QFT Problems

1. **Lorentz transformations of fields**

   Some parts of this problem were worked out in lecture, but this problem fills in the gaps in some of the derivations on the board.

   (a) Consider the infinitesimal Lorentz transformation

$$\Lambda = \mathbb{I} + \omega_{\rho\sigma} M^{\rho\sigma} + O\left(\omega_{\rho\sigma}^2\right). \tag{1}$$

   Under this transformation the coordinate x transforms by an infinitesimal amount δx:

$$\begin{aligned} x \to x' &= \Lambda x \qquad (2)\\ &= x + \delta x + O\left(\omega_{\rho\sigma}^2\right). \qquad (3) \end{aligned}$$

   Show that

$$\delta x = \omega_{\rho\sigma} M^{\rho\sigma} x. \tag{4}$$

   What is this in index notation?

   (b) Now consider a scalar field $\phi(x)$. As discussed in lectures, the field transforms under the Lorentz transformation (1):

$$\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x) + O\left(\omega_{\rho\sigma}^2\right). \tag{5}$$

   Show that

$$\delta\phi(x) = \omega_{\rho\sigma} L^{\rho\sigma} \phi(x), \tag{6}$$

   where

$$L^{\rho\sigma} = -\left(M^{\rho\sigma} x\right) \cdot \frac{\partial}{\partial x}. \tag{7}$$

   What is this in index notation?

   From your answer to the latter question, using:

$$\left(M^{\rho\sigma}\right)^{\mu}{}_{\nu} = \eta^{\sigma\mu} \delta^{\rho}{}_{\nu} - \eta^{\rho\mu} \delta^{\sigma}{}_{\nu}, \tag{8}$$

   show that

$$L^{\rho\sigma} = x^{\sigma} \frac{\partial}{\partial x_{\rho}} - x^{\rho} \frac{\partial}{\partial x_{\sigma}}. \tag{9}$$

   (c) Consider the infinitesimal translation

$$x \to x' = x - a. \tag{10}$$

   This means that the components of the four-vector a are infinitesimal, i.e. $|a^{\mu}| \ll 1$. This induces an infinitesimal transformation in the field ϕ:

$$\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x) + O\left(a^2\right). \tag{11}$$

   Show that

$$\delta\phi(x) = a^{\mu} \frac{\partial}{\partial x^{\mu}} \phi(x).$$

From this we identify the generator of translations to be $P_\mu = \frac{\partial}{\partial x^\mu}$. Show that

$$[P^\mu, L^{\nu\rho}] = \eta^{\mu\rho} P^\nu - \eta^{\mu\nu} P^\rho, \tag{12}$$

$$[P^\mu, P^\nu] = 0. \tag{13}$$

From this we see that, together with (10), $L^{\rho\sigma}$ and P^μ generate the Poincaré algebra!

2. **Working with Noether's theorem**

Part 1. Consider the following action for two real scalar fields ϕ_1 and ϕ_2,

$$S = \int d^4x \left(-\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m^2\phi_1^2 - \frac{1}{2}m^2\phi_2^2 - \lambda\left(\phi_1^2 + \phi_2^2\right)^2 \right). \tag{14}$$

(a) Derive the Euler-Lagrange equations of motion for the fields ϕ_1 and ϕ_2.

(b) Show that the action (15) is invariant under the continuous transformation

$$\phi_1 \to \phi_1' = (\cos\alpha)\,\phi_1 - (\sin\alpha)\,\phi_2, \tag{15}$$

$$\phi_2 \to \phi_2' = (\sin\alpha)\,\phi_1 + (\cos\alpha)\,\phi_2, \tag{16}$$

where α is a constant continuous parameter.

(c) Determine the current and charge associated to the symmetry given in part (b).

(d) Show that the current in part (c) is indeed conserved if the Euler-Lagrange equations of motion are satisfied.

Part 2: Another way to solve the same problem is to use the fact that a theory of two real scalar fields with the same mass can be recast as a theory of a complex scalar field ϕ, which you will show in the following.

We can assemble two real scalar fields ϕ_1 and ϕ_2, with the same mass m, into a single complex scalar field $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$. Show that the action (15) in terms of the complex scalar field ϕ reads

$$S = \int d^4x \left(-\partial_\mu\phi\partial^\mu\phi^* - m^2\phi\phi^* - 4\lambda\left(\phi\phi^*\right)^2 \right). \tag{17}$$

(a) Derive the Euler-Lagrange equations of motion for the field ϕ and its complex conjugate ϕ^*.

(b) Show that the action (18) is invariant under the continuous transformation

$$\phi \to \phi' = e^{i\alpha}\phi, \qquad \phi^* \to (\phi')^* = e^{-i\alpha}\phi^*. \tag{18}$$

where α is a constant continuous parameter. Show that this transformation is equivalent to the transformation (16) considered in part 1. *This question continues on the next page.*

(c) Determine the current and charge associated to the symmetry given in part 2 (c).

(d) Show that the current determined in part 2 (d) is equivalent to the current you obtained in part 1 (c).

3. **Many harmonic oscillators**

(This problem is "half-way" between quantum field theory and quantum mechanics). A system of N decoupled complex simple harmonic oscillators with frequencies $\omega_c,\ c = 1, \ldots, N,$ has action

$$S = \int dt \sum_{c=1}^{N} \left(\dot{q}_c(t)\, \dot{q}_c^\star(t) - \omega_c^2\, q_c(t)\, q_c^\star(t) \right), \tag{19}$$

and Euler-Lagrange equations of motion

$$\ddot{q}_c(t) + \omega_c^2\, q_c(t) = 0, \qquad \ddot{q}_c^\star(t) + \omega_c^2\, q_c^\star(t) = 0, \qquad c = 1, \ldots, N. \tag{20}$$

In the lectures we saw that their general solution can be written as

$$q_c(t) = \frac{1}{\sqrt{2\omega_c}} \left[a_c\, e^{-i\omega_c t} + b_c^\star\, e^{i\omega_c t} \right], \tag{21}$$

$$q_c^\star(t) = \frac{1}{\sqrt{2\omega_c}} \left[b_c\, e^{-i\omega_c t} + a_c^\star\, e^{i\omega_c t} \right]. \tag{22}$$

From the above Lagrangian formulation of the complex Harmonic oscillator, show that the corresponding Hamiltonian takes the form

$$H = \sum_{c=1}^{N} \dot{q}_c(t)\, \dot{q}_c^\star(t) + \omega_c^2\, q_c(t)\, q_c^\star(t). \tag{23}$$

Show that upon canonical quantisation the Hamiltonian takes the following form in terms of a_c, $a_c^\dagger$ and b_c, $b_c^\dagger$:

$$\hat{H} = \sum_{c=1}^{N} \omega_c \left[\hat{a}_c \hat{a}_c^\dagger + \hat{b}_c^\dagger \hat{b}_c \right] = \sum_{c=1}^{N} \omega_c \left[\hat{a}_c^\dagger \hat{a}_c + \hat{b}_c^\dagger \hat{b}_c + 1 \right]. \tag{24}$$

4. **Canonical quantization of the complex scalar field**

From the lectures we saw that the Hamiltonian of the complex scalar field takes the form

$$H = \int d^3\vec{x} \left[\dot{\phi}^\star \dot{\phi} + \vec{\nabla}\phi^\star \cdot \vec{\nabla}\phi + m^2 \phi^\star \phi \right], \tag{25}$$

and ϕ admits the decomposition

$$\phi(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left[a_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x} - i\omega_{\vec{k}} t} + b_{\vec{k}}^\star\, e^{-i\vec{k}\cdot\vec{x} + i\omega_{\vec{k}} t} \right], \tag{26}$$

where $\omega_{\vec{k}}^2 = |\vec{k}|^2 + m^2$. Show that:

$$\phi^{\star}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left[b_{\vec{k}}\,e^{i\vec{k}\cdot\vec{x}-i\omega_{\vec{k}}t} + a^{\star}_{\vec{k}}\,e^{-i\vec{k}\cdot\vec{x}+i\omega_{\vec{k}}t}\right], \tag{27}$$

$$\vec{\nabla}\phi(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left[i\vec{k}\,a_{\vec{k}}\,e^{i\vec{k}\cdot\vec{x}-i\omega_{\vec{k}}t} - i\vec{k}\,b^{\star}_{\vec{k}}\,e^{-i\vec{k}\cdot\vec{x}+i\omega_{\vec{k}}t}\right], \tag{28}$$

$$\vec{\nabla}\phi^{\star}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left[i\vec{k}\,b_{\vec{k}}\,e^{i\vec{k}\cdot\vec{x}-i\omega_{\vec{k}}t} - i\vec{k}\,a^{\star}_{\vec{k}}\,e^{-i\vec{k}\cdot\vec{x}+i\omega_{\vec{k}}t}\right], \tag{29}$$

$$\dot{\phi}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\vec{k}}}{2}}\left[a_{\vec{k}}\,e^{i\vec{k}\cdot\vec{x}-i\omega_{\vec{k}}t} - b^{\star}_{\vec{k}}\,e^{-i\vec{k}\cdot\vec{x}+i\omega_{\vec{k}}t}\right], \tag{30}$$

$$\dot{\phi}^{\star}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\vec{k}}}{2}}\left[b_{\vec{k}}\,e^{i\vec{k}\cdot\vec{x}-i\omega_{\vec{k}}t} - a^{\star}_{\vec{k}}\,e^{-i\vec{k}\cdot\vec{x}+i\omega_{\vec{k}}t}\right]. \tag{31}$$

Promote the above to operators through the procedure of canonical quantisation. Choosing the order of the operators as they appear in the Hamiltonian (26), show that

$$\hat{H} = \int \frac{d^3k}{(2\pi)^3}\,\omega_{\vec{k}}\left[\hat{a}^{\dagger}_{\vec{k}}\hat{a}_{\vec{k}} + \hat{b}_{\vec{k}}\hat{b}^{\dagger}_{\vec{k}}\right], \tag{32}$$

Show that

$$\hat{H} = \int \frac{d^3k}{(2\pi)^3}\,\omega_{\vec{k}}\left[\hat{a}^{\dagger}_{\vec{k}}\hat{a}_{\vec{k}} + \hat{b}^{\dagger}_{\vec{k}}\hat{b}_{\vec{k}} + (2\pi)^3\,\delta^{(3)}(0)\right]. \tag{33}$$

5. **Real scalar fields**

We worked out canonical quantization in excruciating detail for the complex scalar field. However forinteractions we will study the **real** scalar field. Repeat the process of canonical quantization for the real scalar field with action

$$S = \int d^4x\left(-\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi) - \frac{1}{2}m^2\phi^2\right) \tag{34}$$

In particular, verify that the Fourier expansion for the field operator takes the form:

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(a_{\vec{k}}e^{-i\omega_{\vec{k}}t+i\vec{k}\cdot\vec{x}} + a^{\dagger}_{\vec{k}}e^{i\omega_{\vec{k}}t-i\vec{k}\cdot\vec{x}}\right) \tag{35}$$

and prove to yourself that the commutation relation of the creation operator $a_{\vec{k}}$ is:

$$[a_{\vec{k}}, a^{\dagger}_{\vec{k}'}] = (2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}'), \tag{36}$$

and that the normal ordered Hamiltonian is simply

$$:H: = \int \frac{d^3k}{(2\pi)^3}\omega_{\vec{k}}a^{\dagger}_{\vec{k}}a_{\vec{k}} \tag{37}$$

There are no b's any more as there is no longer a conserved charge associated with particle numbers (note that the action no longer has a $U(1)$ symmetry). Thus there is only one type of particle, which is created by $a^{\dagger}_{\vec{k}}$ acting on the vacuum $|0\rangle$.

6. **Momentum of single-particle eigenstates**

The conserved Noether charge associated to symmetry under spatial translations $\vec{x} \to \vec{x} + \vec{b}$ is the momentum of a field configuration, which for a *real* scalar field took the following form.

$$\vec{P} = -\int d^3\vec{x}\, \dot{\phi} \vec{\nabla} \phi. \tag{38}$$

Show that, upon canonical quantisation, the normal ordered expression for the corresponding operator in terms of creation and annihilation operators reads:

$$\vec{P} = \int \frac{d^3\vec{k}'}{(2\pi)^3}\, \vec{k}'\, a^\dagger_{\vec{k}'} a_{\vec{k}'}, \tag{39}$$

Show that a particle created by $a^\dagger_{\vec{k}}$ indeed has momentum $\vec{k}$.

7. **Real scalar fields**

(a) Show that the time ordered product $T(\phi(x_1)\, \phi(x_2))$ and the normal ordered product $:\phi(x_1)\, \phi(x_2):$ are both symmetric under the interchange of x_1 and x_2. Take ϕ to be a real scalar field. Deduce that the Feynman propagator $G_\phi(x_1, x_2)$ has the same symmetry property.

(b) Consider the following action for two real scalar fields $\phi(x)$ and $\Phi(x)$ of mass m and M respectively,

$$S = \int d^4x \left(-\frac{1}{2}\partial_\mu \phi\, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{2}\partial_\mu \Phi\, \partial^\mu \Phi - \frac{1}{2} M^2 \Phi^2 \right). \tag{40}$$

The Feynman propagators for the fields ϕ and Φ are given by:

$$G_\phi(x, y) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \frac{-i}{k\cdot k + m^2 - i\epsilon}, \tag{41}$$

$$G_\Phi(x, y) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \frac{-i}{k\cdot k + M^2 - i\epsilon}. \tag{42}$$

Show that

$$\int d^4z\, G_\phi(x, z)\, G_\phi(z, y) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \left(\frac{-i}{k\cdot k + m^2 - i\epsilon} \right)^2, \tag{43}$$

and similarly for $\Phi(x)$ (You do not need to do this for both fields as the computation is identical). Hence show that the following general result holds:

$$\int d^4z_1\, d^4z_2 \ldots d^4z_{N-1}\, d^4z_N\, G_\phi(x, z_1)\, G_\phi(z_1, z_2) \ldots G_\phi(z_{N-1}, z_N)\, G_\phi(z_N, y)$$
$$= \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \left(\frac{-i}{k\cdot k + m^2 - i\epsilon} \right)^{N+1}. \tag{44}$$

(c) Using Wick's theorem, evaluate the expression

$${}_0\langle 0| T\{\phi(x_1)\, \Phi(x_2)\, \phi(x_3)\, \Phi(x_4)\, \Phi(x_5)\, \Phi(x_6)\} |0\rangle_0, \tag{45}$$

in terms of Feynman propagators $G_\phi(x, y)$ and $G_\Phi(x, y)$. *Hint: note that there is no contraction between ϕ and Φ.*

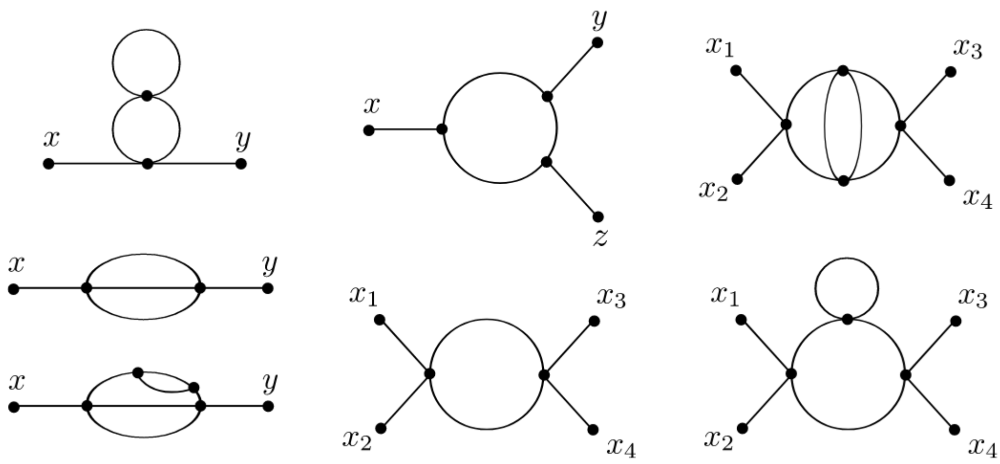
(d) Evaluate the expression:

$$ {}_0\langle 0|T\left\{\phi\left(x_1\right)\phi(x_2)\ldots\phi\left(x_{99}\right)\Phi\left(x_{100}\right)\Phi\left(x_{101}\right)\Phi\left(x_{102}\right)\Phi\left(x_{103}\right)\right\}|0\rangle_0. \qquad (46) $$

Explain how you arrived to the answer.

8. **Diagrammology**

(a) What are the symmetry factors for the following Feynman diagrams?



(b) Consider the vacuum-to-vacuum amplitude in $\lambda\phi^4$ theory:

$$ {}_0\langle 0|T\left\{\exp\left[-\frac{i\lambda}{4!}\int d^4x'\,\phi_I^4\left(x'\right)\right]\right\}|0\rangle_0. \qquad (47) $$

Using Wick's theorem, identify the different contributions up to and including $O\left(\lambda^2\right)$.