# *Progress on two-loop Feynman integrals for $t\bar{t}W$ production*


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based on ongoing work in collaboration with Matteo Becchetti, Vsevolod Chestnov, Tiziano Peraro, Mattia Pozzoli, and Simone Zoia

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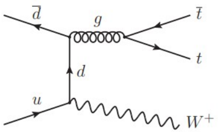
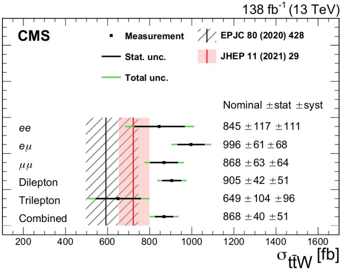
# Introduction

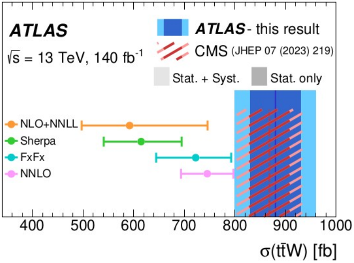
Need for more accurate theoretical predictions for $t\bar{t}W$ production:

- Background to key processes, such as $t\bar{t}H$ and $t\bar{t}t\bar{t}$ production.
- Relevant for many searches of physics beyond the Standard Model.
- Need for better description of experimental data [Calye/Jorge talks].




(a) [CMS, 2023]




(b) [Atlas, 2024]

Validation and improvement of available approximations [Buonocore et al, 2023] [see Chiara's talk].

Required Precision → NNLO!

# *From Cross Section ... to Feynman Integrals*

- Main bottleneck of NNLO prediction is the 2-loop amplitude $\mathcal{M}_n^{(2)}$!
- $\mathcal{M}_n^{(2)}$ is a sum of two-loop Feynman Diagrams → Feynman Integrals (FIs)

$$\mathcal{M}_n^{(2)} = \sum_t C_t F_t \quad \text{with} \quad F_t = \int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \prod_{i \in t} \frac{1}{D_i^{a_i}} \,, \quad D_i = (k_i + p_i)^2 - m_i^2$$

- Work in dimensional regularization: $4 \to d = 4 - 2\epsilon$!
- FIs satisfy integration-by-part relations [K. Chetyrkin, F. Tkachov, 1981; S. Laporta, 2004] → Master Integrals

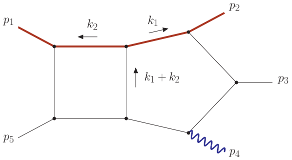
$$\int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \frac{\partial}{\partial k_l^\mu} \prod_{i \in t} \frac{1}{D_i^{a_i}} = 0 \quad \longrightarrow \quad \{\mathcal{J}_i\}$$

- $\mathcal{M}_n^{(2)}$ expressed as a sum of Master Integrals (MIs) multiplied by process-dependent coefficients

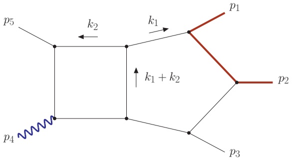
$$\boxed{\mathcal{M}_n^{(2)} = \sum_{i=1}^{N_{\text{MIs}}} \tilde{C}_i \mathcal{J}_i}$$

# *Leading Color $t\bar{t}W$ Integral Families*

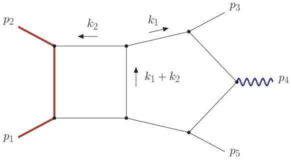
- Group into integral families $\rightarrow$ compute MIs of each family




*(a)* $F_1$ (141 MIs)




*(b)* $F_2$ (122 MIs)




*(c)* $F_3$ (131 MIs)

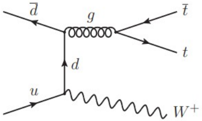
- Modding out by permutations of the external legs: 30 genuinely new sectors containing 85 MIs!

$$I_{\vec{\nu}}^{(F)} = \int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \prod_{i=1}^{11} \frac{1}{D_{F,i}^{\nu_i}}$$

## *Kinematics and Physical Channel*

- We are interested in the process

$$\boxed{0 \to \bar{t}(p_1) + t(p_2) + \bar{d}(p_3) + W(p_4) + u(p_5)}$$




$$p_1^2 = p_2^2 = m_t^2 \qquad p_3^2 = p_5^2 = 0 \qquad \text{and} \qquad p_4^2 = m_w^2$$

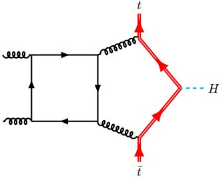
- Five-particle scattering + two different masses → 7 independent invariants → High complexity

$$\vec{x} = \left\{ s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_t^2, m_w^2 \right\} \qquad \text{with} \qquad s_{ij} = (p_i + p_j)^2$$

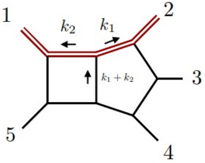
- We focus to the physical channel relevant for $t\bar{t}W$ production ($35 \to 124$)!

# *Current Frontier @ Two-Loop MI Families*

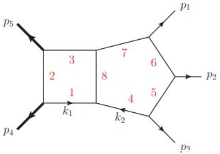
- Five external particles + internal/external masses




*(a)* $t\bar{t}H$ [B. Page et al, 2023]

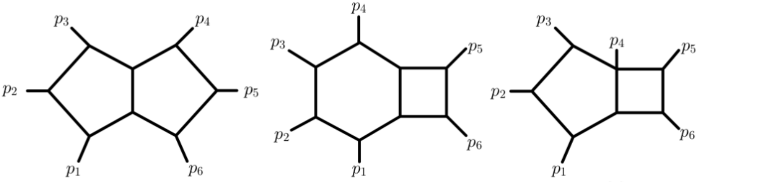



*(b)* $t\bar{t}j$ [S. Badger et al, 2024]




*(c)* $VV'j$ [S. Abreu et al, 2024]

- Six external particles + without internal/external masses




*(d)* 6 partons [S. Abreu et al, 2024] & [J. Henn et al, 2025]

- MIs fulfill differential equations (DEs) w.r.t. $\vec{x}$ [A. Kotikov, 1991; Z. Bern et al, 1994; T. Gehrmann et al, 2000]

$$\mathrm{d}\vec{\mathcal{J}}_F(\vec{x},\epsilon) = \mathrm{d}A^{(F)}(\vec{x},\epsilon)\cdot\vec{\mathcal{J}}_F(\vec{x},\epsilon)$$

- Laurent expansion of MIs around $\epsilon = 0$ $\rightarrow$ Compute up to desired order on $\epsilon$

$$\vec{\mathcal{J}}_F(\vec{x},\epsilon) = \sum_i \epsilon^i \vec{\mathcal{J}}_F^{(i)}(\vec{x}) \quad \text{with} \quad i \geq -2L$$

- Basis not unique $\rightarrow$ Good choice makes DEs easier to solve $\rightarrow$ canonical DEs [J. Henn, 2013]

$$\boxed{\mathrm{d}\vec{\mathcal{J}}_F = \epsilon\ \mathrm{d}A(\vec{x})\ \vec{\mathcal{J}}_F}$$

- If functional space described by poly-logarithmic functions $\rightarrow$ canonical $\mathrm{d}\log$-form

$$\mathrm{d}A(\vec{x}) = \sum_\alpha A_\alpha\ \mathrm{d}\log W_\alpha(\vec{x})$$

where $W_\alpha$ are called letters, and their total set alphabet.

- Iterative solution in $\epsilon$.
- Compact expression of the connection matrices.
- Efficiently solved analytically and numerically.

## *Leading Singularities* *[N. Arkani-Hamed et al, 2012]*

How to construct such a basis?

↓

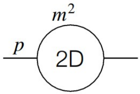
MIs with loop-integrand containing at most simple poles and constant leading singularities

↓

Analyze the integrand

- Example → One-loop massive bubble in 2 dimensions ($x = m^2/p^2$)

$$\mathrm{d}\log(z+c) = \frac{dz}{(z+c)}$$




$$\propto \int \frac{\mathrm{d}a_1 \wedge \mathrm{d}a_2}{p^2(a_1 a_2 - x)[(1+a_1)(1+a_2) - x]}$$

$$= \int \frac{\mathrm{d}a_1}{p^2(a_1 + a_1^2 + x)} [\mathrm{d}\log(a_1 a_2 - x) - \mathrm{d}\log(1 + a_1 + a_2 + a_1 a_2 - x)]$$

$$= \frac{1}{p^2\sqrt{1-4x}} \int \mathrm{d}\log(\ldots) \wedge \mathrm{d}\log(\ldots)$$

Good Choice in this case: $(p^2\sqrt{1-4x}) \times$ (One-loop massive bubble)!

- Square roots can appear in leading singularities → and not only ...

# *Nested Square Root Sector*

- Existence of nested roots in pentagon-triangles observed in [F. Febres Cordero et al, 2023; S. Badger et al, 2024]

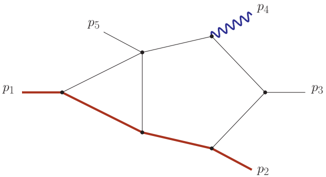



*Figure:* Nested square root sector of $F_1$ (3 MIs)

- Leading singularities of 2 MIs contain nested square roots

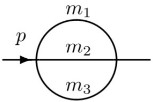
$$\mathrm{NS}_{\pm} = \pm \frac{\sqrt{b + 6am_t^2 \pm c\, r_1}\sqrt{b - 2am_t^2 \pm c\, r_1}}{r_1} \qquad \text{with} \qquad r_1 = \sqrt{\mathrm{G}(p_1, p_2, p_3, p_4)}$$

where $a$ and $b$ are 2nd and 4th degree polynomials in $\vec{x}$, respectively.

- Couples to one of the 4-point elliptic sectors! → Elliptic what? (see next slides)

- Complicated cases → functional space extended beyond poly-logarithmics → elliptic integrals!

[A. Sabry, 1962]




[S. Laporta et al, 2004]

- In such cases elliptic curves appear in the computation of the leading singularities

$$\int \frac{\mathrm{d}z}{\sqrt{\mathcal{P}_4(z)}} \wedge \mathrm{d}\log(\ldots) \quad \text{with} \quad \mathcal{P}_4(z) = (a_1 - z)(a_2 - z)(a_3 - z)(a_4 - z)$$

Can we obtain canonical form?

↓

Yes but hard to do (open problem)

↓

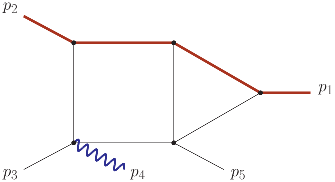
Introduce transcendental functions in the DEs!

↓

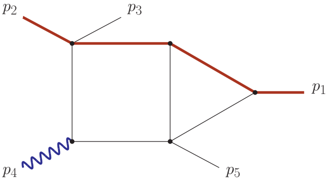
No available package to efficiently evaluate the solution 😢

# *4-point Elliptic Sector*

- In $F_1$ two elliptic sectors appear that depend on four-point kinematics




*(a)* Elliptic of [S. Badger et al, 2024] (3 MIs)




*(b)* 4-point elliptic sector of $F_1$ (3 MIs)

  - First already known from [S. Badger et al, 2024; M. Becchetti et al, 2025] $\rightarrow$ depends on 4 invariants
  - Second is New $\rightarrow$ Same form but a bit more complicated $\rightarrow$ depends on 5 invariants

- Elliptic curve of the form

$$\mathcal{P}_{\text{4-pt.}}(z) = (z + m_t^2)(z - 3m_t^2)\mathcal{P}_2(z)$$

where $\mathcal{P}_2(z)$ a second degree polynomial in $z$.

- Are the two curves identical? $\rightarrow$ $j$-invariants are different $\rightarrow$ No!

## *5-point Elliptic Sector*

- In $F_2/F_3$ a five-point elliptic appears $\to$ 7 invariants! $\to$ First study of such a complicated case!

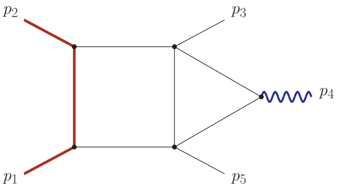



*Figure:* 5-point elliptic sector of $F_2/F_3$ (7 MIs)

- In the leading singularity of the scalar integral one gets

$$\int \left( \frac{\mathrm{d}z \wedge \mathrm{d}\log[\alpha'(z, z_9)]}{\sqrt{\mathcal{P}_{\text{5-pt.}}(z)}} - \frac{\mathrm{d}z \wedge \mathrm{d}\log[\alpha'(z, z_9)^\dagger]}{\sqrt{\mathcal{P}_{\text{5-pt.}}(z)^\dagger}} \right) \quad \text{with} \quad f^\dagger \equiv f|_{r_1 \to -r_1}$$

- Two elliptic curves? $\to$ NO $\to$ same j-invariant!
- $\mathcal{P}_{\text{5-pt.}}$ is of degree 4 polynomial on z, while degree 14 in $\vec{x}$, containing 2787 terms!
- DEs size of $F_2$ and $F_3$ dominated by this sector!

# *Integral Bases Construction*

$$\mathrm{d}\vec{\mathcal{J}}_F(\vec{x},\epsilon) = \mathrm{d}A^{(F)}(\vec{x},\epsilon)\cdot\vec{\mathcal{J}}_F(\vec{x},\epsilon)$$

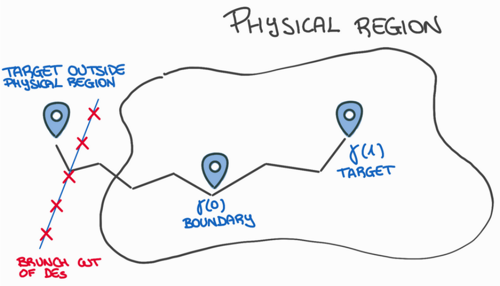
- We construct bases of MIs so that the connection matrix of the DEs satisfies:
  - Blocks coupling MIs not containing nested roots/elliptic curves are $\epsilon$-factorised
  - Other entries are at worst quadratic in $\epsilon$ (no elliptic functions/nested roots introduced)

$$\boxed{\mathrm{d}A^{(F)}(\vec{x},\epsilon) = \sum_{k=0}^{2}\epsilon^k\left[\sum_\alpha c_{k\alpha}^{(F)}\,\mathrm{d}\log\Big(W_\alpha(\vec{x})\Big) + \sum_\beta d_{k\beta}^{(F)}\,\omega_\beta(\vec{x})\right]}$$

  - Free of spurious denominator factors (in $F_2/F_3$ allowing for one yields a better form)
  - Substantially more compact than with an arbitrary basis
- Elliptic MIs chosen to be non-zero starting from the finite part!
- `NeatIBP` [Z. Wu et al, 2023] $\rightarrow$ optimized IBPs + `FiniteFlow` [T. Peraro, 2019] $\rightarrow$ reconstruct DEs!
- How to solve our DEs? $\rightarrow$ Generalized Series Expansion!

# *Generalized Series Expansion Method [F. Moriello, 2019]*

- Semi-numerical method applied for solving DEs [see Renato's talk]




$$\gamma(t): \quad \gamma(0) = \vec{s}_b \quad \text{and} \quad \gamma(1) = \vec{s}_t \quad \text{with} \quad t \in [0,1]$$

- Divide path from $\vec{s}_b$ to $\vec{s}_t$ into segments → Expand and solve DEs therein!
- $\gamma(t)$ in the physical region → no physical singularity crossed → no analytic continuation!
- Mathematica: DiffExp [M. Hidding, 2020], SeaSyde [T. Armadillo et al, 2022], AMFlow Solver [X. Liu et al, 2022]
- C: LINE [R. Prisco et al, 2025]

# *Results*

| | MIs | elliptic | roots | nested | entries | letters | one-forms | one-forms size |
|---|---|---|---|---|---|---|---|---|
| $F_1$ | 141 | 2 | 8 | 1 | 2339 | 101 | 119 | 6.7 MB |
| $F_2$ | 122 | 1 | 11 | 0 | 2027 | 122 | 84 | 311 MB |
| $F_3$ | 131 | 1 | 12 | 0 | 2333 | 137 | 96 | 317 MB |

$$\mathrm{d}A^{(F)}(\vec{x},\epsilon) = \sum_{k=0}^{2} \epsilon^k \left[\sum_{\alpha} c_{k\alpha}^{(F)}\, \mathrm{d}\log\Big(W_\alpha(\vec{x})\Big) + \sum_{\beta} d_{k\beta}^{(F)}\, \omega_\beta(\vec{x})\right]$$

- `DiffExp` implementation with in-house parametrization for obtaining numerical results.
- Boundary values generated by `AMFlow` [X. Liu et al, 2022].
- Cross check against `AMFlow` for 10 physical points.
- We verified that we can integrate between any of these 10 points.

## Outcome

For the integral families contributing on $t\bar{t}W$ production in the leading colour approximation

- Created optimized basis addressing nested square roots and elliptic sectors
- Derived differential equations for this basis
- Provided `DiffExp` implementation for the numerical solution of the DEs

## What's Next

- Creating a special function basis [T. Gehrmann et al, 2018; D. Chicherin et al, 2020; S. Badger et al, 2024]
- Exploring `LINE` [R. Prisco et al, 2025] and searching for $\epsilon$-factorized bases
- Compute the two-loop amplitudes exploiting the good properties of our bases

*Thank you!*

- Choose appropriate candidates that render MC DEs linear on $\epsilon$

$$\partial_\xi \vec{G}^{\text{MC}} = (H_{0,\xi} + \epsilon H_{1,\xi})\vec{G}^{\text{MC}}$$

- $H_0$ can be removed by rescaling MIs a matrix that satisfies the following DEs

$$\partial_\xi \tilde{T}^{\text{MC}} = -\tilde{T}^{\text{MC}} H_{0,\xi}$$

- New candidates defined as $\vec{I}^{\text{MC}} = \tilde{T}^{\text{MC}}\vec{G}^{\text{MC}}$ acquire canonical DEs in MC

$$\partial_\xi \vec{I}^{\text{MC}} = \epsilon A_\xi^{\text{MC}} \vec{I}^{\text{MC}} \qquad \text{with} \qquad A_\xi^{\text{MC}} = \tilde{T}^{\text{MC}} H_{1,\xi} (\tilde{T}^{\text{MC}})^{-1}$$

- Are $\vec{I}$ pure beyond MC? → relax cut conditions → may appear sub-sector entries linear on $\epsilon$

$$\partial_\xi \vec{I} = \epsilon A_\xi^{\text{MC}} \vec{I}^{\text{MC}} + (h_{0,\xi} + \epsilon h_{1,\xi})\vec{I}^{\text{LS}}$$

- To set them in canonical form rotate the lower sector $\epsilon^0$ contributions by integrating out $h_{0,\xi}$

$$\vec{I} = \vec{I}^{\text{MC}} + \tilde{T}^{\text{LS}} \vec{I}^{\text{LS}} \qquad \text{with} \qquad \partial_\xi \tilde{T}^{\text{LS}} = -h_{0,\xi}$$

# *Backup Slides: j-invariant of Elliptic Curves*

- We remind the definition of elliptic curve

$$y^2 = (z - a_1)(z - a_2)(z - a_3)(z - a_4)$$

- From the roots of the elliptic curve the cross ratio is defined

$$\lambda = \frac{(a_1 - a_4)(a_2 - a_3)}{(a_1 - a_3)(a_2 - a_4)}$$

- And from the cross ratio the $j$-invariant is defined

$$j = 256\frac{(1 - \lambda(1 - \lambda))^3}{\lambda^2(1 - \lambda)^2}$$