STFC HEP summer school 2025

Phenomenology Problems

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1 Basic Kinematics

The rapidity y and pseudo-rapidity η are defined as:

$$y = \frac{1}{2}\log\left(\frac{E+p_z}{E-p_z}\right)$$
$$\eta = -\log\left(\tan\left(\frac{\theta}{2}\right)\right)$$

where z is the direction of the colliding beams.

a) Verify that for a particle of mass m

$$E = \sqrt{m^2+p_T^2}\,\cosh y$$
$$p_z = \sqrt{m^2+p_T^2}\,\sinh y$$

with $p_T^2 = p_x^2 + p_y^2$.

b) Prove that $\tanh\eta = \cos\theta$

c) Prove that rapidity equals pseudo-rapidity for a relativistic particle $E \gg m$

d) Prove that the difference of two rapidities is Lorentz invariant.

2 $e^+e^- \to$ hadrons

By studying the shape of the Z-resonance in the R-ratio we can try to see the effect of some parameters in the electro-weak interactions. You can answer the questions qualitatively but you could also plot the function if you have time. The cross-section for fermion annihilation to another fermion pair takes the form:

$$\begin{aligned} d\sigma\left(f\bar{f}\to f'\bar{f}'\right) &= \alpha^2\frac{\pi}{2s}d(\cos\theta) \\ &\left\{\left(1+\cos^2\theta\right)\left(q_f^2 q_{f'}^2 + \frac{g_z^2}{4g_e^2}q_f q_{f'} v_f v_{f'}\chi_1 + \frac{g_z^4}{16g_e^4}(a_f^2+v_f^2)(a_{f'}^2+v_{f'}^2)\chi_2\right)\right. \\ &\left.+\cos\theta\left(\frac{g_z^2}{2g_e^2}q_f q_{f'} v_f v_{f'}\chi_1 + \frac{g_z^4}{2g_e^4}a_f a_{f'} v_f v_{f'}\chi_2\right)\right\} \end{aligned} \tag{1}$$

where

$$\frac{g_Z}{g_e} = \frac{1}{\cos\theta_w\sin\theta_w} \qquad \chi_1 = \frac{s(s-m_Z^2)}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2} \qquad \chi_2 = \frac{s^2}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}$$

	q_f	a_f	v_f
u, c, t	$2/3$	$1/2$	$1/2 - 4/3\sin^2\theta_w$
d, s, b	$-1/3$	$-1/2$	$-1/2 + 2/3\sin^2\theta_w$
e, μ, τ	-1	$-1/2$	$-1/2 + 2\sin^2\theta_w$
ν_e, μ, τ	0	$1/2$	$1/2$

Table 1: EW couplings in the Standard Model.

and

$$g_z^2 = \frac{4\pi\alpha}{\cos^2\theta_w \sin^2\theta_w} \qquad \Gamma_Z \approx \sum_l \Gamma_{Z\to l\bar{l}} + \sum_q N_c \Gamma_{Z\to q\bar{q}} \qquad \Gamma_{Z\to f\bar{f}} = \frac{m_Z \alpha}{12\cos^2\theta_w \sin^2\theta_w}\left(a_f^2 + v_f^2\right)$$

The axial and vector couplings in the Standard Model are given in Table 1.

a) Look at the expression above (1) and identify the physical origin of each term, e.g. draw the relevant Feynman diagrams and discuss the link to the terms in the equation.

b) What happens to the interference between the photon and the Z diagrams if $\sin^2\theta_w = 0.25$?

c) Schematically plot $R(\frac{e^+e^-\to\text{hadrons}}{e^+e^-\to\mu^+\mu^-})$ like we saw in the lectures. If time allows one can use some plotting software to do a more precise job.

3 Jet Kinematics

At the LHC each beam has an energy of 7 TeV. Two partons collide and produce two jets with negligible mass, transverse momentum p_T and rapidities $y_{3,4}$.

a) Show that

$$x_1 = \frac{p_T}{\sqrt{s}}(e^{y_3} + e^{y_4}), x_2 = \frac{p_T}{\sqrt{s}}(e^{-y_3} + e^{-y_4})$$

b) Show that the invariant mass of the dijet system is

$$M_{JJ} = 2p_T \cosh\left(\frac{y_3 - y_4}{2}\right)$$

and the centre of mass scattering angle is:

$$\cos\theta^* = \tanh\left(\frac{y_3 - y_4}{2}\right)$$

c) Discuss the regions of $x_{1,2}, M_{JJ}$ and θ^* probed with a jet trigger of $p_T > 35$ and $|y_{3,4}| < 3$

4 Event shapes

The thrust is defined as

$$T = \max_{\vec{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|},$$

where the sum is over all the particles, the ith particle has 3-momentum $\vec{p}_i$, and $\vec{n}$ is a unit-vector.

Explain why the value of the thrust is given by

a) 1 for back-to-back configurations,

b) $\frac{1}{2}$ for a perfectly spherical event (i.e. uniform ditribution of momenta).

c) Calculate the minimum possible value of the thrust for a $q\bar{q}g$ state. [Hint: This occurs for the 'Mercedes' configuration where all the particles have the same energy and the angle between any two particles is 120^0.]

Consider the two event shape variables

$$S_{\text{lin}} = \left(\frac{4}{\pi}\right)^2 \min_{\vec{n}} \left(\frac{\sum_i |\vec{p}_i \times \vec{n}|}{\sum_i |\vec{p}_i|}\right)^2, \qquad S_{\text{quad}} = \frac{3}{2} \min_{\vec{n}} \frac{\sum_i |\vec{p}_i \times \vec{n}|^2}{\sum_i |\vec{p}_i|^2}.$$

d) Determine whether S_{lin} is infrared safe or not.

e) What are the limiting values of S_{lin} for pencil-like (back-to-back) and spherical events?

f) What is the value for the Mercedes configuration?

5 Infrared safety

Are these observables infrared safe at a hadron collider? If not, how would you modify them to make them infrared safe?

a) Partonic center of mass energy (defined as a the invariant mass of the sum of all final state particles in the event).

b) The sum of the energies of all jets with transverse momentum above a given p_T threshold.

c) The invariant mass of all jets in the event.

d) The number of partons.

6 Higgs physics

a) The cross section for $pp \to X$ at a collider centre of mass energy, $\sqrt{s}$, can be written as a convolution of the partonic contributions $\sigma(ab \to X) = \hat{\sigma}_{ab}$ and the corresponding parton luminosities

$$\sigma_{pp\to X} = \sum_{ab} \int_0^1 \frac{d\tau}{\tau} \cdot \mathcal{L}_{ab}(s,\tau) \cdot \left[\hat{s}\hat{\sigma}_{ab}(\hat{s})\right], \quad \tau = \frac{\hat{s}}{s}.$$

Given the partonic Higgs production cross section from gluon fusion from a quark loop in terms of the amplitude function $A_q(\zeta = 4m_q^2/m_h^2)$

$$\hat{\sigma}_{gg\to h} = \frac{G_F\,\alpha_s^2}{288\sqrt{2}\pi}|A_q(\zeta)|^2, \quad A(\zeta) = \frac{3}{2}\zeta\left[1 + (1-\zeta)f(\zeta)\right], \quad f(\zeta) = \begin{cases} \arcsin^2 1/\sqrt{\zeta} & \zeta \geq 1 \\ \frac{1}{2}\left[\log\frac{1+\sqrt{1-\zeta}}{1-\sqrt{1-\zeta}} - i\pi\right]^2 & \zeta < 1 \end{cases}$$

Calculate the relative correction of the b-quark loop to the gluon fusion Higgs production rate (neglecting all lighter quark loops).

b) Show that the presence of an invisible branching fraction, $\mathrm{BR}_{\mathrm{inv.}}$, in the absence of any other modifications of Higgs couplings, leads to an overall rescaling of the Higgs signal strengths by a factor $(1 - \mathrm{BR}_{\mathrm{inv.}})$.

c) In the high energy limit $(s, t \gg v)$ the amplitude for longitudinal W boson scattering, $W_L W_L \to W_L W_L$ when consider only the gauge boson self-interactions can be shown to be

$$\mathcal{A}^{\mathrm{gauge}}(W_L W_L \to W_L W_L) = \frac{ig^2}{4m_W^2}(s+t) + \mathcal{O}(s^0, t^0)$$

This kind of behaviour for an amplitude is an indication of unitarity violation since the scattering probability blows up in the high energy limit. Show that the Higgs boson unitarises this amplitude.
Recall that the Higgs boson coupling to a pair of W bosons is given by the Feynman rule:

$$h\text{-}W\text{-}W : i\,g\,m_W\,g^{\mu\nu}$$

and that, in the high energy limit, one can replace the longitudinal polarisation vector of the W-boson with

$$\epsilon_L^\mu \to \frac{p^\mu}{m_W}.$$

d) Consider the effect of the Lagrangian term,

$$\mathcal{O}_{HD} = \frac{1}{v^2}(H^\dagger D^\mu H)^*(H^\dagger D^\mu H),$$

after electroweak symmetry breaking. Does it respect custodial symmetry? What will its impact be on the ρ parameter? Recall the covariant derivate acting in the Higgs is given by

$$D^\mu H = (\partial^\mu - i\frac{g}{2}\tau^I W_I^\mu - \frac{g'}{2}B^\mu)H$$

and the ρ parameter is defined as

$$\rho = \frac{m_W^2}{m_Z^2\cos^2\theta_W}$$

and is equal to 1 in the SM.

7 EFT and anomalous couplings

The following dimension-6 operators modify the couplings of the top quark to the weak gauge bosons (Q is the third generation left-handed doublet and t the right handed top quark field, φ is the Higgs field):

$$O_{\varphi Q}^{(3)} = i\frac{1}{2}y_t^2 \left(\varphi^\dagger \overleftrightarrow{D}^I_\mu \varphi\right)(\bar{Q}\gamma^\mu \tau^I Q) \quad (2)$$

$$O_{\varphi Q}^{(1)} = i\frac{1}{2}y_t^2 \left(\varphi^\dagger \overleftrightarrow{D}_\mu \varphi\right)(\bar{Q}\gamma^\mu Q) \quad (3)$$

$$O_{\varphi t} = i\frac{1}{2}y_t^2 \left(\varphi^\dagger \overleftrightarrow{D}_\mu \varphi\right)(\bar{t}\gamma^\mu t) \quad (4)$$

$$O_{tW} = y_t g_w (\bar{Q}\sigma^{\mu\nu}\tau^I t)\tilde{\varphi}W^I_{\mu\nu} \quad (5)$$

$$O_{tB} = y_t g_Y (\bar{Q}\sigma^{\mu\nu} t)\tilde{\varphi}B_{\mu\nu} \quad (6)$$

a) Explain qualitatively which particular top couplings will be modified by each operator.

b) Write down the Feynman rules for the ttZ vertex including the impact of 2-fermion operators listed above: $\mathcal{O}_{tW}, \mathcal{O}_{tB}, \mathcal{O}_{\phi t}, \mathcal{O}^{(3)}_{\phi Q}, \mathcal{O}^{(1)}_{\phi Q}$ e.t.c. and compare with the typical anomalous coupling parametrisation of the ttZ vertex.

$$\mathcal{L}_{ttZ} = e\bar{u}(p_t)\left[\gamma^\mu\left(C^Z_{1,V} + \gamma_5 C^Z_{1,A}\right) + \frac{i\sigma^{\mu\nu}q_\nu}{m_Z}\left(C^Z_{2,V} + i\gamma_5 C^Z_{2,A}\right)\right] v(p_{\bar{t}})Z_\mu$$

c) What are the expressions for: $C^Z_{1,V}, C^Z_{1,A}, C^Z_{2,V}$ and $C^Z_{2,A}$ in terms of the dim-6 Wilson coefficients?

d) Use this to explain why there are degeneracies between operators if one only looks at processes involving the ttZ interaction.

8 DGLAP splitting kernels (Optional)

Show that in the collinear limit $p_3 \to z p_{\tilde{13}}$, $p_1 \to (1-z)p_{\tilde{13}}$ the matrix element

$$\left\langle |\mathcal{M}(a_{e^+}, b_{e^-}, 1_q, 2_{\bar{q}}, 3_g)|^2 \right\rangle = \frac{4e^2 e_q^2 g_s^2 N_c}{s} C_F \frac{s_{a1}^2 + s_{a2}^2 + s_{b1}^2 + s_{b2}^2}{s_{13}s_{23}}$$

factorizes to

$$|\mathcal{M}_{q\bar{q}g}|^2 \to |\mathcal{M}_{q\bar{q}}|^2 \times \frac{2g_s^2}{s_{13}} \times C_F \frac{1+(1-z)^2}{z}$$

$$\left\langle |\mathcal{M}(a_{e^+}, b_{e^-}, 1_q, 2_{\bar{q}}, 3_g)|^2 \right\rangle \overset{3||1}{\to} \left\langle |\mathcal{M}(a_{e^+}, b_{e^-}, (\tilde{13})_q, 2_{\bar{q}})|^2 \right\rangle \frac{2g_s^2 C_F}{s_{13}} \frac{1+(1-z)^2}{z}$$

with

$$\left\langle |\mathcal{M}(a_{e^+}, b_{e^-}, 1_q, 2_{\bar{q}})|^2 \right\rangle = 2e^2 e_q^2 N_c \frac{s_{a1}^2 + s_{a2}^2}{s_{ab}}$$

9 Operators and EOMs (Optional)

Show that the two operators in:

$$\mathcal{O}_{gt} = \bar{t}\, T_A \gamma^\mu D^\nu t\, G^A_{\mu\nu}, \quad (7)$$

$$\mathcal{O}_{gQ} = \overline{Q} T_A \gamma^\mu D^\nu Q G^A_{\mu\nu}, \quad (8)$$

can be written as a sum of four fermion operators