# Amplitudes 2026 at Queen Mary

29 June — 3 July 2026

Local organisers: Andi Brandhuber, Ricardo Monteiro, Gabriele Travaglini, Congkao Wen, Chris White

# Amplitudes 2026 School in Southampton

6 July — 10 July 2026

Local organisers: James Drummond, Ömer Gürdoğan

SAGEX Scattering Amplitudes from Geometry to Experiment

Queen Mary University of London

University of Southampton

# Classical gravity from the Magnus expansion

Gabriele Travaglini

Queen Mary University of London

with

Andi Brandhuber, Graham Brown, Paolo Pichini and Pablo Vives Matasán

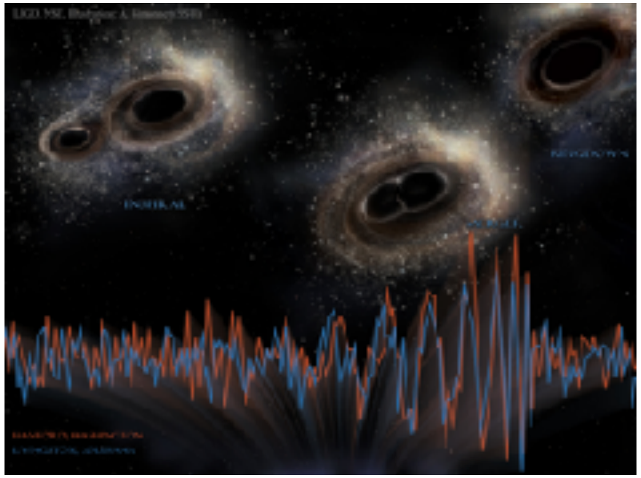
Annual Theory Meeting, Durham University, 15th December 2025

Queen Mary University of London

UKRI Science and Technology Facilities Council

LEVERHULME TRUST

# Motivation

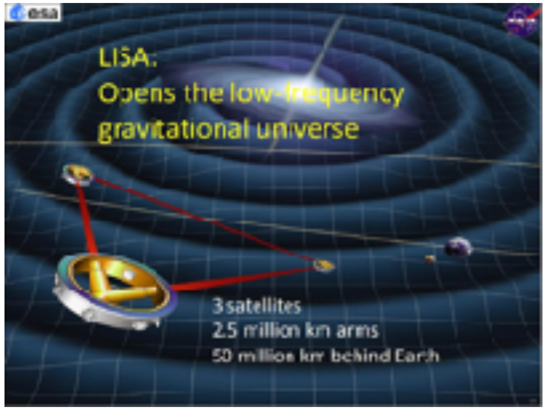


Credit: Contemporary Physics Education Project

- A new era of gravitational observations!
  - LIGO-Virgo-KAGRA: O1,2,3: ~100 binary mergers
    O4: ~250 events
  - Upcoming GW observatories in the 2030s will increase sensitivity and frequency range:

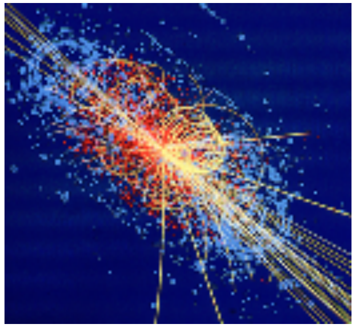
    Advanced LIGO (ground-based), Cosmic Explorer (ground-based), Einstein Telescope (underground), LISA (space-based)

- Need ever more precise theoretical predictions
  - Higher perturbative orders (in Newton's constant $G$)
  - Modifications due to the spin of the black holes
  - Deformations of Einstein-Hilbert theory
  - LSC Observational White Paper



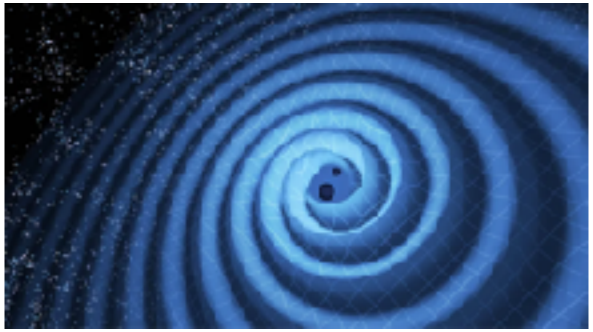

Credit: ESA

# Why amplitudes?



Credit: CMS/CERN

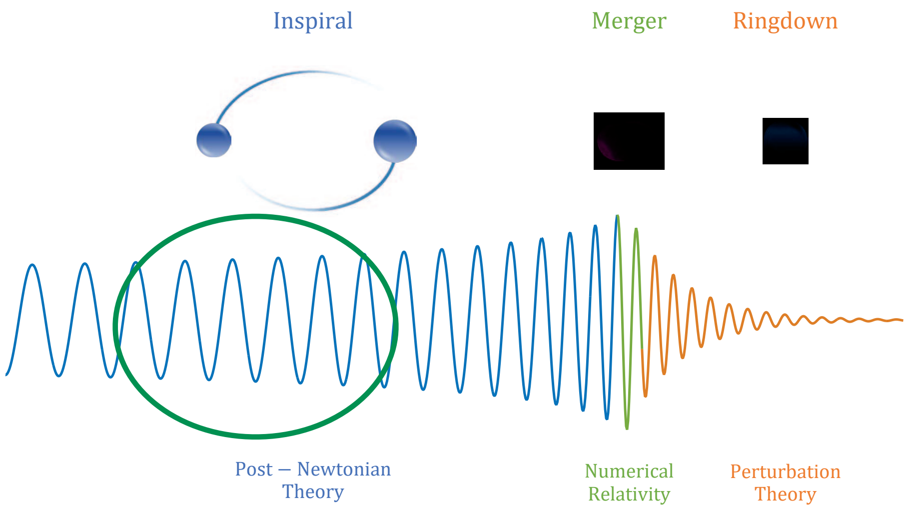
EFT



Credit: Tim Pyle (LIGO)

- Treat black holes/neutron stars as effectively pointlike
  - Describe their structure and interactions using Effective Field Theory
  - Use modern amplitude methods in the perturbative regime (inspiral phase)

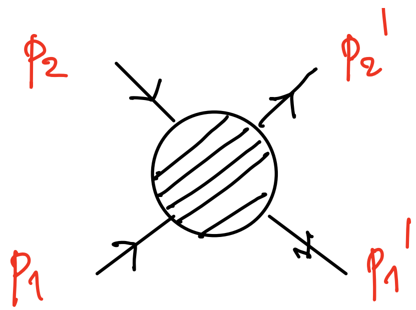
- Merger of 2 BHs:
  - Inspiral
  - Merger
  - Ringdown

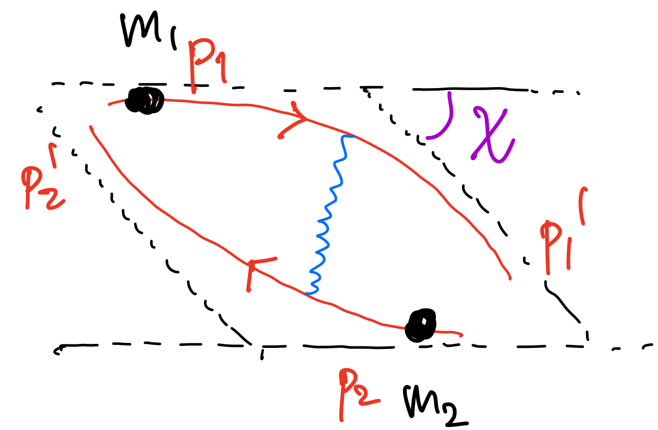



from Antelis and Moreno 1610.03567

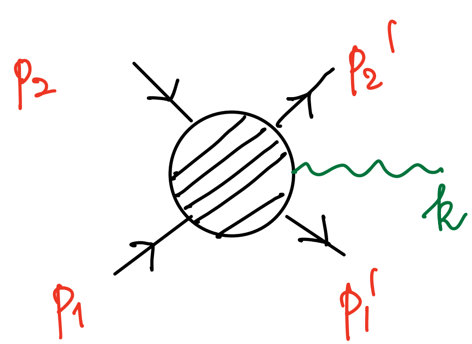
# Binary dynamics from amplitudes

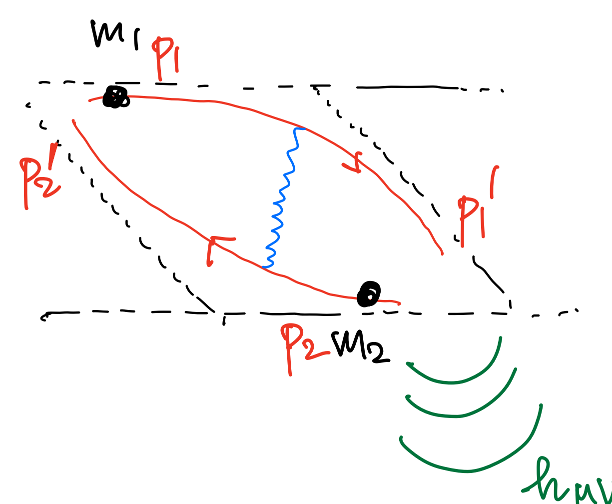
- Conservative: scattering angles
  - From four-point amplitudes




- Dissipative: waveforms
  - From five-point amplitudes

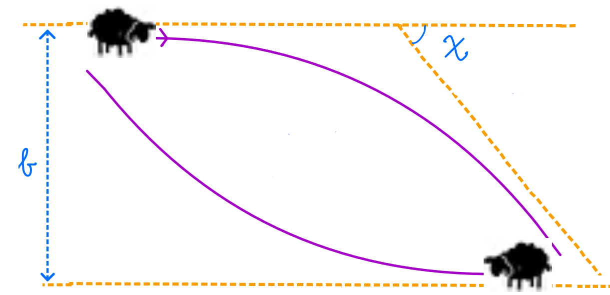


# Menu

- Classical limit and Post-Minkowskian (PM) expansion
- Introducing the $N$-operator $S = e^{iN}$
  - Radial action from matrix elements of the $N$-operator in Impact Parameter Space (IPS)
- The Magnus expansion in relativistic quantum field theory
  - Magnus amplitudes, or matrix elements of the $N$-operator
  - Applications to $\phi^3$ theory
- Classical limit
  - Loops from trees?

# Classical limit and expansions




- Classical limit $GM \gg \hbar/M$ (Schwarzschild radius: $GM$; Compton wavelength: $\hbar/M$)
  - Naive expansion parameter is $GM^2/\hbar \sim 10^{76}$ for $M \sim M_{\rm sun}$! Is this a problem?

- Amplitude eikonal exponentiation in impact parameter space

(Glauber; Levi & Sucher; Amati, Ciafaloni & Veneziano; Muzinich & Soldate; Kabat & Ortiz)

$$\widetilde{\mathcal{A}} \sim e^{i(\delta_0+\delta_1+\cdots)} \quad \text{with} \quad \widetilde{\mathcal{A}}(b) \sim \int \frac{d^D q}{(2\pi)^{D-2}} e^{iq\cdot b}\, \delta(2p_1\cdot q)\delta(2p_2\cdot q)\, \mathcal{A}(q)$$

  - $\delta = \delta_0 + \delta_1 + \cdots$ is the eikonal phase, from which one extracts the deflection angle
  - $\delta_{n+1}/\delta_n \sim GM/b$ hence expansion parameter is $GM/b \sim R_S/b$
  - $GM/b \sim 1/6$ in early inspiral phase for typical LIGO event

- Combine: $\dfrac{\hbar}{M} \ll GM \ll b$

$$3.5 \times 10^{-73}\text{m} \left(\frac{M_\odot}{M}\right) \ll 1.5 \text{ km} \left(\frac{M}{M_\odot}\right) \ll b$$

- Scattering processes vs bound orbits
  - Unbound / highly eccentric orbits
  - Bound trajectories obtained via analytic continuation (Kälin, Porto + Cho)
  - Bound orbits incompatible with $b \gg GM$ in late inspiral phase
- Post-Minkowskian (PM) expansion
  - Exact in velocities, maintains Lorentz invariance at every steps
  - $n\,\mathrm{PM} = G^n$
  - amplitude methods avoid errors from evaluating many separately non gauge-invariant Feynman diagrams
- Contrast with the Post-Newtonian (PN) expansion
  - An expansion in velocity $v$
  - Virial theorem relates the two expansion (bound orbits): $\frac{v^2}{2} \sim \frac{GM}{R}$

# PM vs PN expansions

Newton (1687)
1 PN (1917,1938)
2 PN (1980)
3 PN (2000)
4 PN (2007-2019)
5 PN (2019…)
6 PN (2020…)

- 1 PM $G(1 + v^2 + v^4 + v^6 + v^8 + \cdots)$
- 2 PM $G^2(1 + v^2 + v^4 + v^6 + v^8 + \cdots)$
- 3 PM $G^3(1 + v^2 + v^4 + v^6 + v^8 + \cdots)$
- 4 PM $G^4(1 + v^2 + v^4 + v^6 + v^8 + \cdots)$
- 5 PM $G^5(1 + v^2 + v^4 + v^6 + v^8 + \cdots)$
- 6 PM $G^6(1 + v^2 + v^4 + v^6 + v^8 + \cdots)$

$$n\mathrm{PN} \sim (v^2)^n$$

# Newton potential from amplitudes

- From elastic scattering of two heavy (pointlike!) particles
  - Connection to amplitudes suggested by Yoichi Iwasaki in 1971

Progress of Theoretical Physics, Vol. 46, No. 5, November 1971

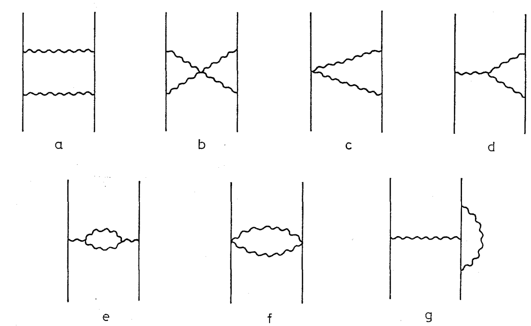
**Quantum Theory of Gravitation vs. Classical Theory**[*)]

——*Fourth-Order Potential*——

Yoichi IWASAKI

*Research Institute for Fundamental Physics, Kyoto University, Kyoto*

(Received May 18, 1971)




  - Classical correction at $O(G^2)$ from one-loop Feynman diagrams

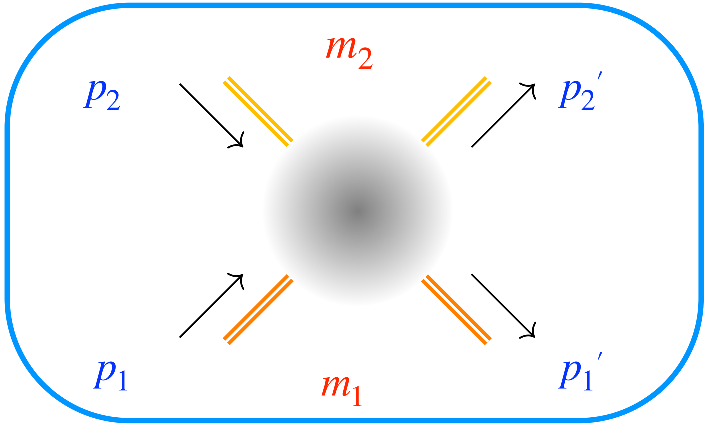
  - Iwasaki pointed out the "erroneous belief" that only tree diagrams contribute to classical processes, e.g. R.P. Feynman, Acta Phys. Polon. 24 (1963), 697

  - He also noted that

"...in spite of the unrenormalisability we can obtain a finite physically meaningful potential"

# From the amplitude to the potential

- Two-to-two scattering:
  - $\vec{q} = \vec{p}_2 - \vec{p}_2'$ momentum transfer
  - $E_1 = E_1', \;\; E_2 = E_2'$ (COM frame)




- Static potential from elastic amplitude:

$$\langle f|S|i\rangle = \delta_{fi} + (2\pi)^4 \delta^{(4)}(p_f - p_i)\, A(\vec{q}) \;=\; -i\,(2\pi)\delta(E_f - E_i)\langle f|\tilde{V}|i\rangle$$

$$V(\vec{x}) \;=\; \frac{i}{4m_1 m_2} \int \frac{d^3q}{(2\pi)^3}\; e^{i\vec{q}\cdot\vec{x}}\; A(\vec{q})$$

  - Potential depends on the choice of coordinates, observables (e.g. scattering angles) better quantities to compute

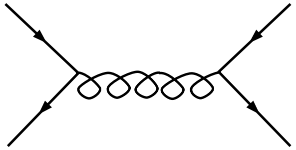
- **Static limit:** $s \simeq (m_1 + m_2)^2\,, \quad t = -\vec{q}^{\,2}, \qquad A \to i\,G\ \dfrac{16\pi\, m_1^2 m_2^2}{\vec{q}^{\,2}}$

- **Static potential:** $\tilde{V}_{\rm static}(\vec{q}) := i\,\dfrac{A(\vec{q})}{4 m_1 m_2}$

    - Finally using $\displaystyle\int \frac{d^3q}{(2\pi)^3}\frac{e^{i\vec{q}\cdot\vec{r}}}{\vec{q}^{\,2}} = \frac{1}{4\pi r}$ we get

$$V_{\rm static} = -\frac{G m_1 m_2}{r}$$

- **Single-graviton exchange gives Newton potential**

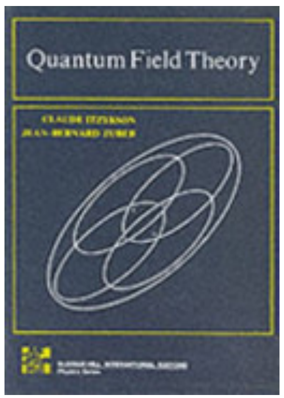


- **Higher orders in $G$ arise from loops**
    - $\hbar \to 0$ (classical) limit receives contributions from loops
    - The loop expansion is not an $\hbar$ expansion!

# Classical physics from loops

(Donoghue & Holstein; Iwasaki…)

- The loop expansion is not an $\hbar$ expansion
  - Itzykson-Zuber, chapter 6.2.1:




  - "The loopwise perturbative expansion, i.e. the expansion according to the increasing number of independent loops of connected Feynman diagrams, may be identified with an expansion in powers of $\hbar$...
  - ...we leave aside the factor of $\hbar$ that gives the mass term a correct dimension. In other words,

    the Klein-Gordon equation should read

$$\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\phi = 0$$

  - ....indicating that the mass is of quantum origin. This phenomenon is disregarded in the sequel."

No $\hbar$ !

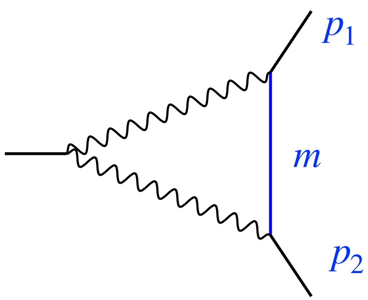
- **Propagator:** $\langle 0|T(\phi(x)\phi(0))|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i\,\hbar}{k^2 - \left(\frac{mc}{\hbar}\right)^2 + i\epsilon}$
  - $k$ is the wave four-vector, so that the loop momentum is $\ell = \hbar\, k$
  - factors of $\hbar$ from masses spoil naïve counting!

- **Example: triangle with one internal mass**

$p_1 = \hbar\, k_2$

$p_2 = \hbar\, k_2$

$p_1^2 = p_2^2 = m^2 c^4$




$$= I_3(k_{12}^2, m) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \left(\frac{mc}{\hbar}\right)^2)(k-k_1)^2(k+k_2)^2}$$

$$= -\frac{i}{32}\Big[\frac{1}{(mc/\hbar)\sqrt{-k_{12}^2}} + \frac{\log(-k_{12}^2/(mc/\hbar)^2)}{\pi^2\,(mc/\hbar)^2}\Big] + \mathcal{O}(\sqrt{k_{12}^2})$$

  - Second term is $O(\hbar)$ compared to the first: classical and quantum
  - $\sqrt{-q^2}$ classical, $\log(-q^2)$ quantum

# What can be reliably computed

- Long-range physics from non-analytic terms
  - GR is non-renormalisable! However…
  - focus on IR theory: low-energy gravitons that propagate long distances
  - From momentum space to position space:

not interested in this!

$$\frac{1}{q^2} \to \frac{1}{r}\,, \quad \frac{1}{q^2}\sqrt{-q^2} \to \frac{1}{r^2}\,, \quad \frac{1}{q^2}\,q^2\,\log(-q^2) \to \frac{1}{r^3}\,, \quad \frac{1}{q^2}q^2 \to \delta^{(3)}(\vec{x})$$

  - Non-analytic terms cannot be reabsorbed with a local counterterm, i.e. they cannot be modified by any high-energy modification to the theory
- Goal: find non-local/non-analytic terms
  - Arise from long-range propagation of massless particles at low energy
  - Ideal for unitarity-based techniques. That is when amplitudes come in!

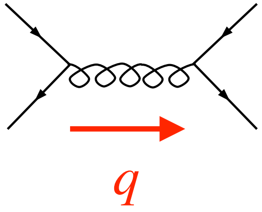
# Extracting the classical limit

(Kosower, Maybee, O'Connell)

- We are (only!) interested in classical physics
  - Take $\hbar \to 0$ limit as soon as possible
  - Write external/internal momenta of massless particles and momentum transfers, $(k, \ell, q)$, in terms of wave-vectors $(\hat{k}, \hat{\ell}, \hat{q})$, to be kept fixed:

$$(k, \ell, q) = \hbar\,(\hat{k}, \hat{\ell}, \hat{q})$$

  - Reinstate $\hbar$ in couplings: $\kappa/\sqrt{\hbar}$ and $G/\hbar$ $(\kappa^2 := 32\pi G)$ $\qquad \alpha_{\text{e.m.}} = \frac{e^2}{4\pi\varepsilon_0\,\hbar c}$
  - Massive particles' momenta and masses are kept fixed
- Example: four-point amplitude




  - Amplitude scales as $A_4 \sim \kappa^2/q^2 \to \hbar^{-3}\kappa^2/q^2$ disconcerting at first?
  - Newton's potential: $V(\vec{x}) = \frac{1}{4m_1 m_2}\int d^3q\, e^{i\vec{q}\cdot\vec{x}} A_4 \longrightarrow \mathcal{O}(\hbar^{3-3}) = \mathcal{O}(\hbar^0)$

# Observables from the $S$-matrix

- Eikonal phase vs radial action
  - Eikonal: $i\delta \sim \log\big(\langle p_1' p_2' | S | p_1, p_2 \rangle_{\rm IPS}\big)$ logarithm of the matrix element
  - Radial action: $iI_r \sim \langle p_1' p_2' | \log(S) | p_1, p_2 \rangle_{\rm IPS}$ matrix element of the logarithm of the $S$-matrix
  - From the radial action one can extract classical observables (Gonzo & Shi; Kim, Kim, Lee)

$$O_{\rm out} = e^{\{\,,\cdot\}} O_{\rm in} = O_{\rm in} + \{I_r, O_{\rm in}\} + \frac{1}{2!}\{I_r,\{I_r,O_{\rm in}\}+\frac{1}{3!}\{I_r,\{I_r,\{I_r,O_{\rm in}\}\}\} + \cdots$$

- Deflection angle $\chi$ in binary encounter:
  - Eikonal: $2\sin\frac{\chi}{2} = -\frac{\partial}{\partial J}{\rm Re}(\delta)$ vs radial action $\chi = -\frac{\partial}{\partial J} I_r$
  - Deceptively similar! Disagree beyond one loop

- Evidence that $I_r \overset{\text{IPS}}{\longleftrightarrow} \langle p_1' p_2' | N | p_1, p_2 \rangle$ with $S = e^{iN}$
  - Deflection angle at two loops from velocity cuts (Damgaard, Planté, Vanhove) and Heavy-mass EFT (HEFT) (Brandhuber, Chen, GT, Wen)
  - Related work with three-loop angle (Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, & Zeng)
  - $N$ is hermitian (since $S$ is unitary) hence $I_r$ is real (unlike the eikonal!)
  - Need a tool to compute the $N$-operator directly

Radial action = HOLY GRAIL!

Credit: Indiana Jones

- Magnus expansion of the $N$-operator
  - Important work of Kim, Kim, Kim & Lee applied to Worldline QFT (WQFT)
  - We discuss applications to relativistic quantum field theories
  - emergence of "Murua coefficients"

# Magnus from Dyson

(Damgaard, Hansen, Planté & Vanhove)

- Derive $N$-operator from the $S$-matrix:

$$S = 1 + i\frac{T}{\hbar} = e^{i\frac{N}{\hbar}}$$

$$N^{(1)} = T^{(1)}$$

$$N^{(2)} = T^{(2)} - \frac{i}{2\hbar}(T^{(1)})^2$$

$$N^{(3)} = T^{(3)} - \frac{i}{2\hbar}\left(T^{(1)}T^{(2)} + T^{(2)}T^{(1)}\right) - \frac{1}{3\hbar^2}(T^{(1)})^3$$

  - is free of hyper-classical terms, $\hbar \to 0$ limit is trivial, however….
  - Amplitudes $T^{(i)}$ contains hyper-classical terms that need to cancel in $N^{(i)}$
  - hyper-classical terms (more dominant than classical in $\hbar$) are iterations of lower loops
  - Lots of cancellation, and in addition RHS is actually quite complicated!

- Can we do better? We need Magnus without Dyson!

# Dyson vs Magnus I.

Freeman Dyson

- Schrödinger equation in the interaction picture
  - $i\hbar\partial_t |t\rangle_I = H_I(t)|t\rangle_I$
  - $H = H_0 + V\,, \quad H_I(t) = e^{\frac{iH_0 t}{\hbar}} V e^{-\frac{iH_0 t}{\hbar}}$
- Evolution operator and $S$-matrix
  - $|t\rangle_I = U(t,t_0)|t_0\rangle_I$ with $U(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, H_I(t')U(t't)$
  - Volterra equation of the 2nd type
- S-matrix: $S = \lim_{\substack{t\to+\infty \\ t_0\to-\infty}} U(t,t_0)$
- Dyson's expansion for the $S$-matrix (Dyson, 1949)

$$S = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n \int dx_1 \cdots dx_n\; T\{\mathcal{H}_I(x_1)\cdots\mathcal{H}_I(x_n)\}$$

- $S = 1 + \frac{i}{\hbar} T$, with $T = T^{(1)} + T^{(2)} + \cdots$, e.g.

$$iT^{(1)} = -i \int d^4x_1 \; \mathcal{H}_1 \, ,$$

$$iT^{(2)} = \frac{(-i)^2}{2!} \frac{1}{\hbar} \int d^4x_1 d^4x_2 \; T\{\mathcal{H}_1 \mathcal{H}_2\}$$

$$iT^{(3)} = \frac{(-i)^3}{3!} \frac{1}{\hbar^2} \int d^4x_1 d^4x_2 d^4x_3 \; T\{\mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3\}$$

$$\mathcal{H}_i = \mathcal{H}(x_i)$$

- Pros
  - Beautiful closed formula: Time-ordered products, Feynman propagators
  - Easy to deal with time-ordered products (Wick theorem, Feynman diagrams)
- Cons
  - Unitarity not manifest unless we sum all perturbative orders
  - Convergence not ideal
  - As mentioned earlier, $T$-matrix elements contain hyper-classical terms

# Dyson vs Magnus 2.

- Magnus: write $S = e^{\frac{i}{\hbar}N}$ and solve for $N$ (Magnus, 1954)

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. VII, 649-673 (1954)

**On the Exponential Solution of Differential Equations for a Linear Operator***

By WILHELM MAGNUS

Wilhelm Magnus

- Advantages:
  - $N = N^\dagger$, unitarity maintained at each order in perturbation theory
  - Faster resummation, particularly useful when Hamiltonian is not small or interaction does not switch off adiabatically
- Cons
  - General solution (e.g. Chen-Strichartz) more involved than Dyson's
  - Leads to advanced/retarded propagators rather than Feynman

- Several applications in physics and chemistry
  - atomic and molecular physics, applications to NMR
  - key in Average Hamiltonian Theory for analysing time-dependent spin dynamics
  - Magnus expansion allows to compute the Floquet effective Hamiltonian (Floquet-Magnus expansion for systems subject to periodic Hamiltonians)

- Connections to mathematics
  - $S$-matrix can be seen as an infinite sequence of steps $e^{-iH_I(t)\Delta t}$
  - Rewriting these as an exponential is the continuous generalisation of the Baker-Campbell-Hausdorff formula
  - naturally gives rise to nested commutators

$$e^X e^Y = \exp\big(X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}[X,[X,Y]] + \tfrac{1}{12}[Y,[Y,X]] - \tfrac{1}{24}[Y,[X,[X,Y]]] + \cdots\big)$$

# Solution (Magnus, 1954)

- Expanding $N = N^{(1)} + N^{(2)} + \cdots$

$$iN^{(1)} = -i \int d^4x_1 \; \mathcal{H}_1 \,,$$

$$iN^{(2)} = \frac{(-i)^2}{2} \frac{1}{\hbar} \int d^4x_1 d^4x_2 \; \theta_{12} \; [\mathcal{H}_1, \mathcal{H}_2] \,,$$

$$iN^{(3)} = \frac{(-i)^3}{6} \frac{1}{\hbar^2} \int d^4x_1 d^4x_2 d^4x_3 \; \theta_{12}\theta_{23} \; ([\mathcal{H}_1, [\mathcal{H}_2, \mathcal{H}_3]] + [\mathcal{H}_3, [\mathcal{H}_2, \mathcal{H}_1]])$$

$$iN^{(4)} = \frac{(-i)^4}{12} \frac{1}{\hbar^3} \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \; \theta_{12}\theta_{23}\theta_{34} \big([\mathcal{H}_1, [\mathcal{H}_2, [\mathcal{H}_3, \mathcal{H}_4]]] + [\mathcal{H}_4, [\mathcal{H}_3, [\mathcal{H}_1, \mathcal{H}_2]]] + [\mathcal{H}_2, [\mathcal{H}_3, [\mathcal{H}_4, \mathcal{H}_1]]] + [\mathcal{H}_1, [\mathcal{H}_4, [\mathcal{H}_3, \mathcal{H}_2]]]\big)$$

- $\theta_{ij} = \theta(t_i - t_j)$, Lorentz invariance less manifest
- $[H_1, [H_2, [\ldots, [H_{n-1}, H_n]\ldots]]]/\hbar^{n-1} \sim O(\hbar^0)$ assuming $[H_i, H_{i+1}] \sim \hbar$
- No hyper-classical terms, hence taking the classical limit is very simple!

# Relativistic invariance

- Second-order term

$$iN^{(2)} = \frac{(-i)^2}{2} \int d^4x_1 d^4x_2 \; \theta_{12} \; [\mathcal{H}_1, \mathcal{H}_2]$$

  - For $x_{12} = x_1 - x_2$ timelike, invariance is manifest
  - the sign of the time component of a timelike vector is Lorentz invariant
  - For $x_{12}$ spacelike we must assume that operators commute $[\mathcal{H}_1, \mathcal{H}_2] = 0$

- $\theta-$functions $\longrightarrow$ retarded/advanced propagators

# Magnum goal

- Compute Magnus amplitudes ($N$-matrix elements):
  - without expressing them in terms of $T$-matrix elements (amplitudes)
  - in a relativistic quantum field theory
  - take the classical limit and apply to gravity $I_r \overset{\text{IPS}}{\longleftrightarrow} \langle p_1' p_2' | N | p_1, p_2 \rangle$
- Toy model:
  - $\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2 - \frac{\lambda}{3!}\phi^3$
- Universality
  - Not so toy! Can express results in a general theory in terms of integrals in a cubic theory
  - Main novelties: changes in $i\varepsilon$ prescriptions, Murua coefficients

# Cast of characters

- Wick theorem $\phi(x_i)\phi(x_j) = :\phi(x_i)\phi(x_j): + \overbrace{\phi(x_i)\phi}(x_j)$
  - Wightman functions $\overbrace{\phi(x_i)\phi}(x_j) = \langle 0|\phi(x_i)\phi(x_j)|0\rangle := i\Delta^{(+)}(x_i - x_j) = i\Delta^{(+)}_{ij}$
  - $\Delta^{(+)}_{ij}(x) = -\,\Delta^{(-)}_{ij}(-x)$
- Commutation/anti-commutation functions:
  - Pauli-Jordan: $i\Delta_{ij} = i\Delta(x_i - x_j) := [\phi(x_i), \phi(x_j)] = i\Delta^{(+)}_{ij} + i\Delta^{(-)}_{ij}$
  - Hadamard: $\Delta^{(1)}_{ij} = \Delta^{(1)}(x_i - x_j) := \{\phi(x_i), \phi(x_j)\} = i\Delta^{(+)}_{ij} - i\Delta^{(-)}_{ij}$
  - Odd/even solutions to the homogeneous Klein-Gordon equation
- Propagators
  - Retarded: $i\Delta^R(x) = i\Delta(x)\,\theta(x_0)$
  - Advanced $i\Delta^A(x) = -\,i\Delta(x)\,\theta(-x_0)$

# Example: five-point tree level

- Goal: compute $i\,N_5^{(3)} = \langle 0\,|\,iN^{(3)}\,|\,\phi(p_1)\cdots\phi(p_5)\rangle$
    - Compute the operator at tree level from Magnus's expansion:

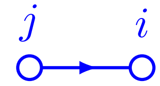
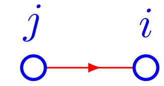
$$iN^{(3)} = \frac{(-i)^3}{6}\int d^4x_1 d^4x_2 d^4x_3\ \theta_{12}\theta_{23}\ \left([\mathcal{H}_1,[\mathcal{H}_2,\mathcal{H}_3]] + [\mathcal{H}_3,[\mathcal{H}_2,\mathcal{H}_1]]\right)$$

    - $N^{(3)} = N_{123} + N_{321}$ with

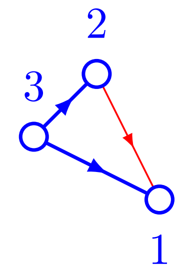
$$iN_{123} = \frac{(-i)^3}{6}\int d^4x_1 d^4x_2 d^4x_3\ (i\Delta^R_{23})\frac{1}{(2!)^2}\Big[(i\Delta^R_{12}) :\phi_1^2\phi_2\phi_3^2: +\theta_{12}(i\Delta^R_{13}) :\phi_1^2\phi_2^2\phi_3:\Big]$$

$$iN_{321} = \frac{(-i)^3}{6}\int d^4x_1 d^4x_2 d^4x_3\ (i\Delta^R_{12})\frac{1}{(2!)^2}\Big[(i\Delta^R_{23}) :\phi_1^2\phi_2\phi_3^2: +\theta_{23}(i\Delta^R_{13}) :\phi_1\phi_2^2\phi_3^2:\Big]$$

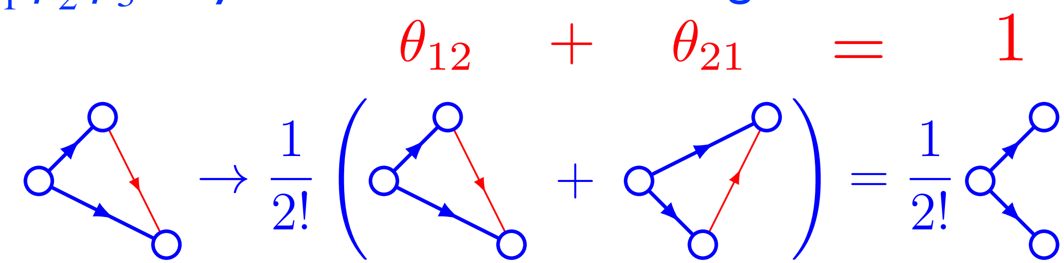
- Unwanted $\theta-$functions
    - Must disappear, by Lorentz invariance
    - They do upon relabelling integration variables!

- Can represent with pictures:  $= i\Delta^R_{ij}$ ,  $= \theta_{ij}$


  - First unwanted term is

$$(i\Delta^R_{23})\,(i\Delta^R_{13})\theta_{12} =$$ 


  - Multiplies $:\phi_1^2\phi_2^2\phi_3:$ , symmetric under exchange $1 \leftrightarrow 2$
  - Can relabel:

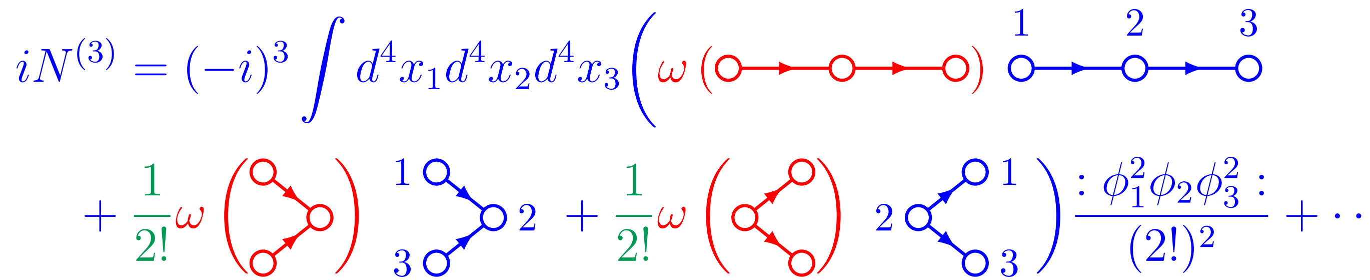



- Result:

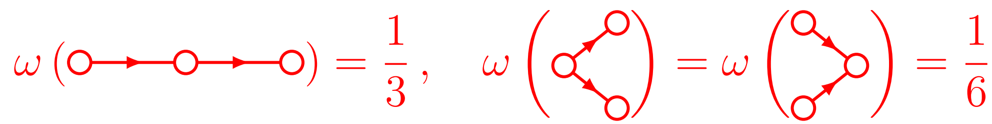
$$iN^{(3)} = (-i)^3 \int d^4x_1 d^4x_2 d^4x_3 \Big[\frac{1}{3}(i\Delta^R_{23})(i\Delta^R_{31}) + \frac{1}{2}\cdot\frac{1}{6}\Big((i\Delta^R_{23})(i\Delta^A_{31}) + (i\Delta^A_{23})(i\Delta^R_{31})\Big)\Big]$$
$$\times\left(\frac{1}{(2!)^2}:\phi_1^2\phi_2^2\phi_3: + \cdots\right)$$

  - Similar relabelling trick led to the introduction of time-ordered products
  - Unusual coefficients $1/3$, $1/6\ldots$what are they?

- **Recast as**




- **Murua coefficients!**




**The Hopf Algebra of Rooted Trees, Free Lie Algebras, and Lie Series**

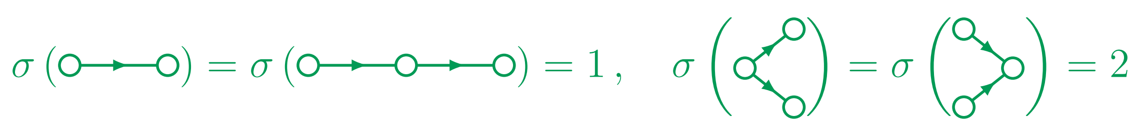

A. Murua
Konputazio Zientziak eta A. A. saila
Informatika Fakultatea
EHU/UPV
Donostia/San Sebastián, Spain
ander@si.ehu.es



**Abstract.** We present an approach that allows performing computations related to the Baker–Campbell–Haussdorff (BCH) formula and its generalizations in an arbitrary Hall basis, using labeled rooted trees. In particular, we provide explicit formulas (given in terms of the structure of certain labeled rooted trees) of the continuous BCH formula. We develop a rewriting algorithm (based on labeled rooted trees) in the dual Poincaré–Birkhoff–Witt (PBW) basis associated to an arbitrary Hall set, that allows handling Lie series, exponentials of Lie series, and related series written in the PBW basis. At the end of the paper we show that our approach is actually based on an explicit description of an epimorphism $\nu$ of Hopf algebras from the commutative Hopf algebra of labeled rooted trees to the shuffle Hopf algebra and its kernel ker $\nu$.


  - Important work of Kim³ & Lee extending to non-rooted trees in WQFT
  - Connected, directed trees $\tau$ appear

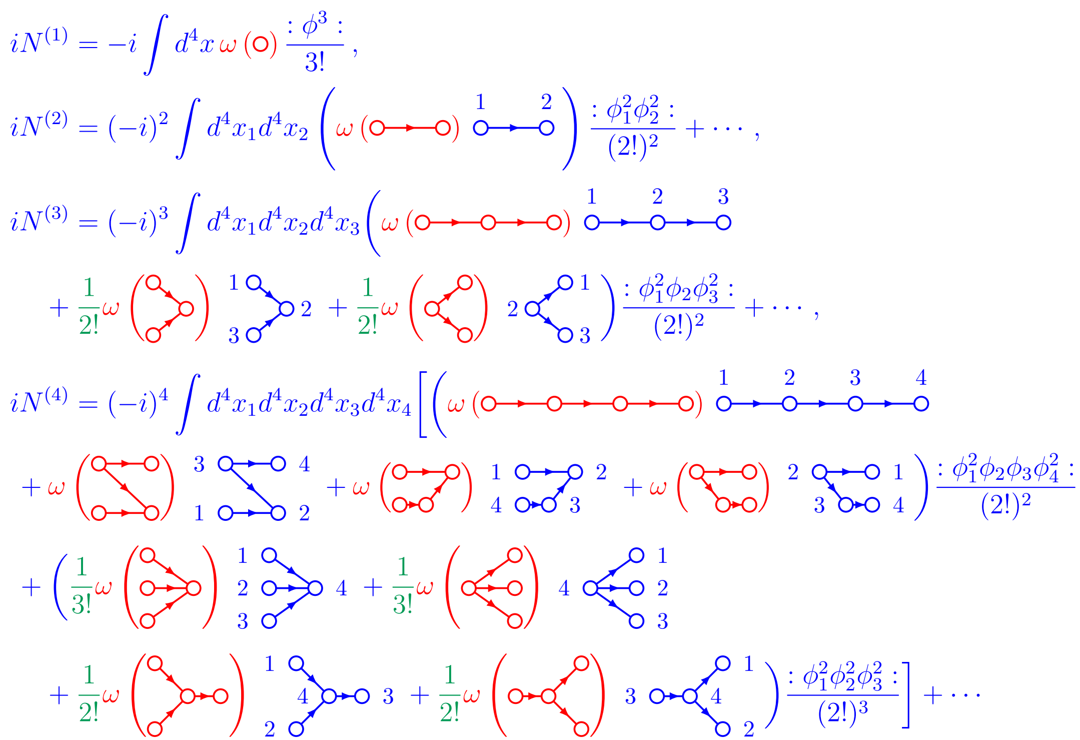
- **Symmetry factors:**



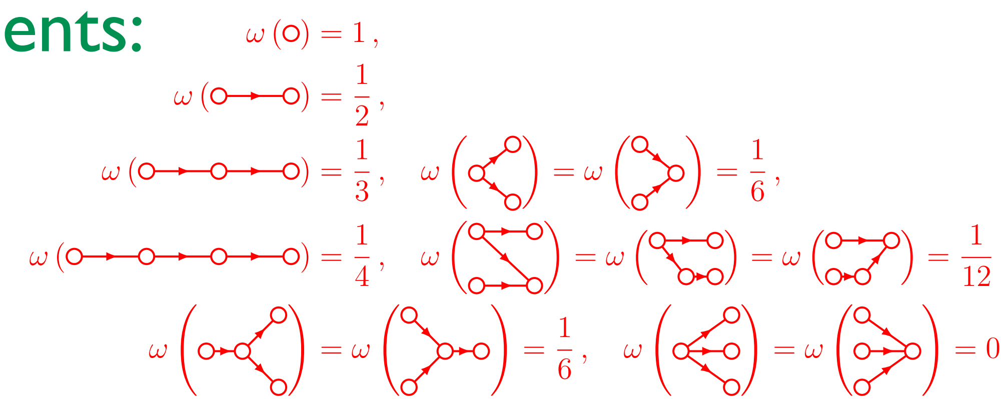

  - $\sigma(\tau)$ = # of permutations of vertices (with the attached edges) that leave $\tau$ invariant (respecting edges's orientations)

# Everything from Murua (tree level)

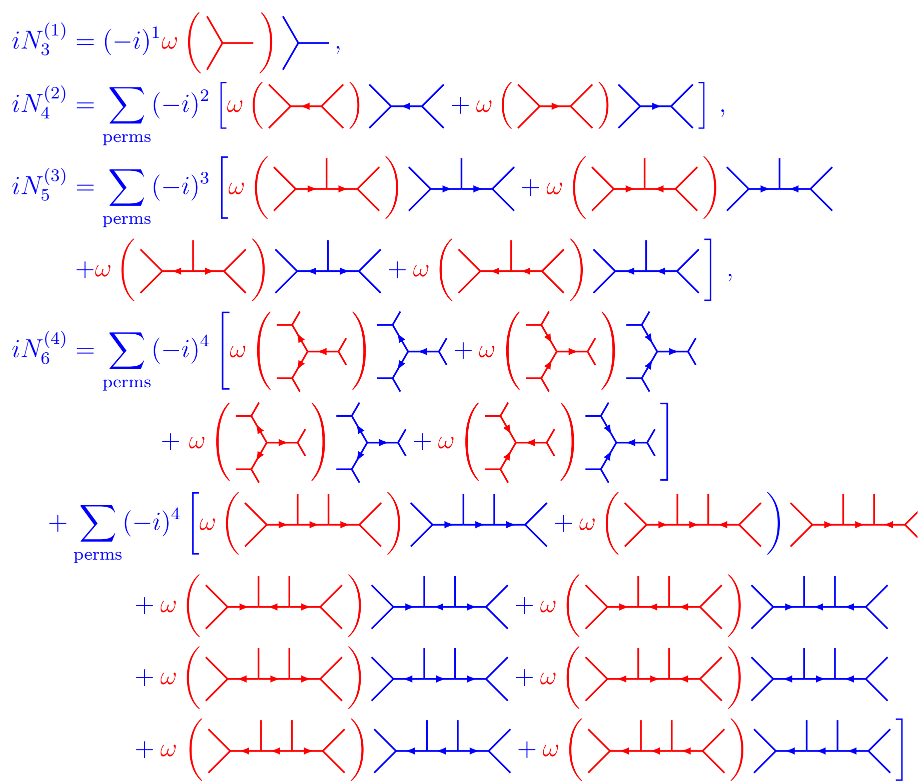
- $N$-operator expansion:



- Murua coefficients:

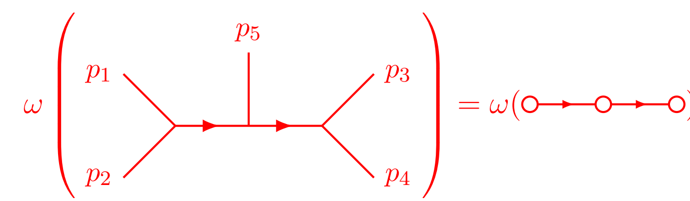


- $N$-matrix elements $\langle 0|N|\phi(p_1)\cdots\phi(p_n)\rangle$ (Magnus amplitudes)

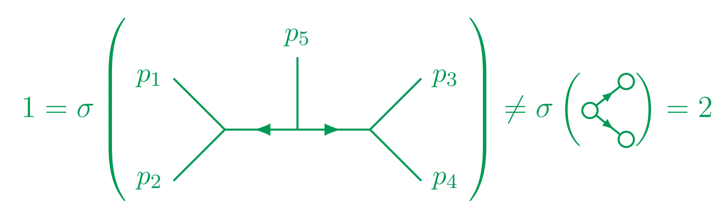



sum over the set of inequivalent permutations of the external momenta (each graph with external momenta should not appear more than once)

- Amplitudes Murua coefficients same as for $N$-operator (depend only on internal propagators). E.g.



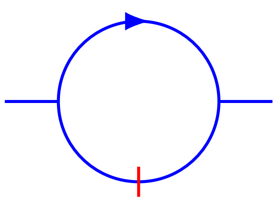

- Symmetry factor for amplitudes all equal to $1$ at tree level (different than $N$-operator. Same as for $i\,T$ vs matrix elements of $i\,T$ ). E.g.:



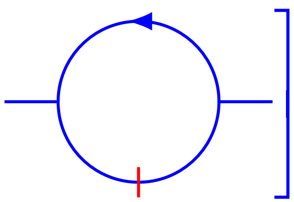
# One-loop

- **Example:**
  - Tree level: $(\Delta_R)^{n-1}$ with $n$ vertices
  - One loop: $(\Delta_R)^{n-1}\Delta^{(1)}$ with $n$ vertices
  - Bubble $\quad iN_2^{(2)} = \hat{\delta}^{(4)}(p_1+p_2)\cdot(-i)^2\cdot\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\Delta^{(1)}(q)\times\frac{1}{2}\left(i\Delta^R+i\Delta^A\right)(q-p_1)$

$$= \hat{\delta}^{(4)}(p_1+p_2)\frac{(-i)^2}{2}\left[\frac{1}{2}\right.$$

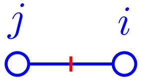


$$+\ \frac{1}{2}$$



$$\left.\phantom{\frac{1}{2}}\right]$$

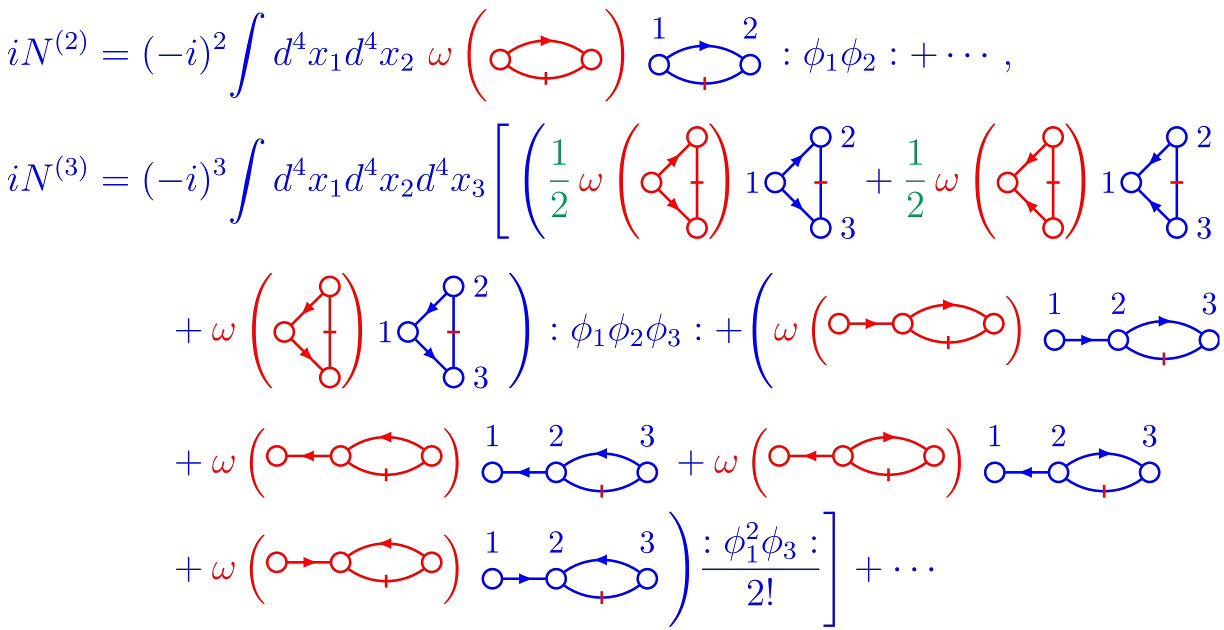
  - Hadamard function is a cut




$$= \Delta_{ij}^{(1)} = \int\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x_i-x_j)}\,(2\pi)\delta(k^2-m^2)$$

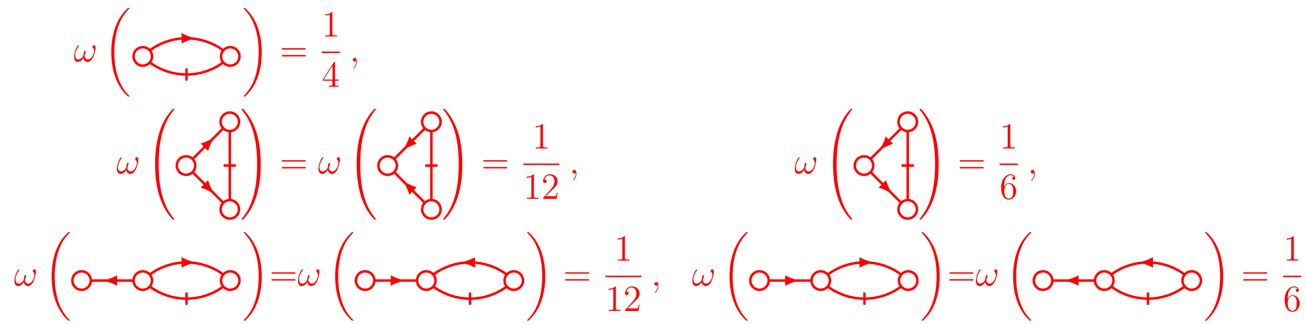
# Everything at one loop

- $N$-operator:

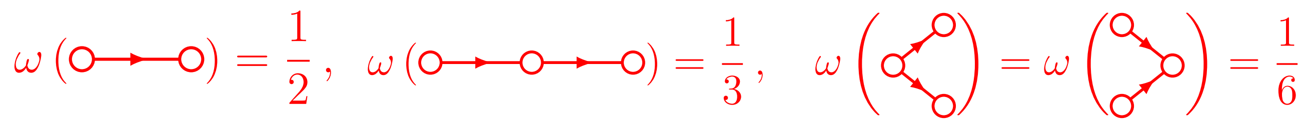



- Key feature: all diagrams have one cut line: loops from trees
  - Similar in spirit (but different from) Feynman tree theorem (Feynman, 1963) and the loop-tree duality (Catani, Gleisber, Krauss, Rodrigo, Winter)

- One-loop Murua coefficients (from Wick contractions):

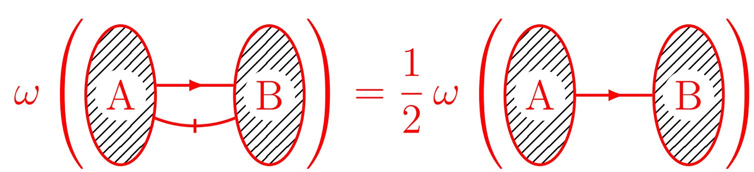



  - Cfr tree level:




- One loop Murua = $1/2$ tree Murua!

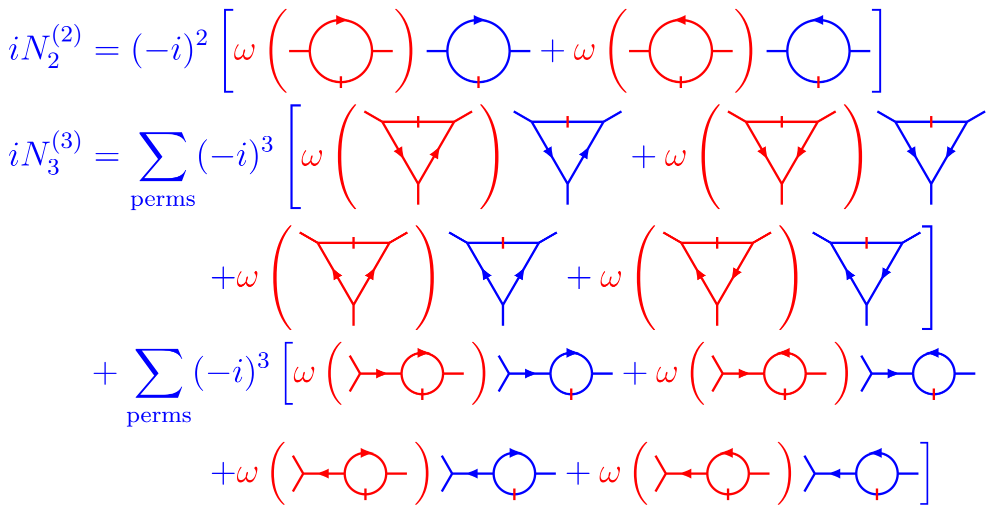
  - General relation:



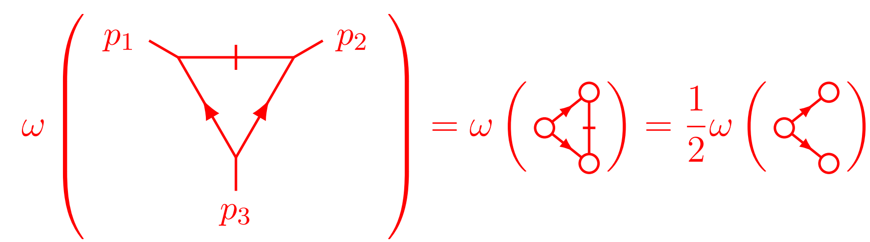

!!!!

  - Factor of $2$ from definition of $\Delta^{(1)}(k) = 2\pi\,\delta(k^2 - m^2)$

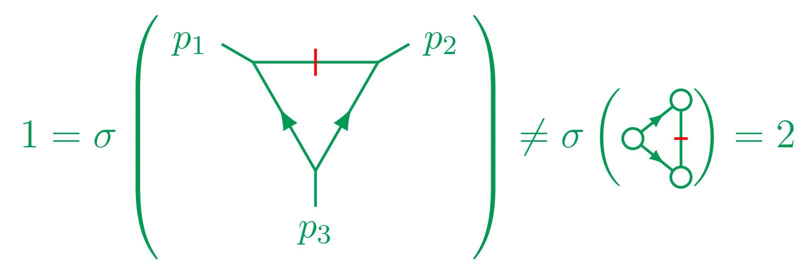
# One-loop Magnus amplitudes:




- As at tree level, amplitudes's Murua coefficients same as for $N$-operator, i.e. coefficients only depend on the internal propagator structure and external lines can be neglected. E.g.



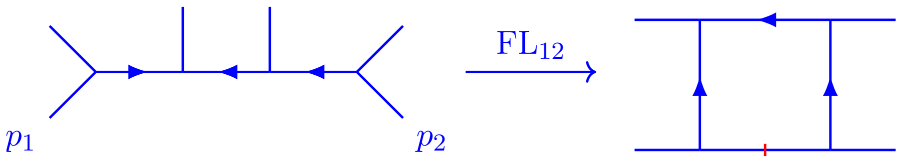

- Symmetry factors not the same as for operator (as a tree level):



- One-loop Magnus amplitudes from forward limits of trees!

$$\mathcal{A}^{(1)} = \frac{1}{4}\int \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2-m^2)\mathcal{A}^{(0)}_{\mathrm{FL}}$$

- Example:

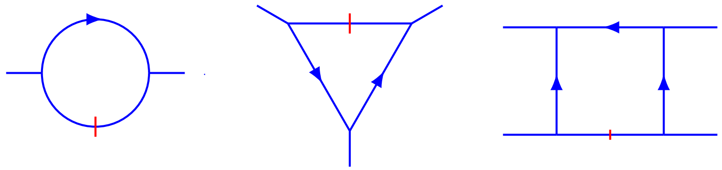


- General structure of Wick contractions:
  - Wick-contract $\phi_k$ with $\phi_{k'}$ within $\mathcal{C}_{1\ldots V} = \Big[\mathcal{H}_1, \big[\mathcal{H}_2, \cdots [\mathcal{H}_{V-1}, \mathcal{H}_V] \cdots \big]\Big]$

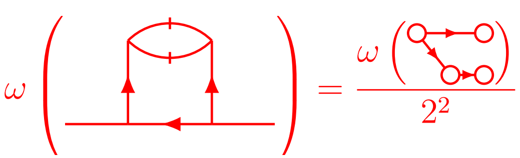
$$\mathcal{C}_{1\ldots V} \to \frac{n_{kk'}}{2} i\Delta_{kk'} \Big[\mathcal{H}_1, \Big[\cdots \Big\{\mathcal{H}_k^{(1)}, \Big[\cdots [\mathcal{H}_{k'}^{(1)}, \cdots [\mathcal{H}_{V-1}, \mathcal{H}_V] \cdots ] \cdots \Big] \cdots \Big\} \cdots \Big] \cdots \Big]$$
$$+ \frac{n_{kk'}}{2} \Delta_{kk'}^{(1)} \Big[\mathcal{H}_1, \Big[\cdots \Big[\mathcal{H}_k^{(1)}, \Big[\cdots [\mathcal{H}_{k'}^{(1)}, \cdots [\mathcal{H}_{V-1}, \mathcal{H}_V] \cdots ] \cdots \Big] \cdots \Big] \cdots \Big] \cdots \Big]$$

- Key observations
  - $\Delta$ flips bracket $[\,\cdot\,]$ into $\{\,\cdot\,\}$, $\Delta^{(1)}$ preserves it, and
  - $:[\,\cdot\,]:= 0$ !
  - $V$ vertices: $V-1$ commutators and $L = I - V + 1$ ($I$=# internal lines, $L$ =# loops)
- Tree level redone:
  - $V-1$ contractions must flip $V-1$ commutators hence $\Delta^{V-1}(\Delta^{(1)})^0$

- One loop redone:
  - $V$ contractions must flip
    $V-1$ commutators hence $\Delta^{V-1}(\Delta^{(1)})^1$



- Higher loops:
  - $L = 2m$ loops: has $2m, 2m-2, \ldots, 0$ cuts $\Delta^{(1)}$ (even $L \to$ even # cuts)
  - $L = 2m+1$ loops has $2m+1, 2m-1, \ldots, 1$ cuts $\Delta^{(1)}$ (odd $L \to$ odd # cuts)
  - max number of cuts is the number of loops, $L$
  - $L$-loop $L$-cut diagram Murua coefficients from tree level! $\omega(\tau^{(L,L)}) = \frac{\omega(\tau^{(0,0)})}{2^L}$
  - Example:

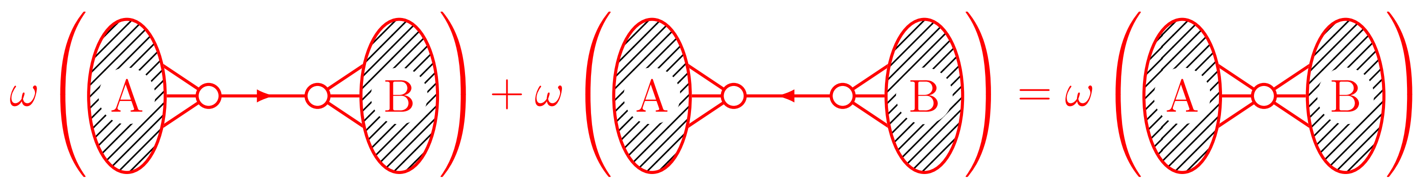


$$= \frac{\frac{1}{12}}{4} = \frac{1}{48}$$

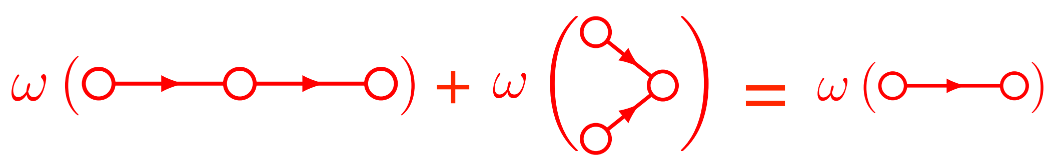
  - lower number of cuts from Wick contraction

# Relations among Murua coefficients

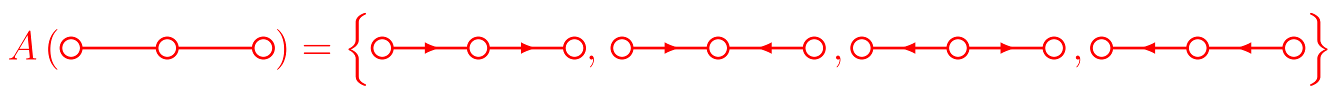
- Edge contraction rule
  - the sum of two graphs which differ only by a single internal edge, where that edge contains both routings of propagators, equals the “collapsed"graph where that edge is deleted:

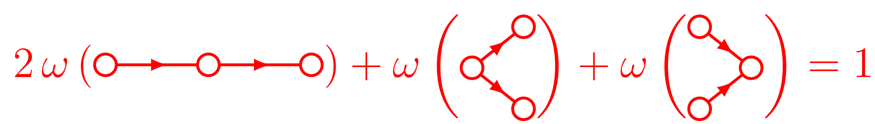



  - Example:




$1/3 \quad + \quad 1/6 \quad = \quad 1/2$

- ## Sum rule: $\sum_{\tau \in A(\rho)} \omega(\tau) = 1$
  - $A(\rho)$ = all directed graphs generated by applying all possible retarded and advanced arrows to the edges of an undirected graph $\rho$
  - Example: for 

    we get 

- ## From Magnus to Feynman:
  - Consequence of sum rule: at tree level, ignoring causality prescriptions (the $i\varepsilon$'s), $N$-matrix elements agree with connected $T$-matrix elements

# From Feynman to Magnus!

- Strategy:
  - 1. Derive Murua coefficients from various relations/codes (or Wick contractions when other routes not available)
  - 2. From each Feynman diagram, generate Magnus amplitudes by changing $i\varepsilon$'s and multiplying by Murua coefficients
  - 3. Compute integrals with retarded/advanced propagators (with Henn's differential equation approach, this amounts to changing boundary conditions)

# On to the classical limit!

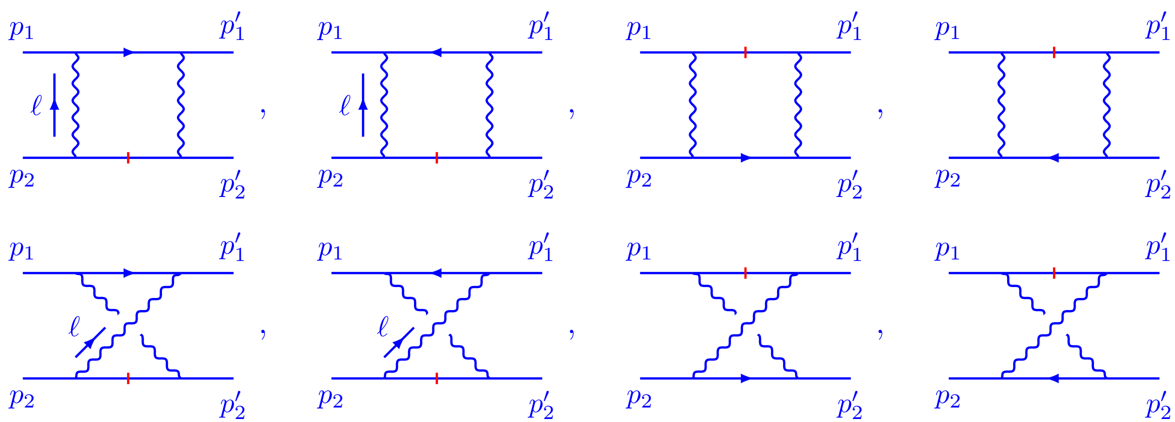
- So far everything applied to the full quantum theory
  - Now take $\hbar \to 0$ limit and find which diagrams survive
- Conjecture
  - Only diagrams with $L$ cuts at $L$ loops survive in the classical limit
  - Can all be obtained from trees:

$$\mathcal{A}^{(L,L)} = \frac{1}{\ell!\,2^{2L}} \int \prod_{k=1}^{L} \left[ \frac{d^D p_k}{(2\pi)^D} (2\pi)\delta(p_k^2 - m^2) \right] \mathcal{A}^{(0,0)}_{L-\mathrm{FL}}$$

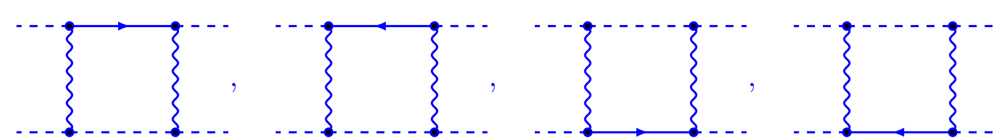
  - Classical $N$-matrix from tree $N$-matrix!
- Comments
  - Parallels the WQFT (Kälin, Porto; Mogull, Plefka, Steinhoff) where all diagrams are trees
  - Connection to the Heavy-mass EFT (HEFT) expansion (for those who know it!) (Brandhuber, Chen, GT, Wen) but details to be understood

# One-loop example

- Toy "gravity" model $S_{\rm QFT} = \int d^4x \left( \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 + \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \frac{\lambda}{2}\phi^2 h \right)$
  - Same scalar integrals as GR
  - WQFT counterpart: $S_{\rm WQFT} = \frac{1}{2}\int d^4x (\partial_\mu h)(\partial^\mu h) - \frac{m}{2}\int d\tau \left(\dot{x}^2 + \lambda h(x)\right)$
    (Capatti, Zeng)
  - Radial action for 2-to-2 scattering at $O(\lambda^2)$:




  - Radial action from WQFT:



  - Results in agreement, hence correct classical limit

# Conclusions

- Exponential representation of the $S$-matrix, $S = e^{iN}$
  - Free of hyper-classical terms
  - Radial action from $N$-matrix element
- Magnus expansion for the $N$-operator
  - Murua coefficients, advanced, retarded and cut propagators
  - Relations intertwining loop orders, classical limit from just trees?
- What's next?
  - Classical limit for GR, connection to Worldline QFT (WQFT), spin (Magnusian?), gravitational wave emission
  - Further applications (not only) to gravity
  - Anomalous dimensions & IR divergences in QFT….
  - …Wilson lines/loops….integrability…?

Thank you!

# Extra slides

# Closed-form solution

- Chen-Strichartz formula (Strichartz, 1987)

$$iN = \sum_{n\geq 1}\sum_{\sigma\in S_n} \frac{1}{\hbar^{n-1}} \frac{(-1)^{d_\sigma}}{n^2\binom{n-1}{d_\sigma}} \int d^4x_1\cdots d^4x_n\ \theta_{12}\theta_{23}\cdots\theta_{n-1\,n}$$
$$\left[\mathcal{H}_{\sigma(1)},\left[\mathcal{H}_{\sigma(2)},\cdots\left[\mathcal{H}_{\sigma(n-1)},\mathcal{H}_{\sigma(n)}\right]\cdots\right]\right]$$

  - $d_\sigma = |D_\sigma|$ descent of the permutation $\sigma$, where
  - $D_\sigma = \left\{ i \mid \sigma(i) > \sigma(i+1), 1 \leq i < n \right\}$ is the "descent set"
  - Example: for the set $\{1,2,3,4\}$,

    $D_{4213} = \{1,2\}\,,\quad D_{3124} = \{1\}\,,\quad D_{4321} = \{1,2,3\}\,,\quad D_{2413} = \{2\}$ and

    $d_{4213} = 2,\qquad d_{3124} = 1\,,\qquad d_{4321} = 3\,,\qquad d_{2413} = 1$
  - CS formula contains $n!$ terms

## New formula:

$$iN^{(n)} = \int d^4x_1 \cdots d^4x_n \Big[\mathcal{H}_1, \big[\mathcal{H}_2, \cdots [\mathcal{H}_{n-1}, \mathcal{H}_n] \cdots \big]\Big] \sum_{D \subseteq \{1,\ldots,n-1\}} \frac{(-1)^{|D|}}{n^2 \binom{n-1}{|D|}} \Theta_D$$

$$\Theta_D = \sum_{\sigma \in S_n^D} \sigma^{-1}(\theta_{12} \cdots \theta_{n-1\ n}) = \Big(\prod_{i \in D} \theta_{i+1\ i}\Big)\Big(\prod_{j \in A} \theta_{j\ j+1}\Big)$$

- sum over the set of permutations of $n$ objects with descent set $D = \{i_1, \ldots, i_k\}$ (and hence with ascent set $A = \{j_1, \ldots, j_{(n-1)-k}\}$, such that $A \cup D = \{1, \ldots, n-1\}$
- Example: for the set $\{1,2,3,4\}$:

| $d$ | $D_\sigma$ | $\sigma$ | $\Theta_D$ |
|---|---|---|---|
| 0 | $\{\}$ | 1234 | $\theta_{12}\theta_{23}\theta_{34}$ |
| | $\{1\}$ | 2134, 3124, 4123 | $\theta_{21}\theta_{23}\theta_{34}$ |
| 1 | $\{2\}$ | 1324, 1423, 2314, 2413, 3412 | $\theta_{12}\theta_{32}\theta_{34}$ |
| | $\{3\}$ | 1243, 1342, 2341 | $\theta_{12}\theta_{23}\theta_{43}$ |
| | $\{1,2\}$ | 3214, 4213, 4312 | $\theta_{21}\theta_{32}\theta_{34}$ |
| 2 | $\{1,3\}$ | 2143, 3142, 3241, 4132, 4231 | $\theta_{21}\theta_{23}\theta_{43}$ |
| | $\{2,3\}$ | 1432, 2431, 3421 | $\theta_{12}\theta_{32}\theta_{43}$ |
| 3 | $\{1,2,3\}$ | 4321 | $\theta_{21}\theta_{32}\theta_{43}$ |

## Comments

- Only one ordering of Hamiltonians appear
- Sum over descent sets, not permutations
- Fewer terms ($2^{n-1}$ compared to $n!$ of the CS formula)

# More on Lorentz invariance

- Third-order term $iN^{(3)} = \frac{(-i)^3}{6}\int d^4x_1 d^4x_2 d^4x_3\ \theta_{12}\theta_{23}\theta_{13}\ ([\mathcal{H}_1,[\mathcal{H}_2,\mathcal{H}_3]] + [\mathcal{H}_3,[\mathcal{H}_2,\mathcal{H}_1]])$
  - For $x_{12}$ and $x_{23}$ timelike ($x_{13}$ automatically timelike since $x_{12}$ and $x_{23}$ are future pointing): invariance is manifest
  - For $x_{12}$ spacelike and $x_{23}$ timelike: first use Jacobi

$$[\mathcal{H}_1,[\mathcal{H}_2,\mathcal{H}_3]] + \underbrace{[\mathcal{H}_3,[\mathcal{H}_2,\mathcal{H}_1]]}_{0} = [\mathcal{H}_1,[\mathcal{H}_2,\mathcal{H}_3]] = -[\mathcal{H}_2,[\mathcal{H}_3,\mathcal{H}_1]] - \underbrace{[\mathcal{H}_3,[\mathcal{H}_1,\mathcal{H}_2]]}_{0} = -[\mathcal{H}_2,[\mathcal{H}_3,\mathcal{H}_1]]$$

  - Then relabel integration variables

$$\theta_{12}\theta_{23}\theta_{13}[\mathcal{H}_2,[\mathcal{H}_3,\mathcal{H}_1]] \to \frac{1}{2}\theta_{23}\theta_{13}\Big(\theta_{12}[\mathcal{H}_2,[\mathcal{H}_3,\mathcal{H}_1]] + \theta_{21}[\mathcal{H}_1,[\mathcal{H}_3,\mathcal{H}_2]]\Big) = \frac{1}{2}\theta_{23}\theta_{13}[\mathcal{H}_2,[\mathcal{H}_3,\mathcal{H}_1]]$$

Symmetrise by relabelling integration variables $x_1 \leftrightarrow x_2$

  - For $x_{13}$ timelike, invariance is manifest. For $x_{13}$ spacelike $[\mathcal{H}_3,\mathcal{H}_1] = 0$
  - Finally, when $x_{12}, x_{23}, x_{13}$ are all spacelike, all commutators vanish