\to n-2} = \sum_{a,b} \int dx_a dx_b f_{a/h_1}(x_a, \mu_F)\, f_{b/h_2}(x_b, \mu_F)\, \hat{\sigma}_{ab\to n-2}(x_a, x_b, \mu_F, \mu_R)\,,$$

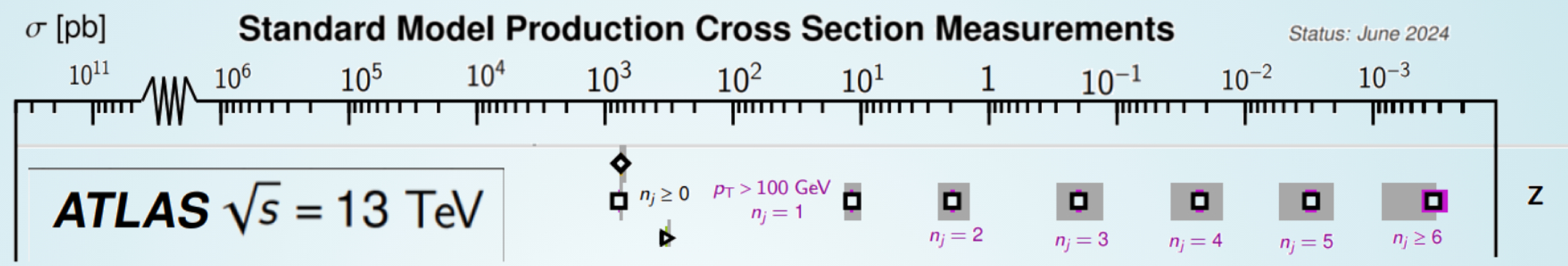
$$\hat{\sigma}_n = \frac{1}{2\hat{s}} \int d\Pi_{n-2}\, (2\pi)^4 \delta^4\Big(\sum_{i=1}^{n} p_i\Big)\, \overline{|\mathcal{A}(p_i, h_i, a_i, \mu_F, \mu_R)|^2}\,.$$

at least to the extent with which we can compute $\mathcal{A} = \mathcal{A}^{(0)} + \alpha_{(s)}\mathcal{A}^{(1)} + \alpha_{(s)}^2\mathcal{A}^{(2)} + \ldots$

Precision Physics Requires Compact Amplitudes

- Theoretical uncertainties already larger than experimental ones, especially at higher points

$\sigma^{\text{tot.}}_{pp \to Z + n\, j}$:

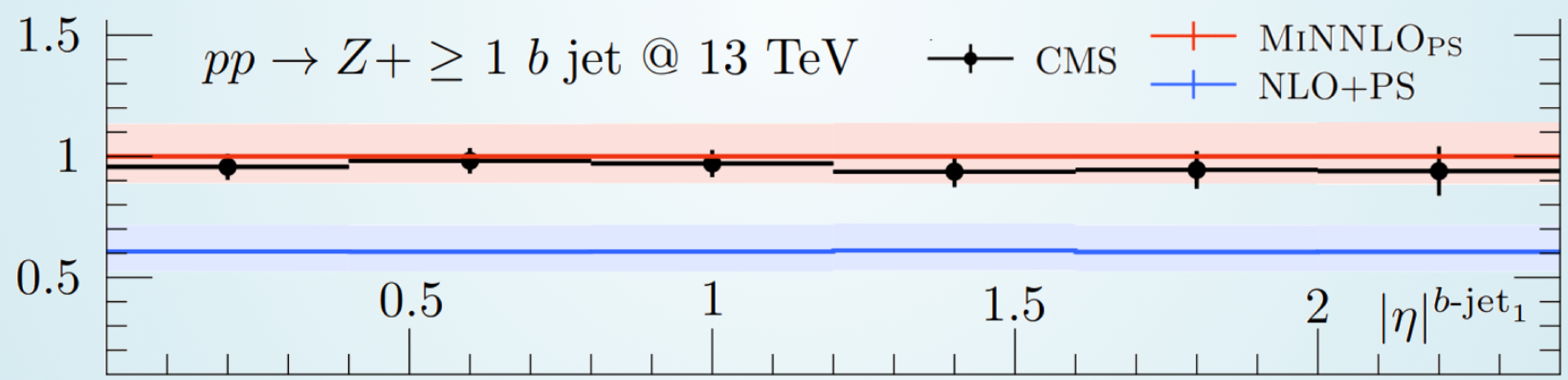


ATLAS Collab. '24

- NNLO is essential for agreement with experiment, e.g.

Mazzitelli, Sotnikov, Wiesemann '24

$\frac{d\sigma_{pp \to Z + \geq 1\, b\text{ jet}}}{d|\eta|^{b-\text{jet}_1}}$:



- Besides this, only two other cross-section studies at NNLO, only for the process $q\bar{q}' \to Wb\bar{b}$

Hartanto, Poncelet, Popescu, Zoia '22; Buonocore, Devoto, Kallweit, Mazzitelli, Rottoli, Savoini '22;

- Phenomenology can be hindered by complexity of results. It's hard to do Monte Carlo integration and verify IR cancellations when you have to evaluate >1GB of files in higher precision.

NUMERICAL COMPUTATION

Partial Amplitudes & Finite Remainders

- Amplitude (integrands) can be written as (for a suitable choice of master integrals)

$$A(\lambda,\tilde\lambda,\ell) = \sum_{\substack{\Gamma,\\ i\in M_\Gamma\cup S_\Gamma}} c_{\Gamma,i}(\lambda,\tilde\lambda,\epsilon)\,\frac{m_{\Gamma,i}(\lambda\tilde\lambda,\ell)}{\prod_j \rho_{\Gamma,j}(\lambda\tilde\lambda,\ell)} \xrightarrow{\int d^D\ell} \sum_{\substack{\Gamma,\\ i\in M_\Gamma}} \frac{\sum_{k=0}^{\text{finite}} c^{(k)}_{\Gamma,i}(\lambda,\tilde\lambda)\,\epsilon^k}{\prod_j(\epsilon - a_{ij})}\, I_{\Gamma,i}(\lambda\tilde\lambda,\epsilon)$$

  - Γ: topologies
  - M_Γ: master integrands
  - S_Γ: surface terms

- <u>All physical information</u> is contained in the *finite remainders*, at two loops

Weinzierl ('11)

$$\underbrace{\mathcal{R}^{(2)}}_{\text{finite remainder}} = \mathcal{A}^{(2)}_R \underbrace{- \; I^{(1)}\mathcal{A}^{(1)}_R \; - \; I^{(2)}\mathcal{A}^{(0)}_R}_{\text{divergent + convention-dependent finite part}} + \mathcal{O}(\epsilon)$$

Catani ('98)
Becher, Neubert ('09)
Gardi, Magnea ('09)

$\mathcal{A}^{(1)}_R$ to order ϵ^2 is still needed to build $\mathcal{R}^{(2)}$, but there is no real reason to reconstruct it.

- Finite remainder as a weighted sum of *pentagon functions*

Chicherin, Sotnikov ('20)
Chicherin, Sotnikov, Zoia ('21)

$$\mathcal{R}(\lambda,\tilde\lambda) = \textstyle\sum_i r_i(\lambda,\tilde\lambda)\, h_i(\lambda\tilde\lambda)$$

- Goal: reconstruct $r_i(\lambda,\tilde\lambda)$ from numerical samples in a field $\mathbb{F}$

$\mathbb{F}_p$: von Manteuffel, Schabinger ('14); Peraro ('16)
$\mathbb{C}$: GDL, Maitre ('19); $\mathbb{Q}_p$: GDL, Page ('22)

Setting up the Calculation

○ Original computation [1] was performed with `Caravel`

$$\sum_{\text{states}} \prod_{\text{trees}} A^{\text{tree}}(\lambda, \tilde{\lambda}, \ell)\Big|_{\text{cut}_\Gamma} = \sum_{\substack{\Gamma' \geq \Gamma, \\ i \in M'_\Gamma \cup S'_\Gamma}} c_{\Gamma',i}(\lambda, \tilde{\lambda}) \frac{m_{\Gamma',i}(\lambda\tilde{\lambda}, \ell)}{\prod_{j \in P_{\Gamma'}/P_\Gamma} \rho_j(\lambda\tilde{\lambda}, \ell)}\Bigg|_{\text{cut}_\Gamma}$$

C++ code

Caravel

Abreu, Dormans, Febres Cordero, Ita Kraus, Page, Pascual, Ruf, Sotnikov ('20)

★ Numerical Berends-Giele recursion for LHS, solve for coeffs. in RHS.

★ IBP reduction = decomposition on RHS, $m_{\Gamma,i} \in M_\Gamma \cup S_\Gamma$

○ This computation started from the ancillaries files of [1] Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov

1. Wrote a Python script to split the 1.4 GB ancillaries into >10k files
2. Compile into 18.2 GB of C++ binaries (for reference `Caravel` compiles into approx. 5 GB)
3. Obtain $\mathbb{F}_p$ evaluations of the form factors (each takes approx. 1 sec per point)
4. Recombine triplets of form factors into helicity amplitudes

○ Assemble helicity amplitudes into 3 categories: $\mathcal{R}^{\text{NMHV}}_{\bar{q}Q\bar{Q}qV}$, $\mathcal{R}^{\text{MHV}}_{\bar{q}ggqV}$, $\mathcal{R}^{\text{NMHV}}_{\bar{q}ggqV}$

Guiding Principles

- Amplitude should be gauge and Lorentz invariant, and little group covariant
  - ✗ gauge dependence, e.g. through reference vectors
  - ✗ tensor decompositions $\epsilon_\mu T^\mu$, polarizations are needed for simplifications
  - ✓ $\epsilon_\mu \to \epsilon_{\alpha\dot{\alpha}}$, $P^\mu \to \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$; all $\alpha, \dot{\alpha}$ indices contracted
- The singularity structure should be manifest in $\mathbb{C}$ (exprs will then be better behaved in $\mathbb{R}$ too)
  - ✗ Rational reparametrisations of the kinematics change the denominator structure
  - ✗ If a function is neither even nor odd, forcing the split misses cancellations
  - ✓ Chiral cancellations yield true Least Common Denominator
  - ✓ Work off the real slice: $P^\mu \in \mathbb{C}^4$, $\lambda_\alpha \neq \tilde{\lambda}^\dagger_{\dot{\alpha}}$. In practice, $P^{\mu=y} \in i\mathbb{Q} \Rightarrow \lambda_\alpha \in \mathbb{F}_p$ or $\mathbb{Q}_p$
- Focus only on final physical expressions
  - ✗ Unphysical intermediate steps may be unnecessarily complicated
  - ✓ Bypass all intermediate steps with numerical evaluations

ANALYTIC & GEOMETRIC STRUCTURE

see algebro-geometric formulation in:

GDL, Page (JHEP 12 (2022) 140)

Least Common Denominator

(i.e. geometry at codimension one)

- Polynomials belong to the the covariant quotient ring of spinors,

$$R_n = \mathbb{F}\big[|1\rangle, [1|, \ldots, |n\rangle, [n|\big] / \big\langle \sum_i |i\rangle[i| \big\rangle$$

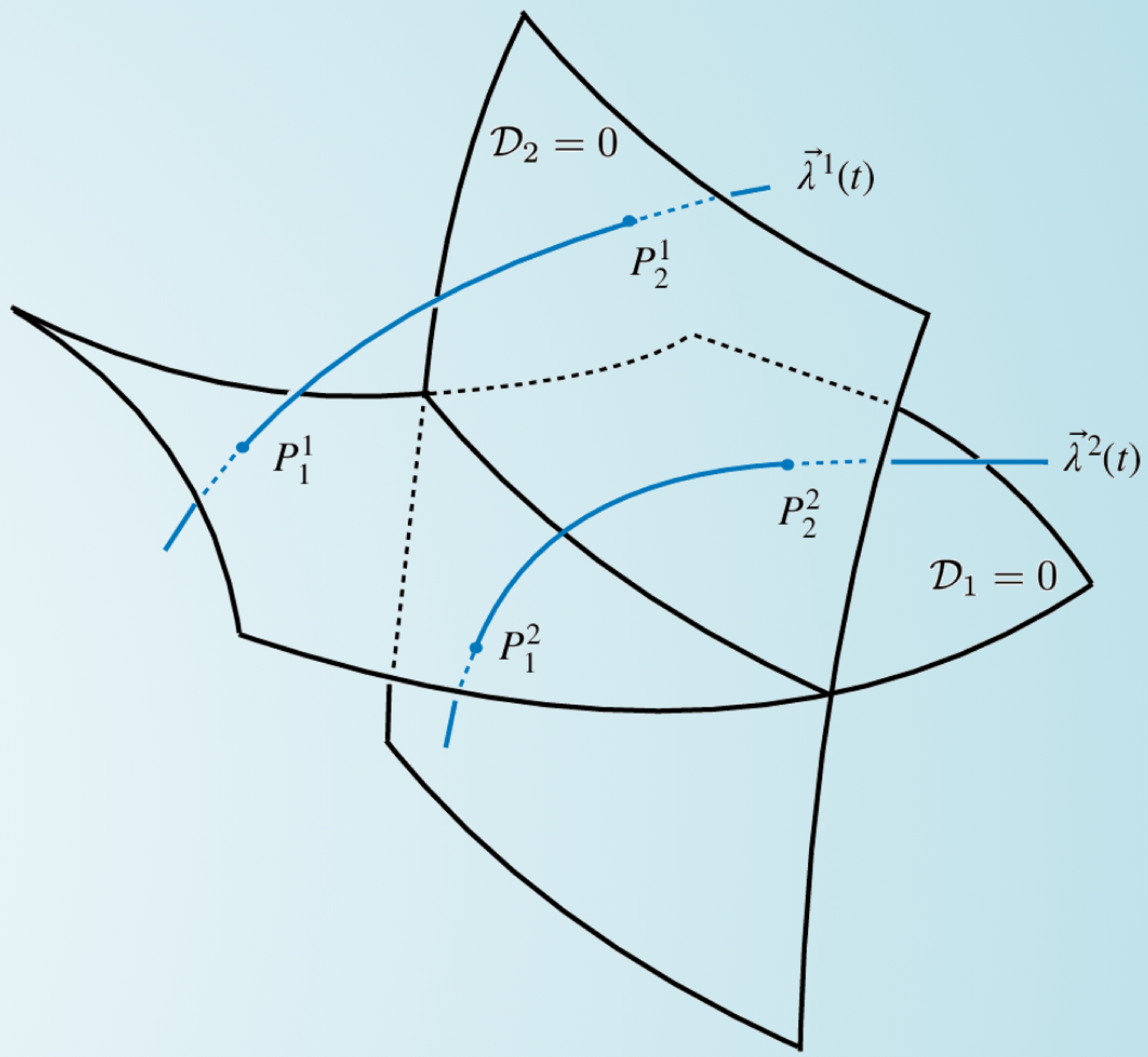
- The rational function r_i belong to the field of fractions of R_n,

$$r_i(|i\rangle, [i|) = \frac{\mathcal{N}(|i\rangle, [i|)}{\prod_j \mathcal{D}_j^{q_{ij}}(|i\rangle, [i|)}$$

we obtain q_{ij} from a univariate slice $\vec{\lambda}(t)$.

- The $\mathcal{D}_j$ are related to the letters of the symbol alphabet

Abreu, Dormans, Febres Cordero, Ita, Page ('18)



Space has dimension $4n-4$,
$\mathcal{D}_j = 0$ have dimension $4n-5$,
$\vec{\lambda}(t)$'s have dimension 1.

$$\{D_j\} \subset \bigcup_{\sigma \in \mathrm{Aut}(R_6)} \sigma \circ \big\{\langle 12\rangle, \langle 1|2+3|1], \langle 1|2+3|4], s_{123}, \Delta_{12|34|56}, \langle 3|2|5+6|4|3] - \langle 2|1|5+6|4|2]\big\}$$

New letter!

Poles & Zeros ⇔	Irreducible Varieties ⇔	Prime Ideals
Physics	*Geometry*	*Algebra*

Basis Change from Pole Residues

- Change basis from a subset of the pentagon coefficients $r_{i\in\mathcal{B}}$ to $\mathbb{Q}$-linear combinations $\tilde{r}$,

$$R = r_j h_j = r_{i\in\mathcal{B}} M_{ij} h_j = \tilde{r}_i \, O_{ii'} M_{i'j} \, h_j \, , \qquad O_{ii'}, M_{i'j} \in \mathbb{Q}$$

Rational-Function	Reference [6] (Mandelstam Variables)		Max Spinor-Helicity Ansatz Size (LCD)	
Vector Space	Common Denominator	Partial Fractioning	Before Basis Change	After Basis Change
$B_{\mathrm{g}}(1^+, 2^-, 3^+, 4^-)$	50 000 k	1 100 k	24 800 k	620 k

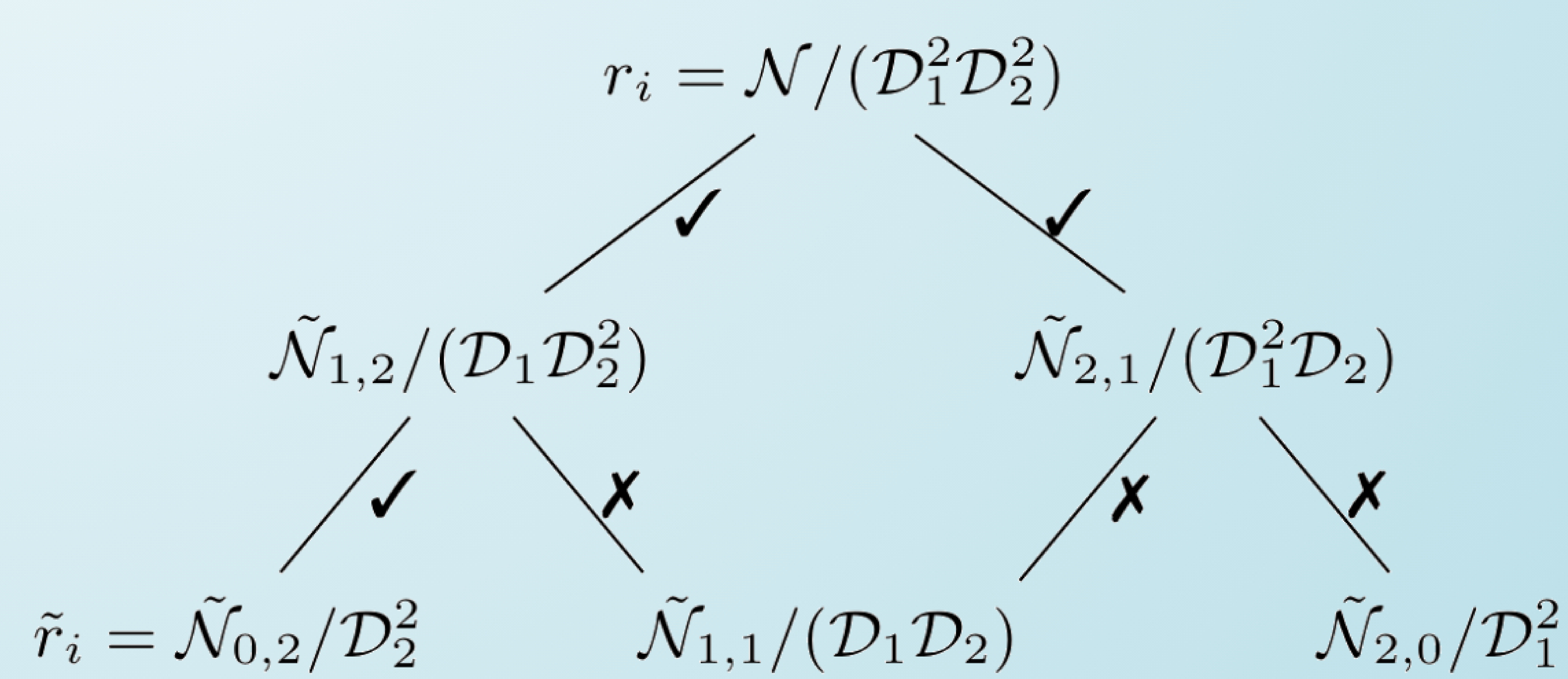
[6] Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov '21

- By Gaussian elimination, partition the space:

$$\mathrm{span}(r_{i\in\mathcal{B}}) = \underbrace{\mathrm{column}(\mathrm{Res}(r_{i\in\mathcal{B}}, \mathcal{D}_k^m))}_{\text{functions with the singularity}} \oplus \underbrace{\mathrm{null}(\mathrm{Res}(r_{i\in\mathcal{B}}, \mathcal{D}_k^m))}_{\text{functions without the singularity}}$$

- Search for linear combinations that remove as many singularities as possible

$$O_{i'i} = \bigcap_{k,m} \mathrm{nulls}(\mathrm{Res}(r_{i\in\mathcal{B}}, \mathcal{D}_k^m))$$



ANALYTIC RECONSTRUCTION

Reconstruction from Conjectured Properties

(for planar five-point one-mass amplitudes - all properties checked a posteriori)

- Denominator pairs $\{\mathcal{D}_i, \mathcal{D}_j\}$ can be *cleanly separated*:

$$\frac{\mathcal{N}}{\mathcal{D}_i^{q_i}\mathcal{D}_j^{q_j}\mathcal{D}_{\text{rest}}} \to \frac{\mathcal{N}_i}{\mathcal{D}_i^{q_i}\mathcal{D}_{\text{rest}}} + \frac{\mathcal{N}_j}{\mathcal{D}_j^{q_j}\mathcal{D}_{\text{rest}}}$$

  Examples of $\{\mathcal{D}_i, \mathcal{D}_j\}$ are:

  - ⋆ Any pairs of s_{ijk} or $\Delta_{ij|kl|mn}$ or $\langle i|j|p_V|k|i] - \langle j|l|p_V|k|j]$
  - ⋆ Any conjugate pair $\{\langle i|j+k|l], \langle l|j+k|i]\}$ or cyclic $\{\langle i|j\rangle, [i|j]\}$
  - ⋆ Pairs of the form $\{\Delta_{ij|kl|mn}, \langle c|a+b|d]$ or $\langle ab\rangle$ or $[ab]\}$ unless $\{ab\}$ are $\{ij\}$ or $\{kl\}$ or $\{mn\}$

- Other denominator pairs $\{\mathcal{D}_i, \mathcal{D}_j\}$ can be *separated to order* κ

$$\frac{\mathcal{N}}{\mathcal{D}_i^{q_i}\mathcal{D}_j^{q_j}\mathcal{D}_{\text{rest}}} \to \sum_{\kappa-q_j\leq m\leq q_i} \frac{\mathcal{N}_i}{\mathcal{D}_i^{m}\mathcal{D}_j^{\kappa-m}\mathcal{D}_{\text{rest}}}$$

  - ⋆ E.g. $\Delta^4_{ij|kl|mn}, \langle i|k+l|j]^5$ are separable to order 5.

✓ Reconstruction only requires $\mathbb{F}_p$ samples ✓ Already simpler than original ones (∼20MB)

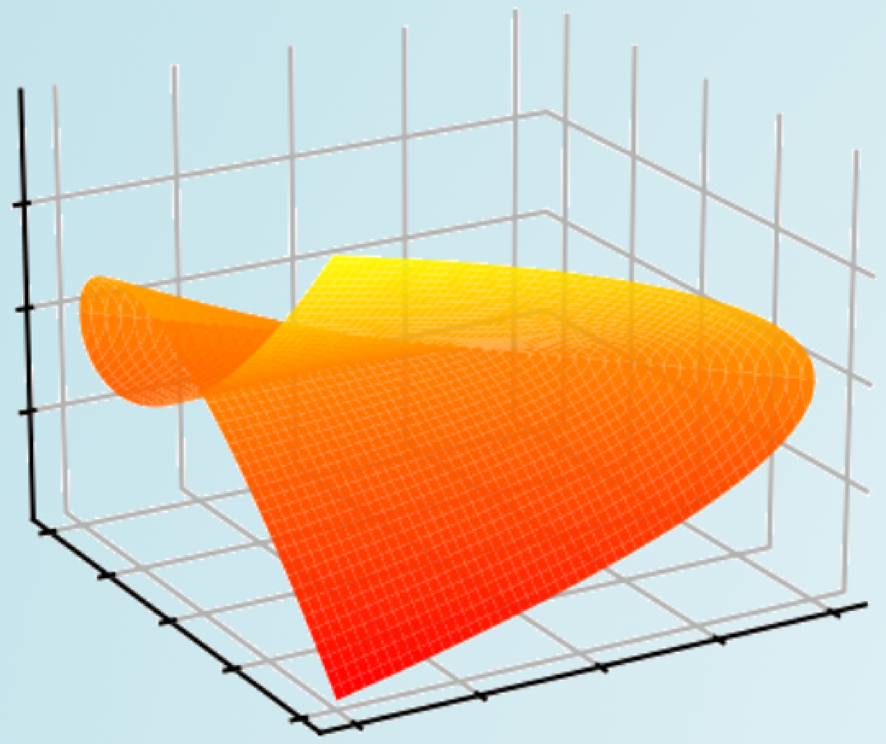
✗ Results are unstable and sub-optimal, e.g. numbers like this appeared

127187555379407704220939486282289348327703498501718808908391691454242601886997968263623652083189652150273

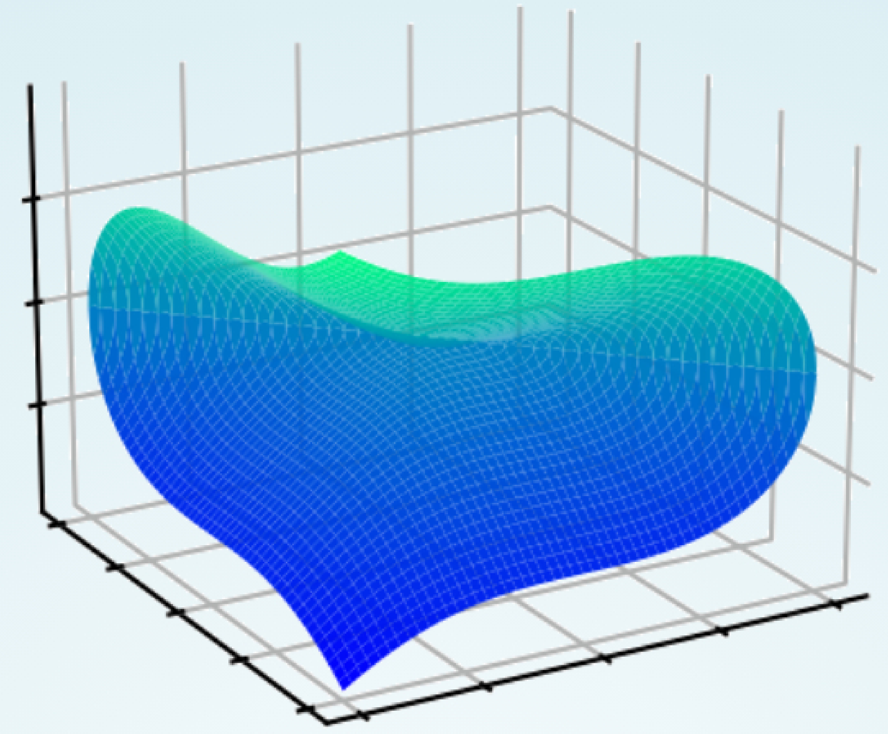
Iterated Pole Subtraction

(i.e. geometry at codimension greater than one)

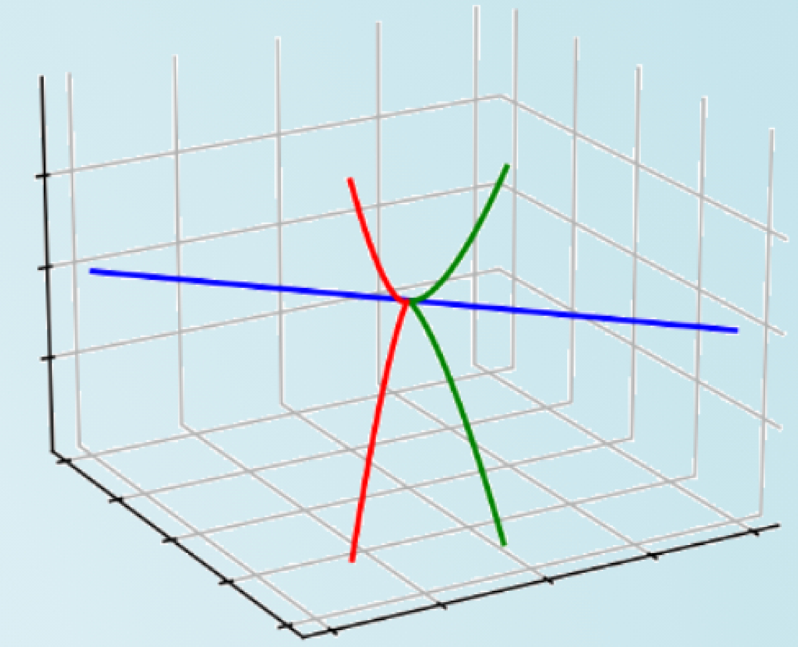
- Multivariate partial fraction decompositions follow from varieties where pairs of denominator factors vanish



$\langle xy^2 + y^3 - z^2 \rangle$ $\cap$



$\langle x^3 + y^3 - z^2 \rangle$ $=$



$\langle 2y^3 - z^2, x - y \rangle \cap \langle y^3 - z^2, x \rangle \cap \langle z^2, x + y \rangle$

- Retain control by iteratively fitting residues on varieties (using p-adic numbers, $\mathbb{Q}_p$)

$$r^{(139 \text{ of } 139)}_{\bar{u}^+ g^+ g^- d^- (V \to \ell^+ \ell^-)} =$$

$$+ \frac{7/4(s_{24} - s_{13})\langle 6|1+4|5] s_{123}(s_{124} - s_{134})}{\langle 1|2+3|4]\langle 2|1+4|3]^2 \Delta_{14|23|56}}$$

$$- \frac{49/64 \langle 3|1+4|2]\langle 6|1+4|5] s_{123}(s_{123} - s_{234})(s_{124} - s_{134})}{\langle 1|2+3|4]\langle 2|1+4|3]\Delta^2_{14|23|56}} + \dots$$

Variety (scheme?) to isolate term(s)

$$\Big\langle \langle 2|1+4|3]^2, \Delta_{14|23|56} \Big\rangle$$

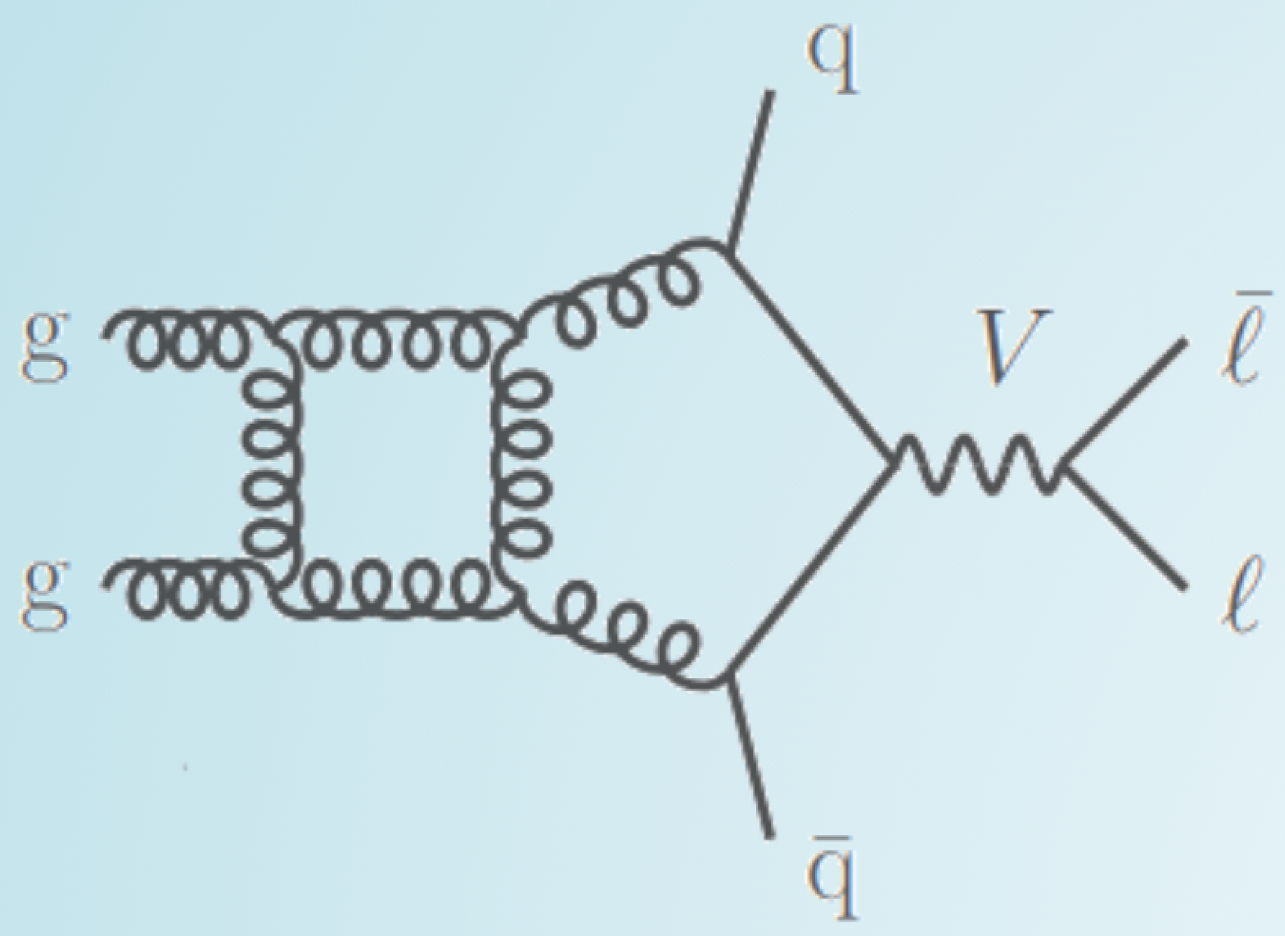
$$\Big\langle \Delta_{14|23|56} \Big\rangle$$

- Partial fraction decomposition and numerator insertions from e.g. (see appendix of paper):

$$\sqrt{\Big\langle \langle 2|1+4|3], \Delta_{14|23|56} \Big\rangle} = \Big\langle s_{124} - s_{134}, \langle 2|1+4|3] \Big\rangle ,$$

$$\Big\langle \langle 1|2+3|4], \langle 2|1+4|3] \Big\rangle = \Big\langle \langle 1|2+3|4], \langle 2|1+4|3], (s_{13} - s_{24}) \Big\rangle \cap \Big\langle \langle 12 \rangle, [34] \Big\rangle$$

For a fleshed out example with open-source code see GDL (ACAT '22)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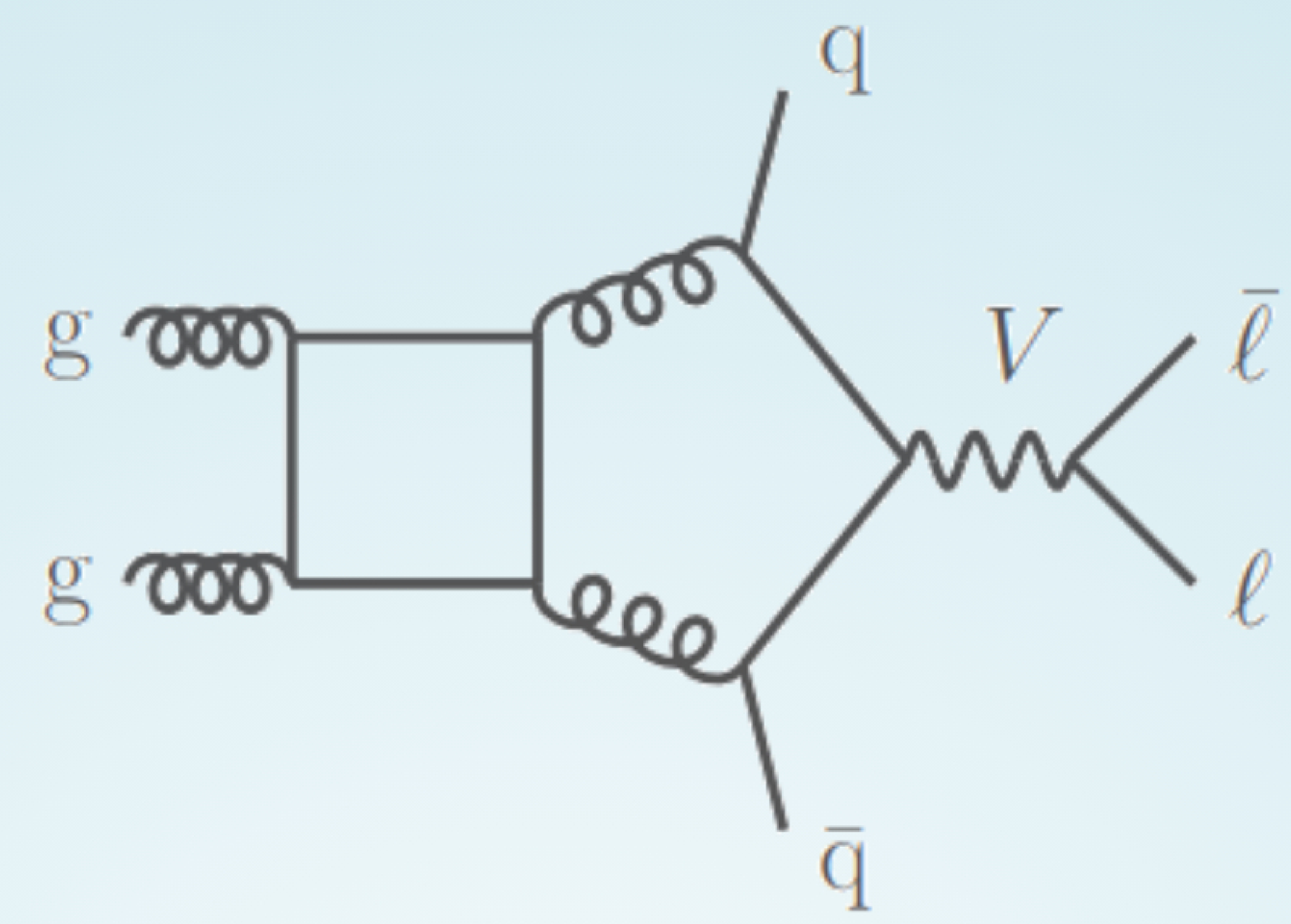
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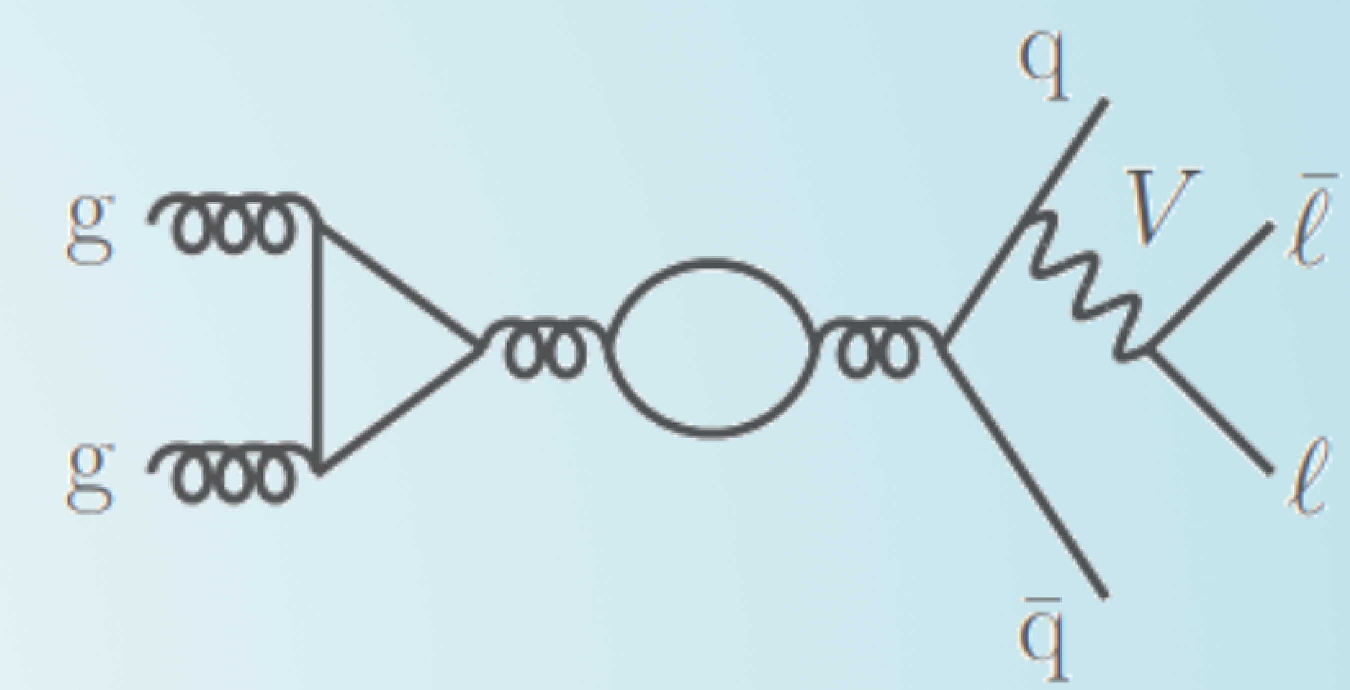
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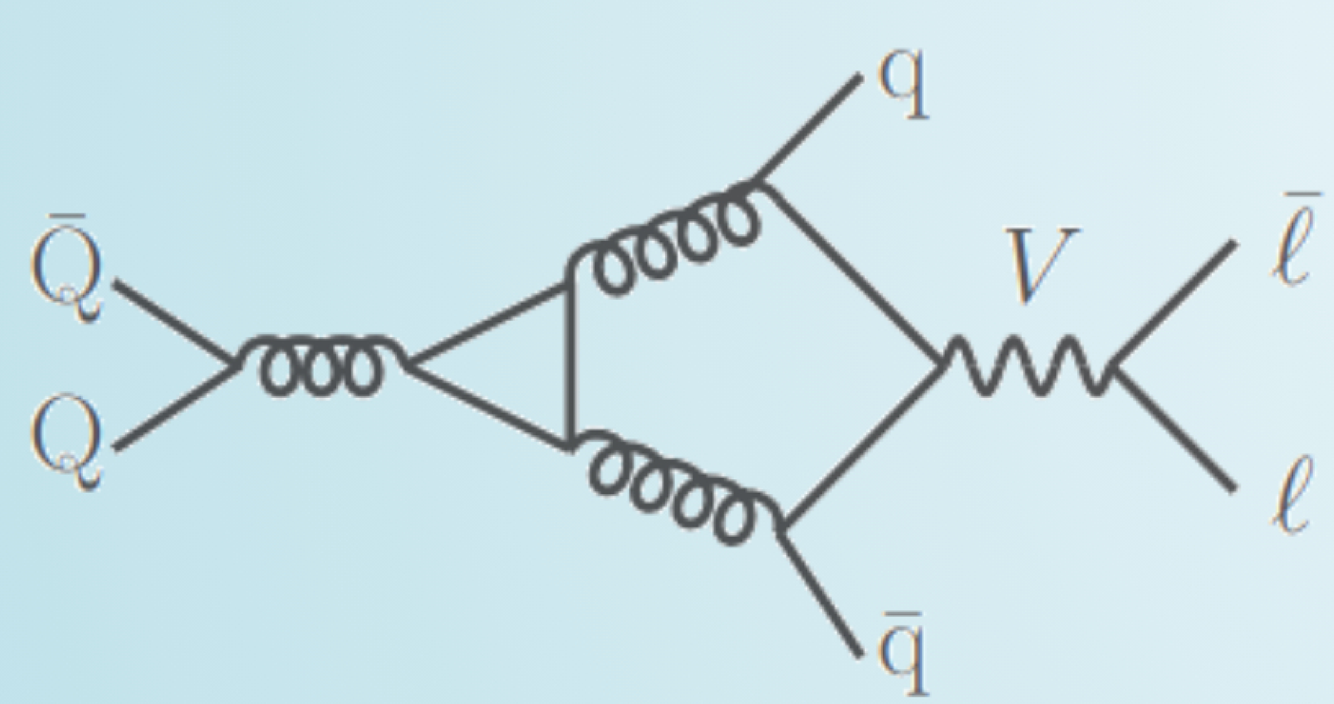
CONCLUSION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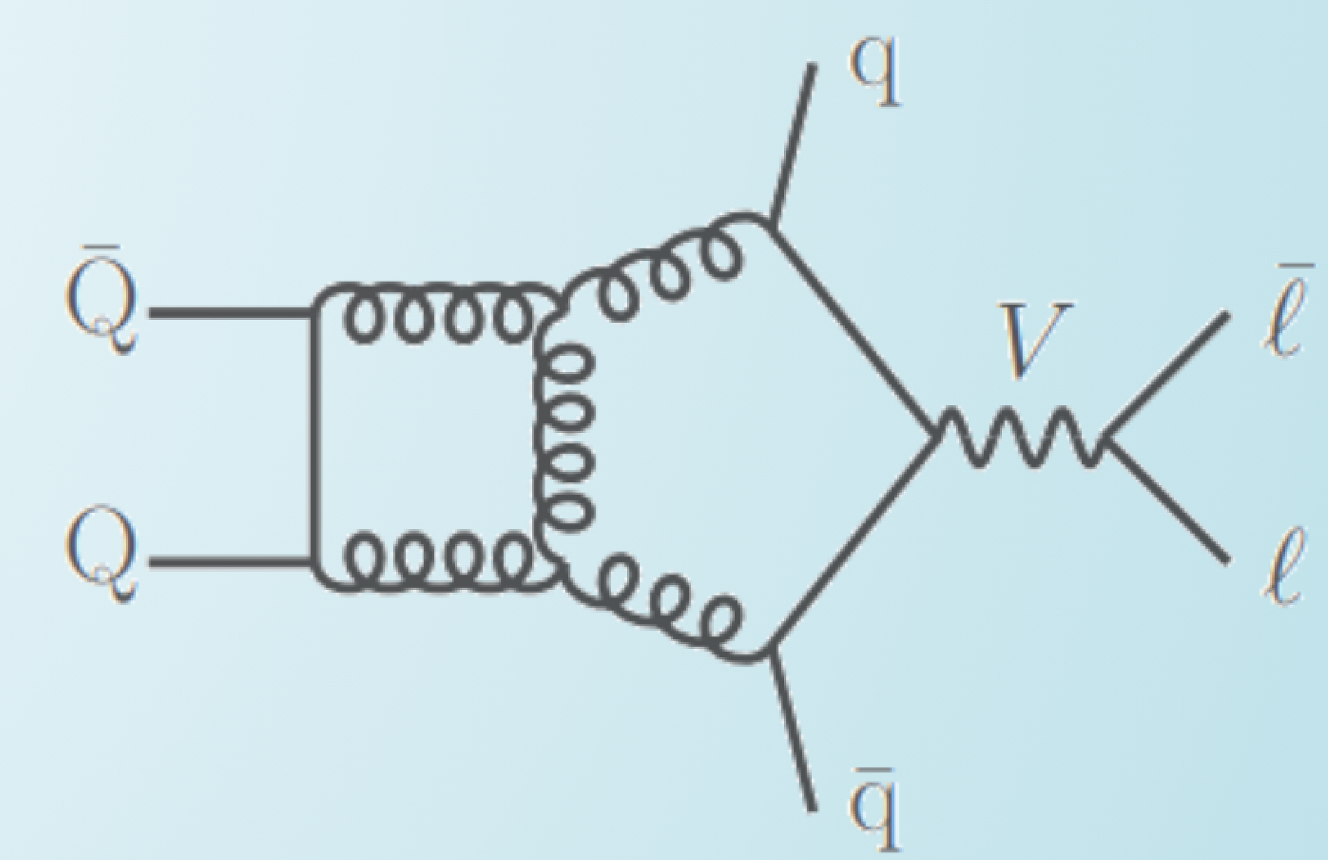
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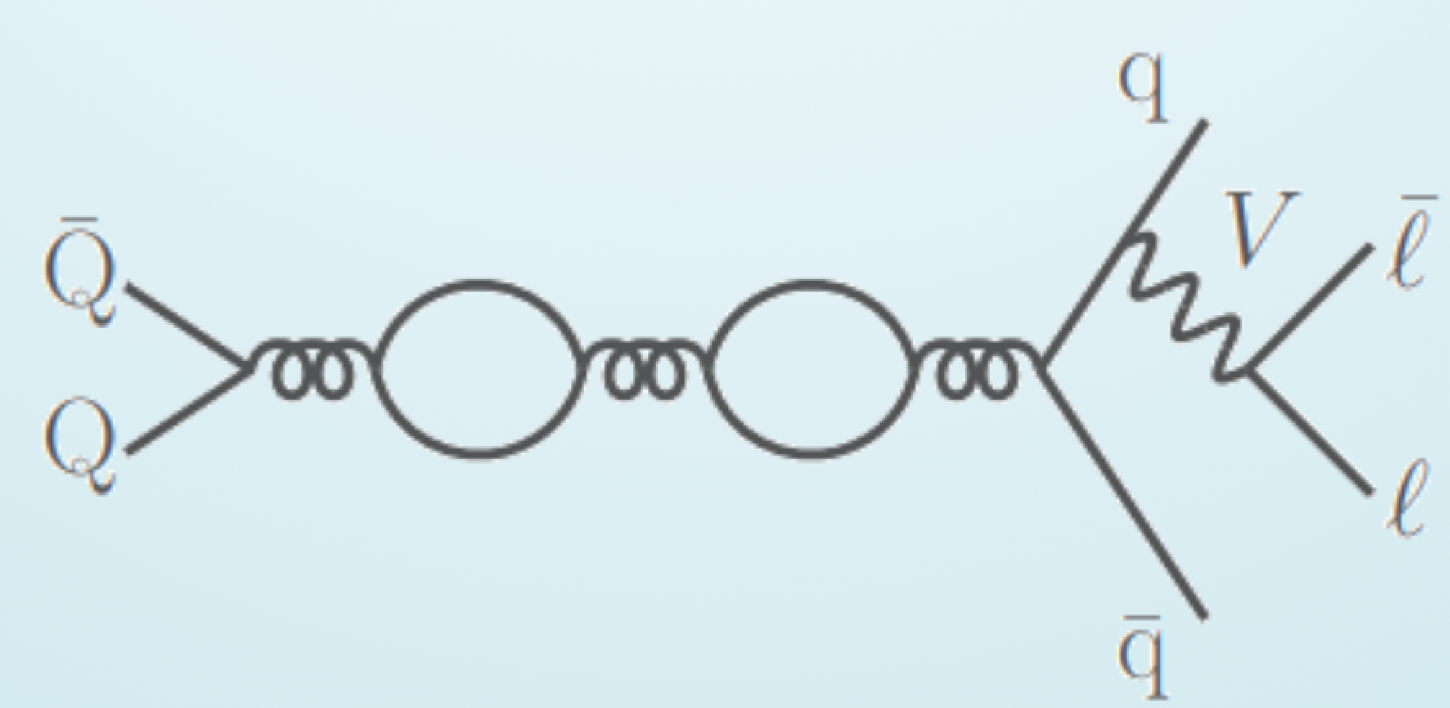
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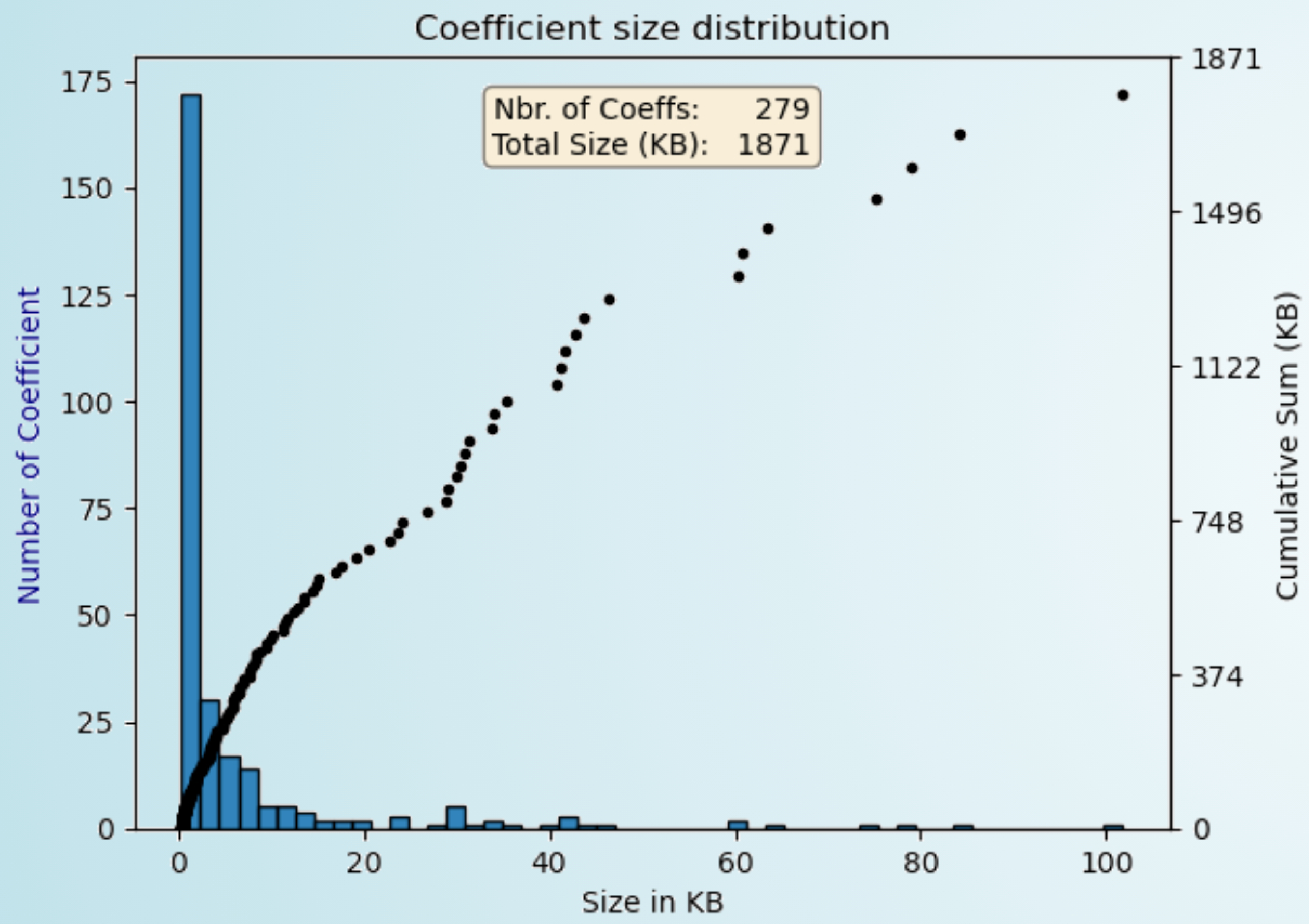
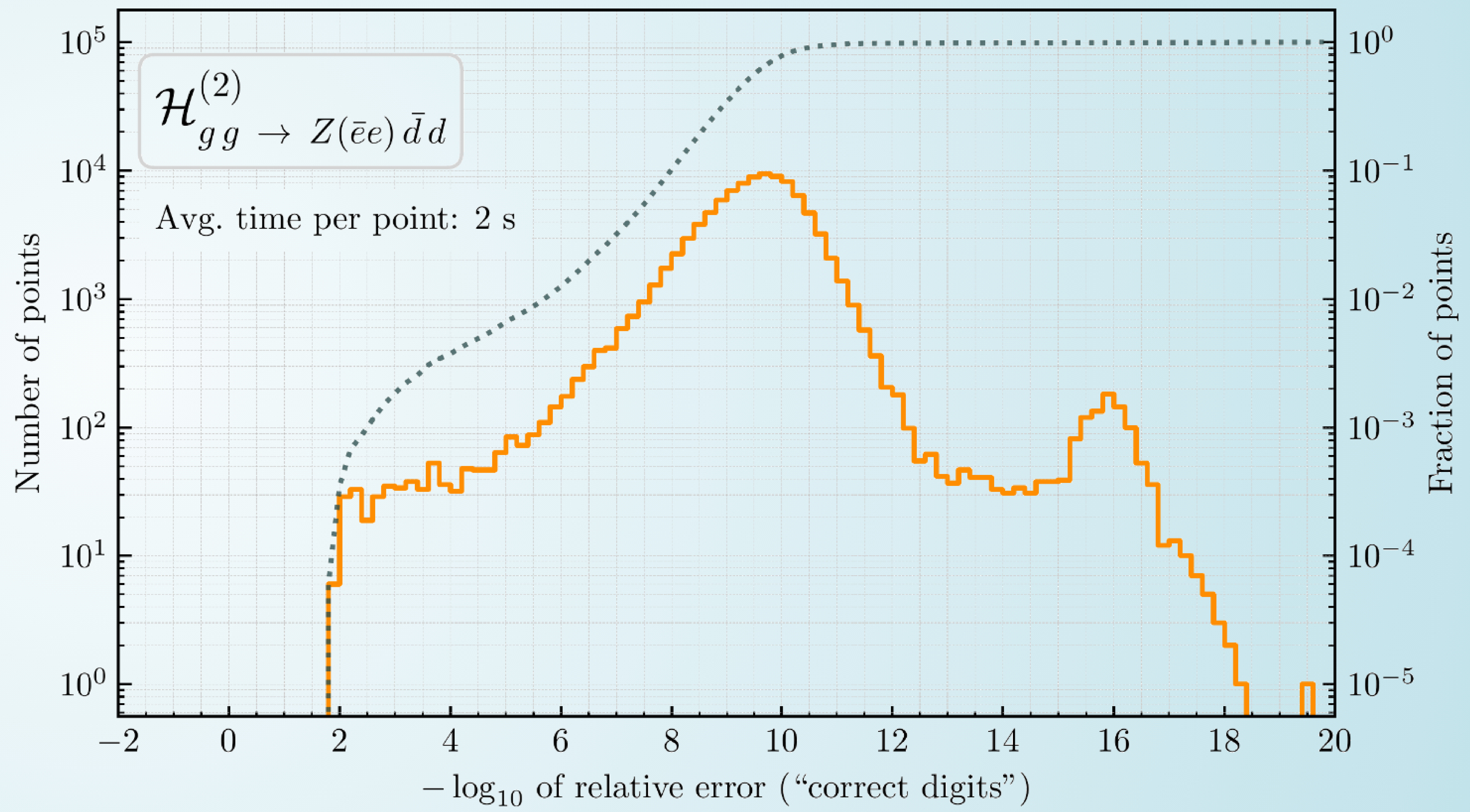
Spinor-Helicity Amplitudes Results

- The $pp \to Vjj$ coefficient functions are now 1.9 MB (from 1.4 GB), fast and stable. Matrices M_{ij} account for another 2 MB overall. Transcendental basis at PentagonFunctions++.





- The complexity split is: quarks NMHV: 100 KB, gluons MHV: 200 KB, gluons NMHV: 1.6 MB.
- The largest numbers are: quarks NMHV and gluons MHV: 3-digit, gluons NMHV: 12 digits.
- Pheno ready results for the hard functions are available at FivePointAmplitudes.
- Amplitudes at antares-results, with human readable expr., and CI tests for full amplitude in real kinematics

Thank you for your attention!

These slides are powered by:

markdown, html, revealjs, hugo, mathjax, github

BACKUP SLIDES

The Numerator Ansatz

- The numerator Ansatz takes the form

GDL, Maître ('19)

$$\text{Num. poly}(\lambda, \tilde{\lambda}) = \sum_{\vec{\alpha},\vec{\beta}} c_{(\vec{\alpha},\vec{\beta})} \prod_{j=1}^{n} \prod_{i=1}^{j-1} \langle ij \rangle^{\alpha_{ij}} [ij]^{\beta_{ij}}$$

subject to constraints on $\vec{\alpha}, \vec{\beta}$ due to: 1) mass dimension; 2) little group; 3) linear independence.

- Construct the Ansatz via the algorithm from Section 2.2 of GDL, Page ('22)

Linear independence = irreducibility by the Gröbner basis of a specific ideal.

- Efficient implementation using open-source software only

Singular

Gröbner bases $\longrightarrow$ constrain $\vec{\alpha}, \vec{\beta}$

Decker, Greuel, Pfister, Schönemann

Google OR-Tools

Integer programming $\longrightarrow$ enumerate sols. $\vec{\alpha}, \vec{\beta}$

Perron and Furnon (Google optimization team)

- Linear systems solved w/ CUDA over $\mathbb{F}_{2^{31}-1}$ ($t_{\text{solving}} \ll t_{\text{sampling}}$) w/ linac (coming soon-ish)